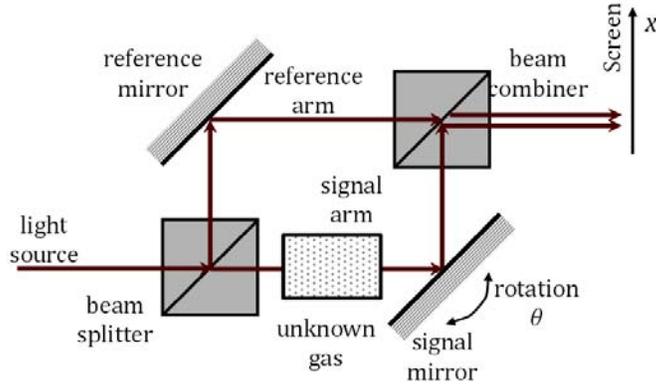## 1. Mach-Zehnder Interferometer

a) In a Mach-Zehnder interferometer, the light source is split into two beams ($E_1 = E\,e^{i\varphi_1} = E\,e^{ik_0 \text{OPL}_1}e^{i\omega t}$,

$E_2 = E\,e^{i\varphi_2} = E\,e^{ik_0 \text{OPL}_2}e^{i\omega t}$). Since the two beams come from the same source, they have equal wavelength λ

(therefore equal wavenumber $k_0 = 2\pi/\lambda$) and equal time dependent phase $e^{i\omega t}$. The magnitude of the fields do

not change the interference pattern except for its brightness, and thus can be set as a constant value (E).

The two beams go through two different light paths with a slightly different optical path length, and then they are joined in the end to produce an interference pattern on the screen. The interference pattern on the screen can be controlled by the rotation of the signal mirror and by a gradual change of the refractive index inside the pressurized cell.

In this problem, the brightness change on the screen is produced by the gradual pressurization of the gas cell, and all measurements are taken at x=0 on the screen. A difference in the optical phase between the two beams arriving at x=0 can be calculated from:

$$\Delta\varphi = \varphi_2 - \varphi_1$$
$$= k_0(\text{OPL}_2 - \text{OPL}_1)$$
$$= k_2 \left(\int n_2\, dl_2 - \int n_1\, dl_1 \right)$$
$$= \tfrac{2\pi}{\lambda} L(n_2 - n_1).$$

When the cell is evacuated, $n_2 = n_1 = 1$ and there is no phase difference between the two signals, resulting in a constructive interference between two beams (brightness). As the cell is being pressurized, 10 brightness cycles were recorded (m=10). When the cell is fully pressurized, the refractive index reaches *n,* and the phase difference is equal to $\Delta\varphi = \tfrac{2\pi}{\lambda} L(n-1) = 2\pi \cdot m$. Therefore, the refractive index can be calculated from:

$$\Delta\varphi = \tfrac{2\pi}{\lambda} L(n-1) = 20\pi$$
$$\therefore n = 1 + \tfrac{m\lambda}{L} = 1.0000633$$

b) The interference pattern for λ_1 and λ_2 can be considered separately, since two beams with distinct wavelengths are independent and therefore do not interfere with each other. (Recall that two sinusoidal function are orthogonal when the period is not exactly equal.) For each wavelength, the interference pattern can be computed from:

$$\begin{aligned}
I_\lambda &= |E_1 + E_2|^2 = (E_1 + E_2)\cdot(E_1{}^* + E_2{}^*) \\
&= E^2(e^{i\varphi_1} + e^{i(\varphi_1+\Delta\varphi)})\cdot(e^{-i\varphi_1} + e^{-i(\varphi_1+\Delta\varphi)}) \\
&= E^2(1 + e^{i\Delta\varphi})\cdot(1 + e^{-i\Delta\varphi}) \\
&= E^2(2 + 2\cos(\Delta\varphi)).
\end{aligned}$$

Using the relationship $\Delta\varphi = \frac{2\pi}{\lambda}L(n-1)$, we obtain:

$$\begin{aligned}
I_1 &= E^2(2 + 2\cos(\tfrac{2\pi}{\lambda_1}L(n-1))) \\
I_2 &= E^2(2 + 2\cos(\tfrac{2\pi}{\lambda_2}L(n-1))).
\end{aligned}$$

Recall that the variable in the above equation is *n*, since the gas cell is being pressurized in time. The total intensity is calculated by summing the two intensities $I_{\lambda_1} + I_{\lambda_2}$:

$$\begin{aligned}
I_{tot} = I_{\lambda_1} + I_{\lambda_2} &= 2E^2\left[2 + \cos\left(2\pi L(n-1)\tfrac{1}{\lambda_1}\right) + \cos\left(2\pi L(n-1)\tfrac{1}{\lambda_2}\right)\right] \\
&= I_0\left[1 + \cos\left(2\pi L(n-1)(\tfrac{1}{\lambda_1}+\tfrac{1}{\lambda_2})\right)\cos\left(2\pi L(n-1)(\tfrac{1}{\lambda_1}-\tfrac{1}{\lambda_2})\right)\right] \\
&= I_0\left[1 + \cos\left(2\pi L(n-1)\tfrac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}\right)\cos\left(2\pi L(n-1)\tfrac{\lambda_2-\lambda_1}{\lambda_1\lambda_2}\right)\right].
\end{aligned}$$

The result is a beating interference: a high frequency oscillation, $\cos\left(2\pi L(n-1)\frac{\lambda_1+\lambda_2}{\lambda_1\lambda_2}\right)$, is modulated by a slowly varying envelope, $\cos\left(2\pi L(n-1)\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2}\right)$. The plot of I_{tot} as a function of n, which is a function of time, looks like the following figure.

Image removed due to copyright restrictions. Image found at http://hyperphysics.phy-astr.gsu.edu/hbase/sound/imgsou/beat4.gif.

How a beat is formed (http://hyperphysics.phy-astr.gsu.edu/hbase/sound/beat.html)

Thus, the intensity at x=0 on the screen varies periodically in time as the cell is gradually pressured. Since the change in the refractive index is very small, it is likely that only the high frequency oscillation can be counted, rather than the envelope. The number of cycles for the high frequency oscillation can be counted to measure the total change in the optical phase:

$$\Delta\varphi = 2\pi L(n-1)\frac{\lambda_2-\lambda_1}{\lambda_1\lambda_2} = 2\pi m$$

$$\therefore n = 1 + \frac{1}{L}\frac{\lambda_1\lambda_2}{\Delta\lambda}m \cong 1 + \frac{\bar{\lambda}^2}{L\Delta\lambda}m.$$

The accuracy of the measurement can be determined by how much index difference is measurable from a single brightness cycle (m=1), which gives

$$\Delta n = \frac{\bar{\lambda}^2}{L\Delta\lambda} \ .$$

b) Alternate interpretation for the calculation of accuracy

Let us consider a comparison between two separate interference measurements with λ_1 and λ_2. While this situation is physically different from the first interpretation and is also hard to implement in reality, the accuracy of the refractive index measurement can be derived in a different way. For the same change in the refractive index, two different number of cycles can be measured for λ_1 and λ_2.

$$m_1 = L(n-1)\frac{1}{\lambda_1}$$
$$m_2 = L(n-1)\frac{1}{\lambda_2}$$

The smallest change in the refractive index can be measured when m1 and m2 differ by one:

$$\Delta m = m_2 - m_1 = L(n-1)\left(\frac{1}{\lambda_2} - \frac{1}{\lambda_1}\right) = L(n-1)\left(\frac{\lambda_1 - \lambda_2}{\lambda_2 \lambda_1}\right)$$
$$\Delta m = 1 \rightarrow \Delta n = \left(\frac{\bar{\lambda}^2}{L\Delta\lambda}\right)$$

2. Fabry-Perot Interferometer

The relationship between the frequency and the wavelength is $f = c/\lambda$, where c is the speed of light. Using $\lambda_1 = 435.8\ nm, \lambda_2 = 546.1\ nm, c = 3\cdot 10^8 m/s$, we get:

$$f_1 = \frac{3\cdot 10^8}{435.8\cdot 10^{-9}} = 6.9\cdot 10^{14}\ \text{Hz},$$

$$f_2 = \frac{3\cdot 10^8}{546.1\cdot 10^{-9}} = 5.5\cdot 10^{14}\ \text{Hz}.$$

The thickness d of the etalon can now be calculated from:

$$2d = m\lambda$$

$$m_1 = \frac{2d}{\lambda_1}$$

$$m_2 = \frac{2d}{\lambda_2} = m_1 - 150$$

$$d = \frac{150}{2}\left(\frac{1}{435.8nm} - \frac{1}{546.1nm}\right)^{-1} = 161.8\mu m.$$

Problem 3. Fourier Transforms

I assume here that each function is defined as follows (**Section 2.1.6 in Goodman**):

$$\text{rect}(x) = \begin{cases} 1 & |x| < \tfrac{1}{2} \\ \tfrac{1}{2} & |x| = \tfrac{1}{2} \\ 0 & \textit{otherwise} \end{cases} \qquad \text{circ}(\sqrt{x^2 + y^2}) = \begin{cases} 1 & \sqrt{x^2 + y^2} < 1 \\ \tfrac{1}{2} & \sqrt{x^2 + y^2} = 1 \\ 0 & \textit{otherwise} \end{cases}$$

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \qquad \text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$$

Part (a)

Separability Theorem: $g(x) \cdot h(y) \xleftrightarrow{\;F\;} G(f_X) \cdot H(f_Y)$ **Eq. 2-21 in Goodman**

Scaling Theorem: $g(ax) \xleftrightarrow{\;F\;} \dfrac{1}{|a|} G\left(\dfrac{f_X}{a}\right)$ **Eq. 2-12 in Goodman**

FT Pair: $\text{rect}(x) \xleftrightarrow{\;F\;} \text{sinc}(f_X)$ **Table 2.1 in Goodman**

<u>Apply</u>

$$g(x,y) = \text{rect}\left(\frac{x}{2W}\right) \cdot \text{rect}\left(\frac{y}{2W}\right)$$

$$g(x,y) = \text{rect}(ax) \cdot \text{rect}(ay) \qquad\qquad \text{where } a = \frac{1}{2W}$$

$$g(x,y) = h(ax) \cdot h(ay) \qquad\qquad \text{where } h(x) = \text{rect}(x)$$

$$G(f_X, f_Y) = \frac{1}{a^2} H\left(\frac{f_X}{a}\right) \cdot K\left(\frac{f_Y}{a}\right) \qquad \text{Separability \& Scaling Theorems}$$

$$G(f_X, f_Y) = \frac{1}{a^2} \text{sinc}\left(\frac{f_X}{a}\right) \cdot \text{sinc}\left(\frac{f_Y}{a}\right) \qquad \text{FT Pair}$$

$$G(f_X, f_Y) = 4W^2 \text{sinc}(2Wf_X) \cdot \text{sinc}(2Wf_Y)$$

$$\boxed{\text{rect}\left(\frac{x}{2W}\right) \cdot \text{rect}\left(\frac{y}{2W}\right) \xleftrightarrow{\;F\;} 4W^2 \cdot \text{sinc}(2Wf_X) \cdot \text{sinc}(2Wf_Y)}$$

Part (b)

Separability Theorem: $g(x) \cdot h(y) \xleftrightarrow{\;F\;} G(f_X) \cdot H(f_Y)$ **Eq. 2-21 in Goodman**

Scaling Theorem: $g(ax) \xleftrightarrow{\;F\;} \dfrac{1}{|a|} G\!\left(\dfrac{f_X}{a}\right)$ **Eq. 2-12 in Goodman**

FT Pair: $\exp(-\pi x^2) \xleftrightarrow{\;F\;} \exp(-\pi f_X^2)$ **Table 2.1 in Goodman**

Apply

$$g(x,y) = \exp\!\left(-\frac{x^2 + y^2}{2W^2}\right)$$

$$g(x,y) = \exp\!\left(-\frac{x^2}{2W^2}\right) \cdot \exp\!\left(-\frac{y^2}{2W^2}\right)$$

$$g(x,y) = \exp\!\left(-\left(\frac{x}{\sqrt{2}W}\right)^2\right) \cdot \exp\!\left(-\left(\frac{y}{\sqrt{2}W}\right)^2\right)$$

$$g(x,y) = \exp\!\left(-\pi\left(\frac{x}{\sqrt{2\pi}W}\right)^2\right) \cdot \exp\!\left(-\pi\left(\frac{y}{\sqrt{2\pi}W}\right)^2\right)$$

$$g(x,y) = \exp\!\left(-\pi(ax)^2\right) \cdot \exp\!\left(-\pi(ay)^2\right) \qquad \text{where } a = \frac{1}{\sqrt{2\pi}W}$$

$$g(x,y) = h(ax) \cdot h(ay) \qquad\qquad \text{where } h(x) = \exp\!\left(-\pi x^2\right)$$

$$G(f_X,f_Y) = \frac{1}{|a|} H\!\left(\frac{f_X}{a}\right) \cdot \frac{1}{|a|} H\!\left(\frac{f_Y}{a}\right) \qquad \text{Separability \& Scaling Theorems}$$

$$G(f_X,f_Y) = \frac{1}{a^2} H\!\left(\frac{f_X}{a}\right) \cdot H\!\left(\frac{f_Y}{a}\right)$$

$$G(f_X,f_Y) = \frac{1}{a^2} \exp\!\left(-\pi\left(\frac{f_X}{a}\right)^2\right) \cdot \exp\!\left(-\pi\left(\frac{f_Y}{a}\right)^2\right) \qquad \text{FT Pair}$$

$$G(f_X,f_Y) = \frac{1}{a^2} \exp\!\left(-\pi\frac{f_X^2 + f_Y^2}{a^2}\right)$$

$$G(f_X,f_Y) = 2\pi W^2 \exp\!\left(-2\pi^2 W^2 (f_X^2 + f_Y^2)\right)$$

$$\boxed{\exp\!\left(-\frac{x^2 + y^2}{2W^2}\right) \xleftrightarrow{\;F\;} 2\pi W^2 \exp\!\left(-2\pi^2 W^2 (f_X^2 + f_Y^2)\right)}$$

Part (c)

Scaling Theorem for Circular Symmetry:

$$g(ar) \xleftrightarrow{F} \frac{1}{a^2} G\left(\frac{\rho}{a}\right) \qquad \text{Eq. 2-34 in Goodman}$$

$$\text{where } r = \sqrt{x^2 + y^2} \text{ and } \rho = \sqrt{f_X^2 + f_Y^2}$$

FT Pair: $\text{circ}(r) \xleftrightarrow{F} \dfrac{J_1(2\pi\rho)}{\rho}$ **Eq. 2-35 in Goodman**

where $r = \sqrt{x^2 + y^2}$, $\rho = \sqrt{f_X^2 + f_Y^2}$, and J_1 is a Bessel function of the 1st kind, order 1

<u>Apply</u>

$$g(x,y) = \text{circ}\left(\frac{\sqrt{x^2 + y^2}}{W}\right)$$

$$g(r) = \text{circ}\left(\frac{r}{W}\right) \qquad \text{where } r = \sqrt{x^2 + y^2}$$

$$g(r) = \text{circ}(ar) \qquad \text{where } a = \frac{1}{W}$$

$$G(\rho) = \frac{1}{a^2} G\left(\frac{\rho}{a}\right) \qquad \text{Scaling Theorem}$$

$$G(\rho) = \frac{1}{a^2} \frac{J_1(2\pi\frac{\rho}{a})}{\rho/a} \qquad \text{FT Pair}$$

$$G(\rho) = \frac{1}{a} \frac{J_1(2\pi\frac{\rho}{a})}{\rho}$$

$$G(\rho) = W \frac{J_1(2\pi W\rho)}{\rho}$$

$$\boxed{\text{circ}\left(\frac{\sqrt{x^2 + y^2}}{W}\right) \xleftrightarrow{F} W\frac{J_1(2\pi W\rho)}{\rho} = W\frac{J_1\left(2\pi W \sqrt{f_X^2 + f_Y^2}\right)}{\sqrt{f_X^2 + f_Y^2}}}$$

Part (d)

Separability Theorem: $g(x) \cdot h(y) \xleftrightarrow{F} G(f_X) \cdot H(f_Y)$ **Eq. 2-21 in Goodman**

Scaling Theorem: $g(ax) \xleftrightarrow{F} \dfrac{1}{|a|} G\left(\dfrac{f_X}{a}\right)$ **Eq. 2-12 in Goodman**

Convolution Theorem: $g(x,y) \otimes h(x,y) \xleftrightarrow{F} G(f_X, f_Y) \cdot H(f_X, f_Y)$ **Eq. 2-15**

FT Pair: $\mathrm{sinc}(x) \xleftrightarrow{F} \mathrm{rect}(f_X)$ **Table 2.1 in Goodman**

$\mathrm{comb}(x) \xleftrightarrow{F} \mathrm{comb}(f_X)$ **Table 2.1 in Goodman**

<u>Apply</u>

$$g(x,y) = \mathrm{sinc}(x) \cdot \mathrm{comb}(y) \otimes \mathrm{comb}\left(\frac{x}{2}\right) \cdot \mathrm{sinc}\left(\frac{y}{4}\right)$$

$$g(x,y) = h(x)k(y) \otimes k(ax)h(by) \qquad \text{where } a = \frac{1}{2}, b = \frac{1}{4}, h(x) = \mathrm{sinc}(x), k(x) = \mathrm{comb}(x)$$

$$G(f_X, f_Y) = H(f_X) \cdot K(f_Y) \cdot \frac{1}{|a|} K\left(\frac{f_X}{a}\right) \cdot \frac{1}{|b|} H\left(\frac{f_Y}{b}\right) \qquad \text{Convolution \& Scaling Theorems}$$

$$G(f_X, f_Y) = \mathrm{rect}(f_X) \cdot \mathrm{comb}(f_Y) \cdot \frac{1}{|ab|} \mathrm{comb}\left(\frac{f_X}{a}\right) \cdot \mathrm{rect}\left(\frac{f_Y}{b}\right) \qquad \text{FT Pair}$$

$$G(f_X, f_Y) = 8 \cdot \mathrm{rect}(f_X) \cdot \mathrm{comb}(f_Y) \cdot \mathrm{comb}(2f_X) \cdot \mathrm{rect}(4f_Y)$$

$$\mathrm{sinc}(x) \cdot \mathrm{comb}(y) \otimes \mathrm{comb}\left(\frac{x}{2}\right) \cdot \mathrm{sinc}\frac{y}{4} \xleftrightarrow{F} 8 \cdot \mathrm{rect}(f_X) \cdot \mathrm{comb}(f_Y) \cdot \mathrm{comb}(2f_X) \cdot \mathrm{rect}(4f_Y)$$

Part (e)

Linearity Theorem: $g(x, y) + h(x, y) \xleftrightarrow{F} G(f_X, f_Y) + H(f_X, f_Y)$ **page 8 in Goodman**

Shift Theorem: $g(x-a, y-b) \xleftrightarrow{F} G(f_X, f_Y) \cdot \exp(-j2\pi(af_X + bf_Y))$ **Eq. 2-13**

FT Pair: $\text{circ}(r) \xleftrightarrow{F} \dfrac{J_1(2\pi\rho)}{\rho}$ **Eq. 2-35 in Goodman**

where $r = \sqrt{x^2 + y^2}$ and $\rho = \sqrt{f_X^2 + f_Y^2}$ and J_1 is a Bessel function of the 1$^{\text{st}}$ kind, order 1

<u>Apply</u>

$$g(x, y) = \text{circ}(\sqrt{(x-a)^2 + y^2}) + \text{circ}(\sqrt{(x+a)^2 + y^2})$$

$$g(x, y) = h(x-a, y) + h(x+a, y) \qquad\qquad \text{where } h(x, y) = \text{circ}(\sqrt{x^2 + y^2})$$

$$G(f_X, f_Y) = H(f_X, f_Y) \cdot \exp(-j2\pi af_X) + H(f_X, f_Y) \cdot \exp(+j2\pi af_X)$$

$$G(f_X, f_Y) = H(f_X, f_Y) \cdot (\exp(-j2\pi af_X) + \exp(+j2\pi af_X))$$

$$G(f_X, f_Y) = H(f_X, f_Y) \cdot (\cos(-2\pi af_X) + j\sin(-2\pi af_X) + \cos(2\pi af_X) + j\sin(2\pi af_X))$$

$$G(f_X, f_Y) = H(f_X, f_Y) \cdot (\cos(2\pi af_X) - j\sin(2\pi af_X) + \cos(2\pi af_X) + j\sin(2\pi af_X))$$

$$G(f_X, f_Y) = H(f_X, f_Y) \cdot 2\cos(2\pi af_X)$$

$$G(f_X, f_Y) = 2\frac{J_1(2\pi\rho)}{\rho} \cdot \cos(2\pi af_X)$$

$$\text{circ}(\sqrt{(x-a)^2 + y^2}) + \text{circ}(\sqrt{(x+a)^2 + y^2})$$
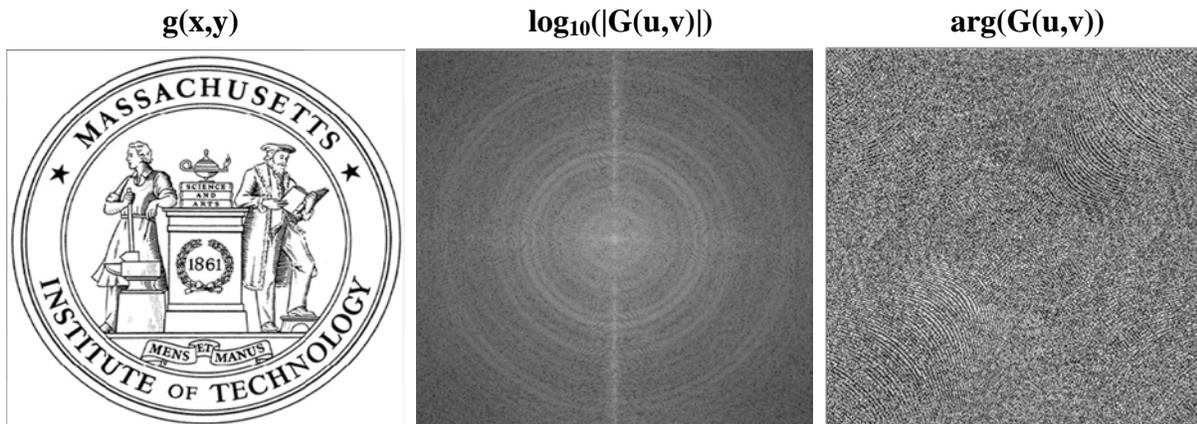
$$\xleftrightarrow{F}$$

$$2\frac{J_1(2\pi\rho)}{\rho}\cos(2\pi af_X) = 2\frac{J_1\left(2\pi\sqrt{f_X^2 + f_Y^2}\right)}{\sqrt{f_X^2 + f_Y^2}}\cos(2\pi af_X)$$

Problem 4. Fourier Transform with MIT Seal

Part (a)

Matlab Commands
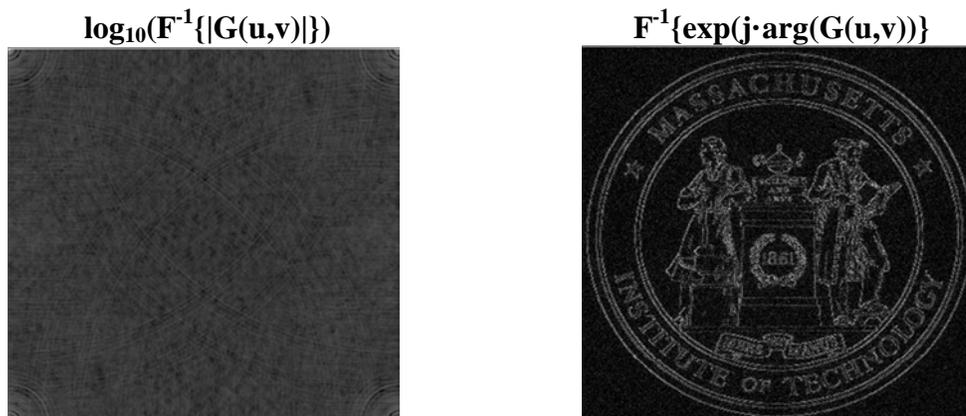```
g   = imread('MIT Seal','png');   %g(x,y)
G   = fftshift(fft2(g));   %G(u,v)
magG = abs(G);   %|G(u,v)|
argG = atan2(imag(G),real(G));   %arg(G(u,v))
```

| **g(x,y)** | **log₁₀(\|G(u,v)\|)** | **arg(G(u,v))** |
|---|---|---|



### Part (b)

**Matlab Commands**
```
G_MagOnly = magG;   %|G(u,v)|
g_MagOnly = ifft2(ifftshift(G_MagOnly));   %F⁻¹{|G(u,v)|}
G_ArgOnly = exp(sqrt(-1)*argG);   %exp(j·arg(G(u,v)))
g_ArgOnly = ifft2(ifftshift(G_ArgOnly));   %F⁻¹{exp(j·arg(G(u,v)))}
```

| **log₁₀(F⁻¹{\|G(u,v)\|})** | **F⁻¹{exp(j·arg(G(u,v)))}** |
|---|---|



The image recovered from the phase information is closer to the original image, because the phase information determines the spatial location of frequencies in the image.
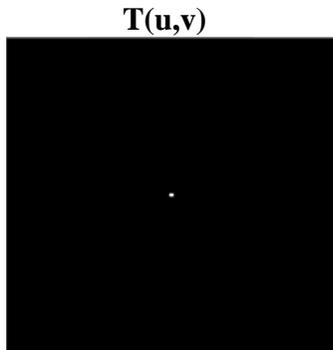
**Part (c)**

Here, we applied a low-pass filter (LPF) to the image. The result, **g1(x,y)**, is a blurry image containing low frequencies only.

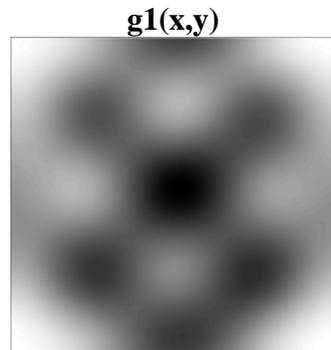**Matlab Commands**
```
N   = size(g,1);   %size of image
mid   = ceil(N/2);   %middle index of image
S   = 5;   %filter size
mid_inds   = (0:S-1)+mid-floor(S/2); %indices of filter
T   = zeros(size(g));   %initialize T
T(mid_inds,mid_inds) = 1;   %T(u,v) → Low-Pass Filter
g1   = ifft2(ifftshift(G.*T)); %g1(x,y) = F⁻¹{G(u,v)T(u,v)}
```

**T(u,v)**

**g1(x,y)**
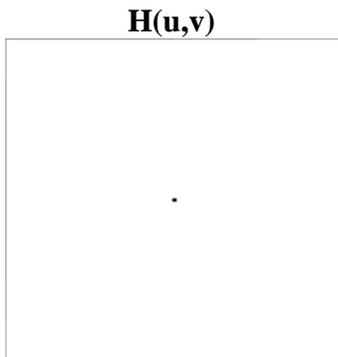


**Low Pass Filter (LPF)**

**Filtered Image**

**Part (d)**

Here, we applied a high-pass filter (HPF) to the image. The result, **g2(x,y)**, is a more detailed image with high frequency edges and features.

**Matlab Commands**
```
H = 1 - T;   %H(u,v) → High-Pass Filter
g2 = ifft2(ifftshift(G.*H));   %g2(x,y) = F⁻¹{G(u,v)H(u,v)}
```

**H(u,v)**

**g2(x,y)**



**High Pass Filter (HPF)**

**Filtered Image**

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Spring 2014