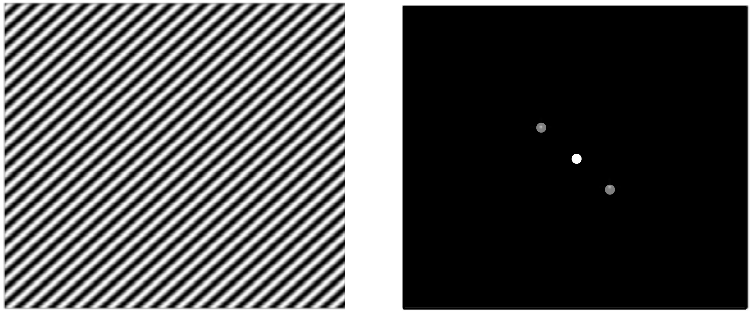
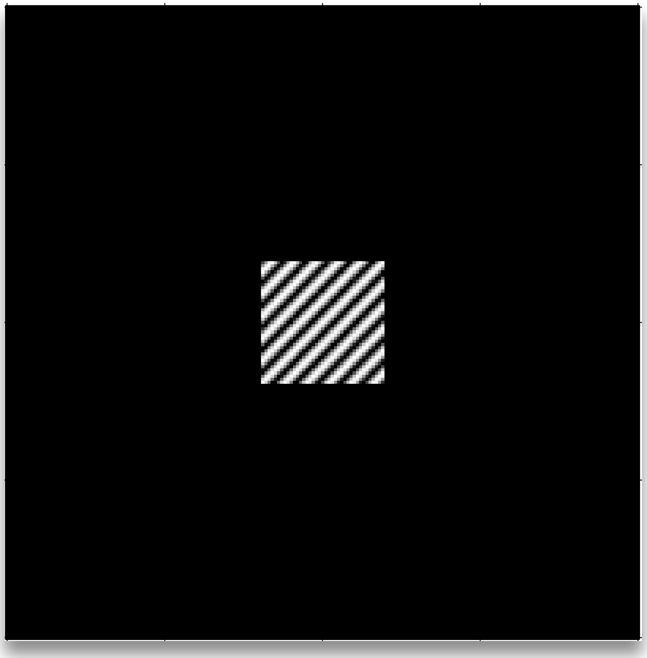
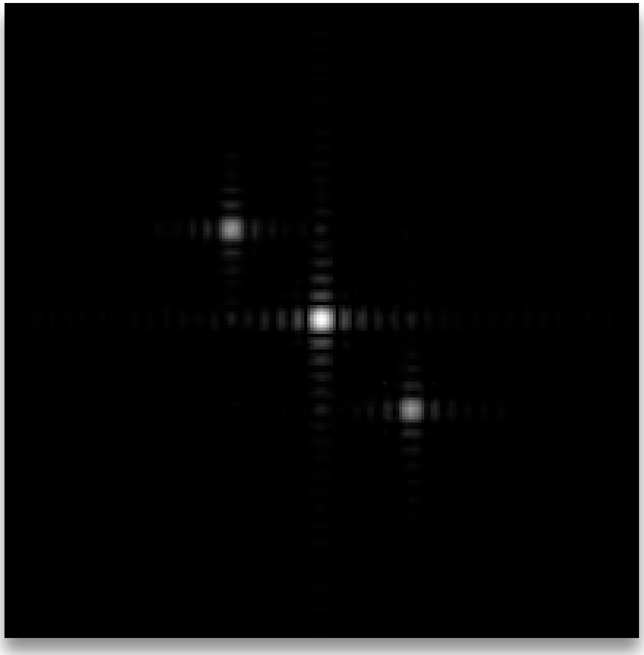
The convolution theorem



multiplication



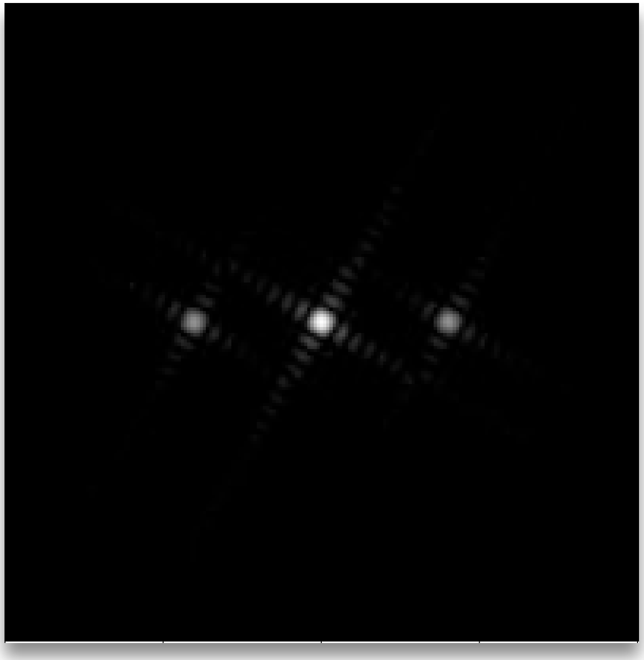
convolution



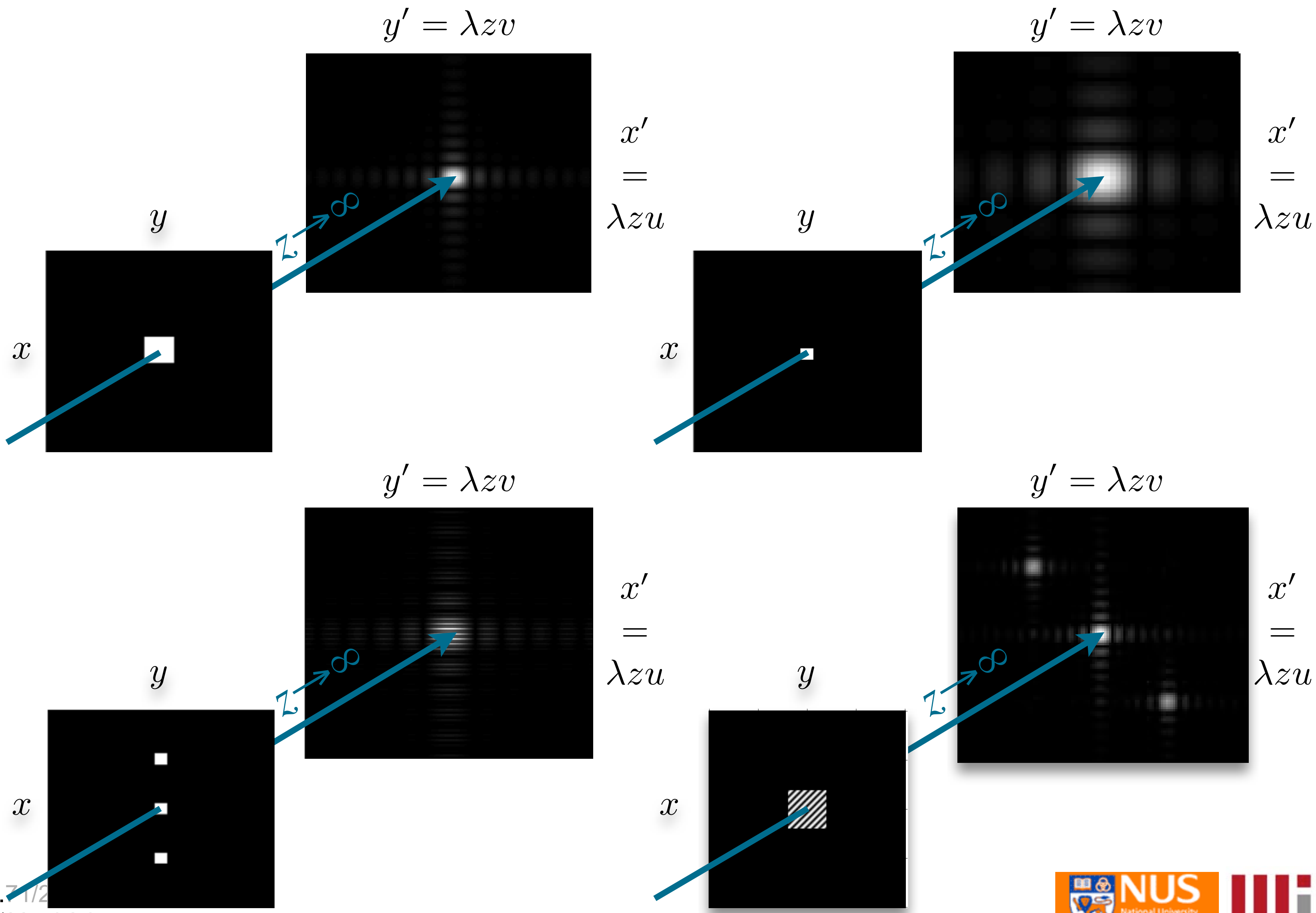
The convolution theorem

multiplication

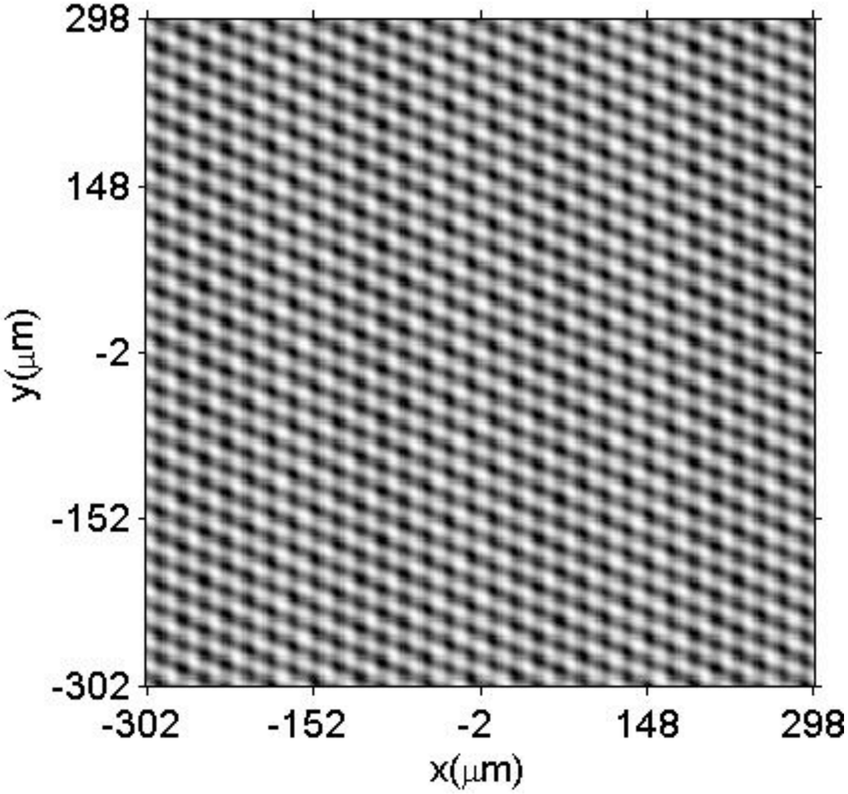
convolution



Fraunhofer diffraction patterns

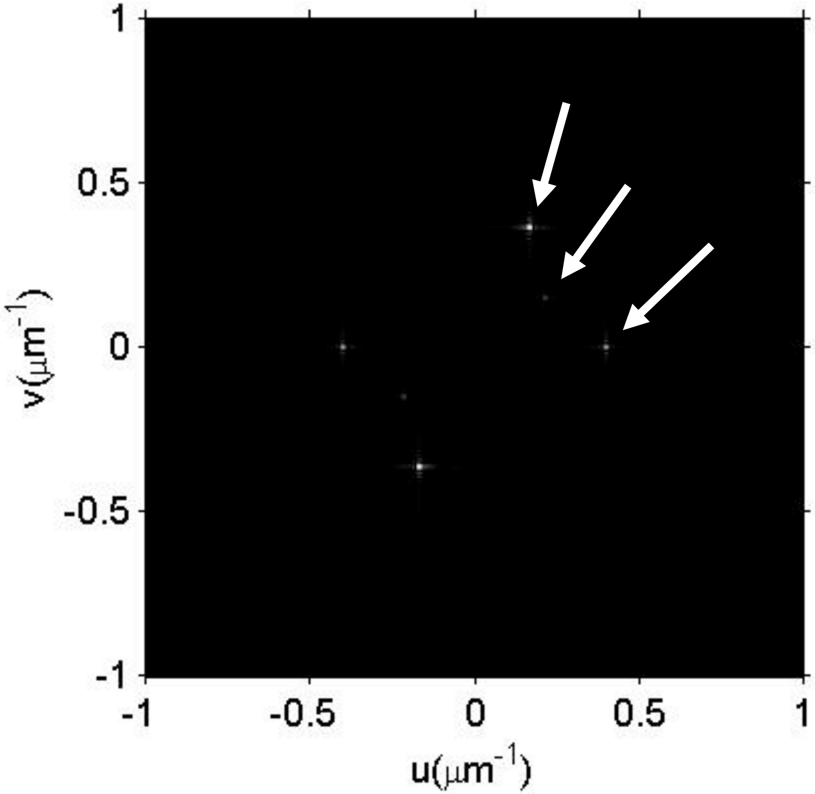


Spatial filtering



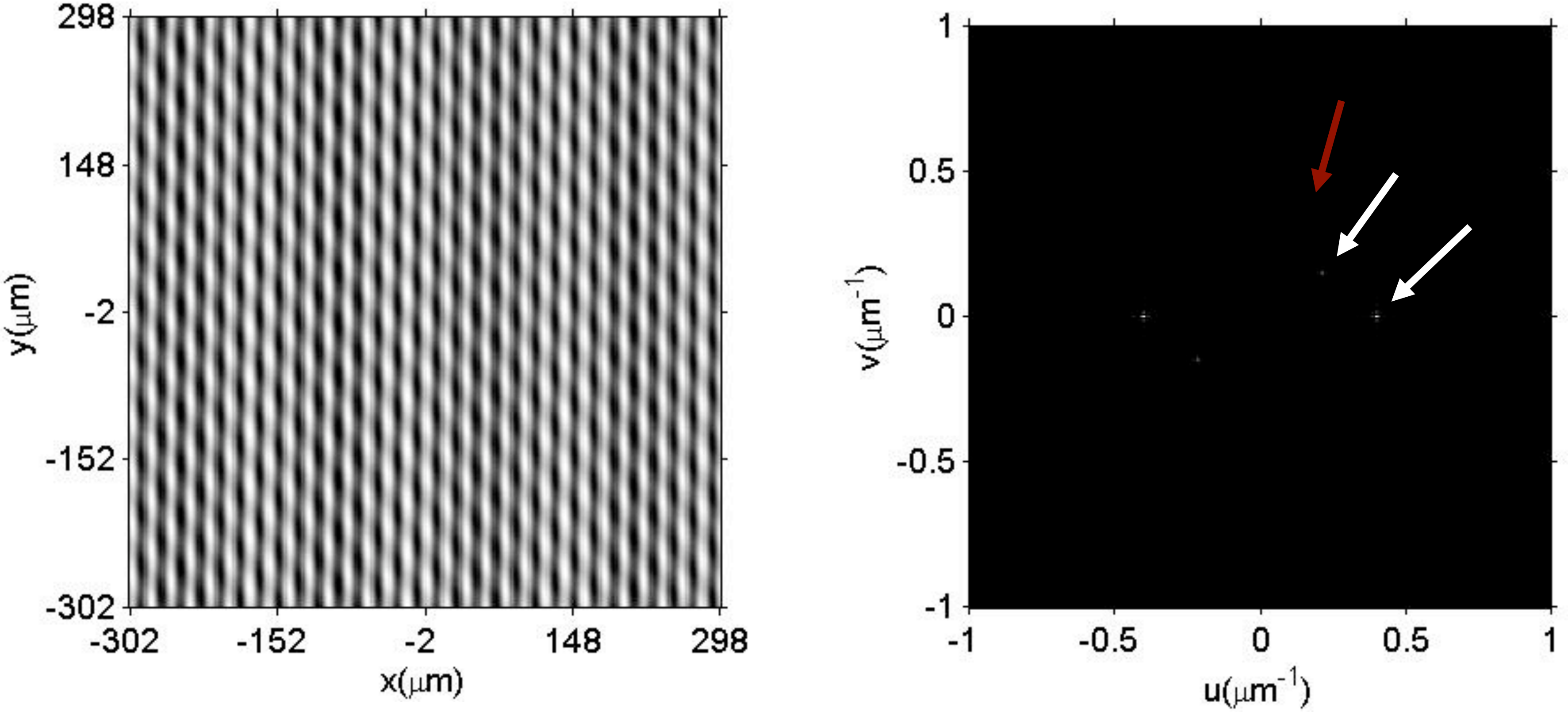
space domain

3 sinusoids



Fourier domain

Spatial filtering



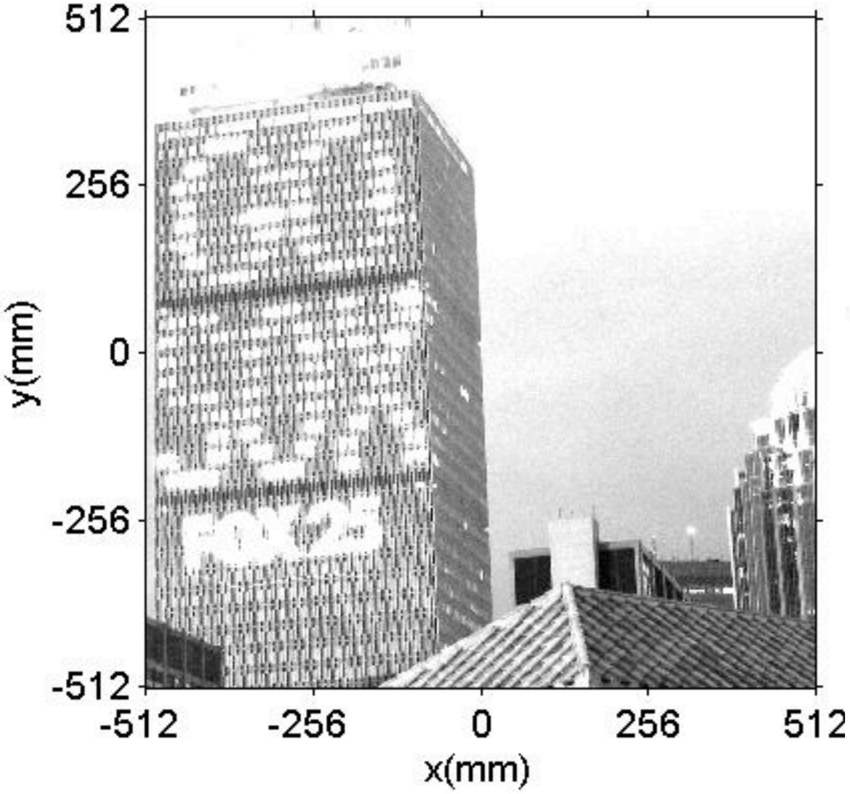
space domain

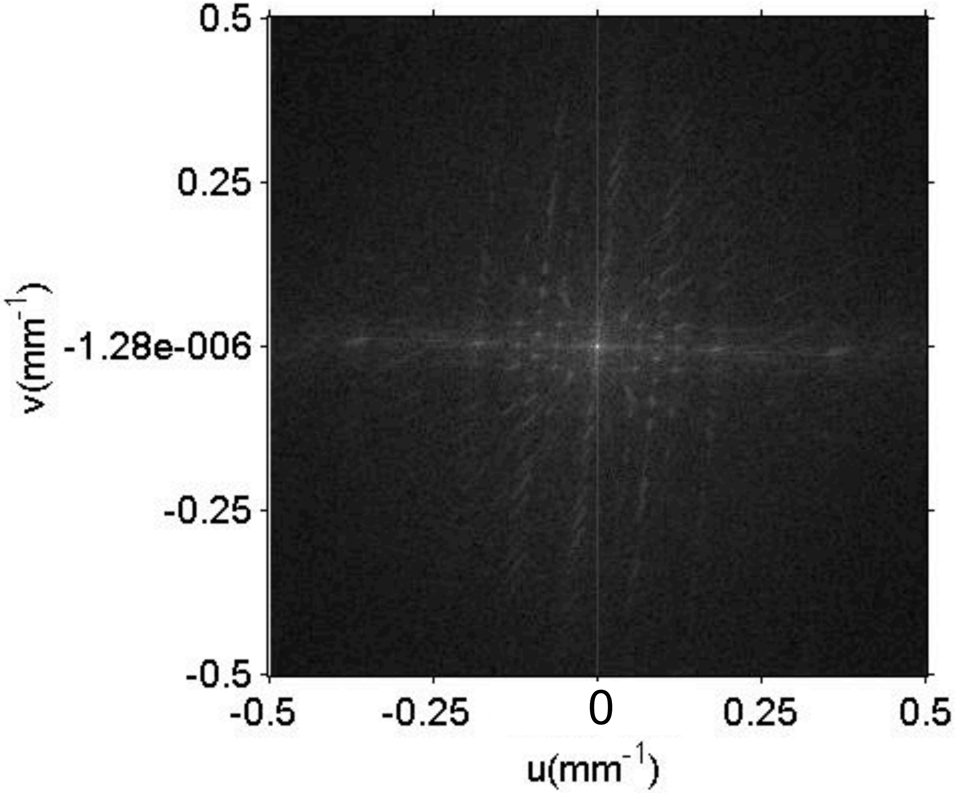
2 sinusoids (one removed)

Fourier domain

Spatial filtering of a scene

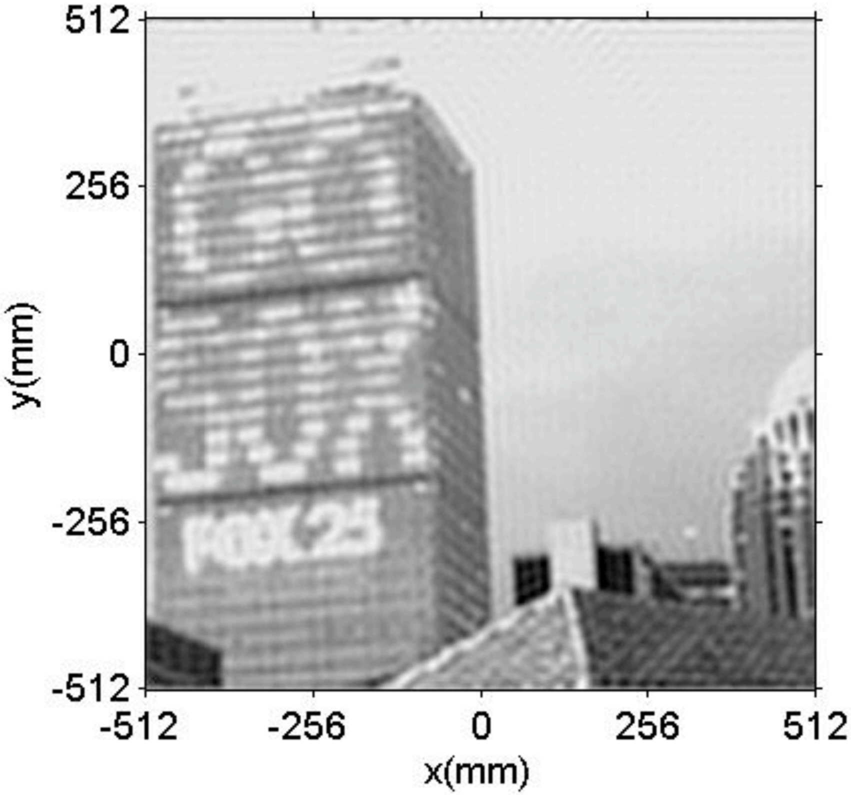
Unfiltered: all spatial frequencies present

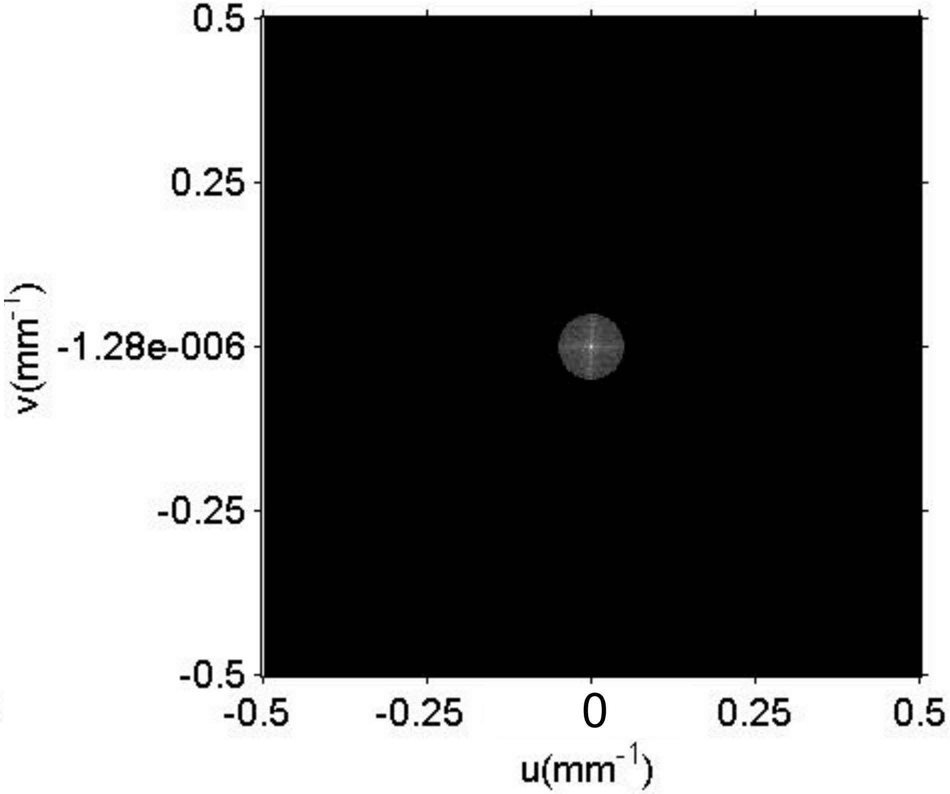




Spatial filtering of a scene

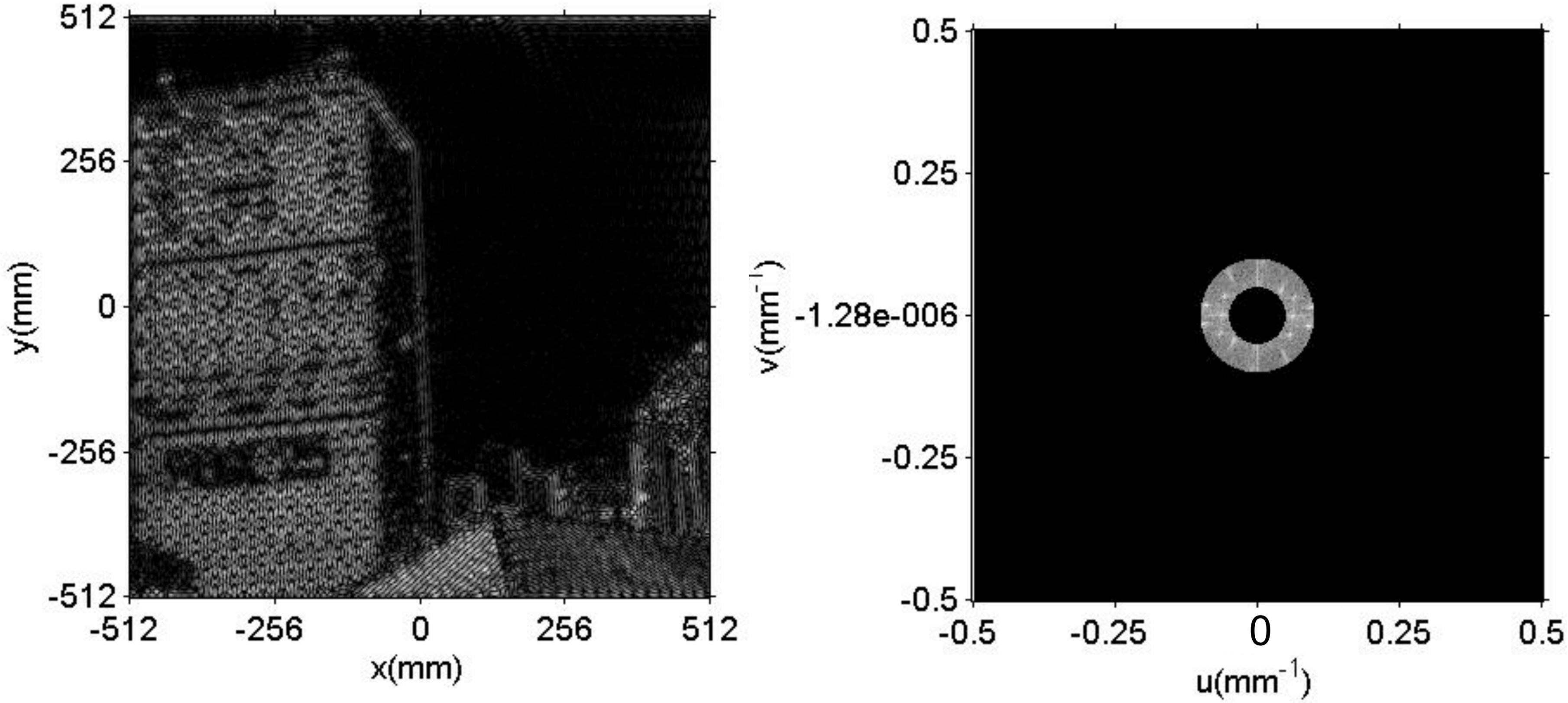
Low-pass filtered: high spatial frequencies removed



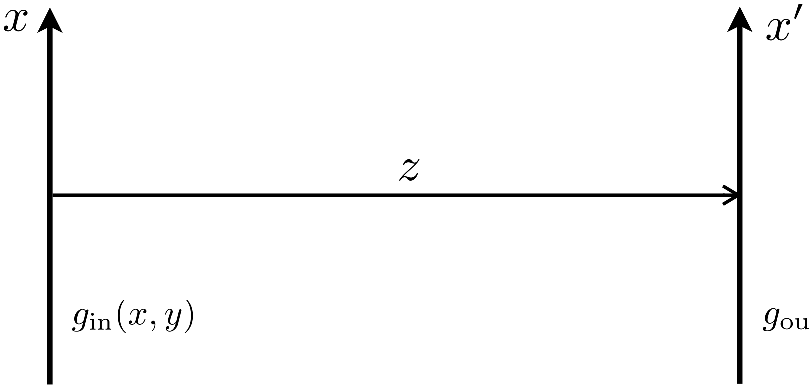


Spatial filtering of a scene

Band-pass filtered: high & low spatial frequencies removed



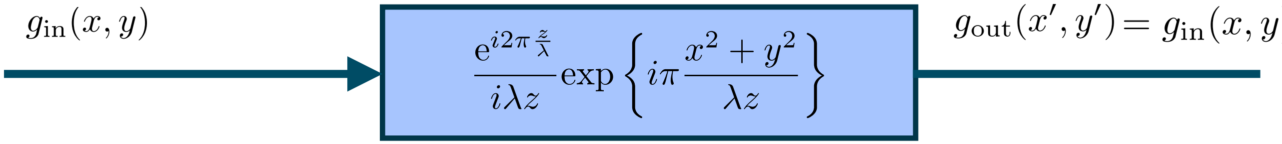
The transfer function of Fresnel propagation



Fresnel (free space) propagation
may be expressed as a
convolution integral

$$g_{\text{out}}(x',y') = g_{\text{in}}(x,y) \star \left(\frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z} \exp\left\{ i\pi \frac{x^2+y^2}{\lambda z} \right\} \right)$$

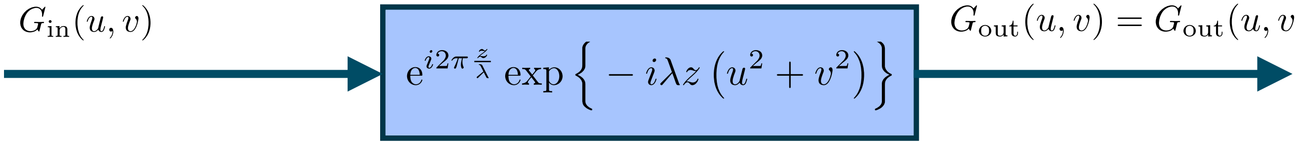
$$g_{\text{out}}(x',y';z) = \frac{1}{i\lambda z} \exp\left\{ i2\pi \frac{z}{\lambda} \right\} \iint g_{\text{in}}(x,y) \exp\left\{ i\pi \frac{(x'-x)^2+(y'-y)^2}{\lambda z} \right\} \mathrm{d}x\mathrm{d}y$$



$g_{\text{in}}(x,y)$

$$\frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z} \exp\left\{ i\pi \frac{x^2+y^2}{\lambda z} \right\}$$

$$g_{\text{out}}(x',y') = g_{\text{in}}(x,y) \star \left(\frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z} \exp\left\{ i\pi \frac{x^2+y^2}{\lambda z} \right\} \right)$$



$G_{\text{in}}(u,v)$

$$e^{i2\pi\frac{z}{\lambda}} \exp\left\{ -i\lambda z \left(u^2+v^2 \right) \right\}$$

$$G_{\text{out}}(u,v) = G_{\text{out}}(u,v) \cdot \left(e^{i2\pi\frac{z}{\lambda}} \exp\left\{ -i\lambda z \left(u^2+v^2 \right) \right\} \right)$$

$$h(x,y) = \frac{e^{i2\pi\frac{z}{\lambda}}}{i\lambda z} \exp\left\{ i\pi \frac{x^2+y^2}{\lambda z} \right\}$$

Point–Spread Function
(Impulse Response)

$$H(u,v) = e^{i2\pi\frac{z}{\lambda}} \exp\left\{ -i\lambda z \left(u^2+v^2 \right) \right\}$$

Transfer Function

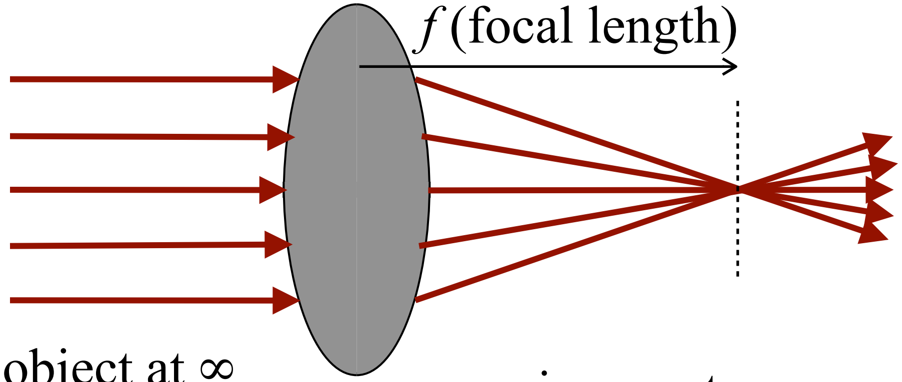
Today

- Fourier transforming properties of lenses
- Spherical - plane wave duality
- The telescope (4F system) revisited
  - imaging as a cascade of Fourier transforms
  - spatial filtering by a pupil plane transparency

Wednesday

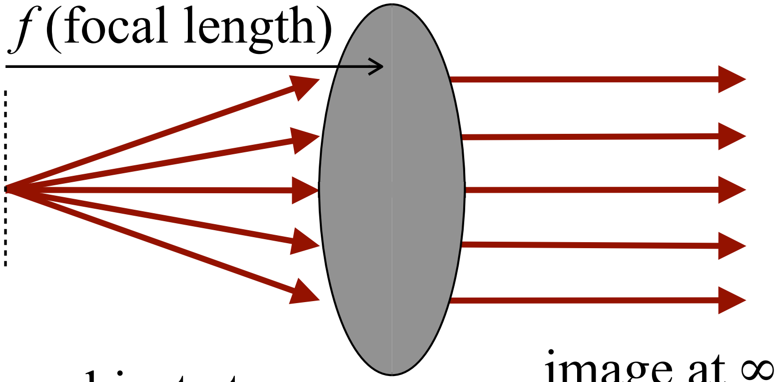
- Spatial filtering in the 4F system
- Point-Spread Function (PSF)
  and Amplitude Transfer Function (ATF)

Reminder: thin lens (geometrical optics)



object at ∞
(plane wave)

image at
back focal plane

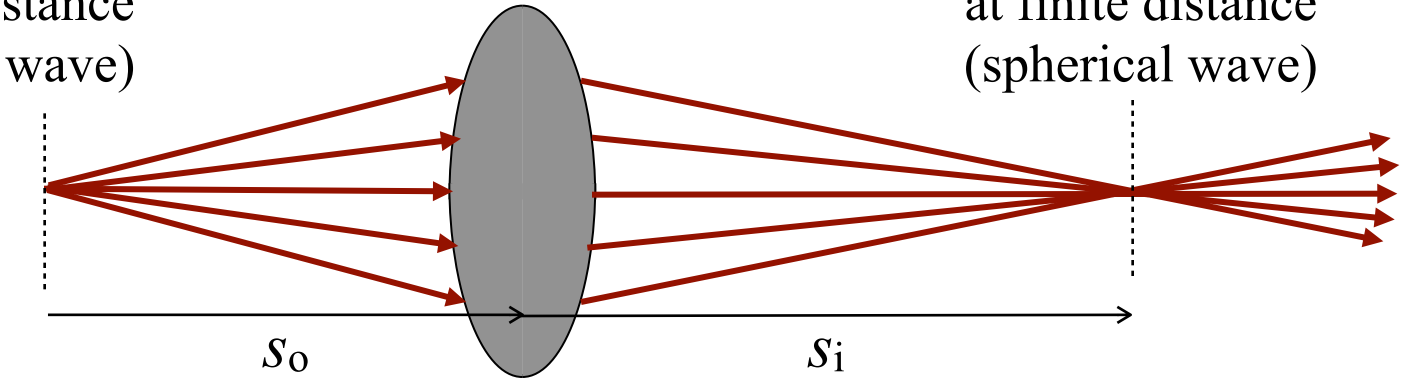


object at
front focal plane

image at ∞
(plane wave)

amount of ray bending is proportional to the distance from the optical axis

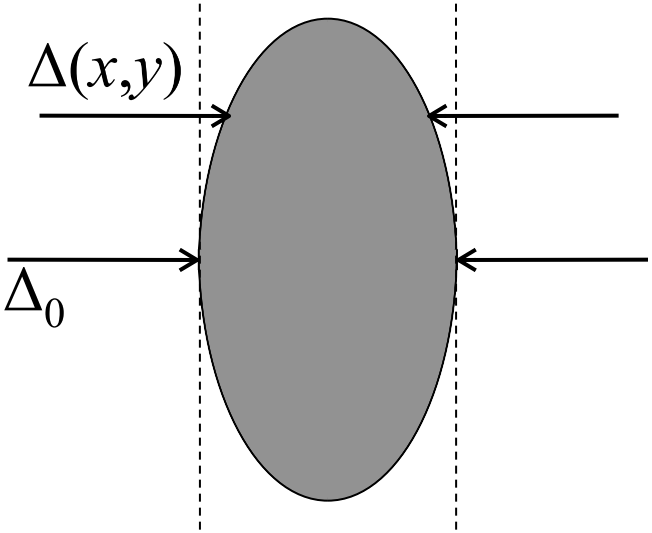
point object
at finite distance
(spherical wave)



point image
at finite distance
(spherical wave)

$$\frac{1}{s_o} + \frac{1}{s_i} = \frac{1}{f}$$

Thin lens complex transmittance



$\Delta(x,y)$: glass thickness
at coordinate (x,y)
n: index of refraction
R_1, R_2: radii of curvature

$$\Delta(x,y)=\Delta_0-R_1\left(1-\sqrt{1-\frac{x^2+y^2}{R_1}}\right)+R_2\left(1-\sqrt{1-\frac{x^2+y^2}{R_2}}\right)$$

$$\Delta(x,y)\approx\Delta_0-R_1\left(1-\left[1-\frac{x^2+y^2}{2R_1}\right]\right)+R_2\left(1-\left[1-\frac{x^2+y^2}{2R_2}\right]\right)$$

$$\Delta(x,y)\approx\Delta_0-\frac{x^2+y^2}{2}\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$$

$$a_{\text{lens}}(x,y)=\exp\left\{i\frac{2\pi}{\lambda}\Delta_0+\frac{2\pi}{\lambda}(n-1)\Delta(x,y)\right\}$$

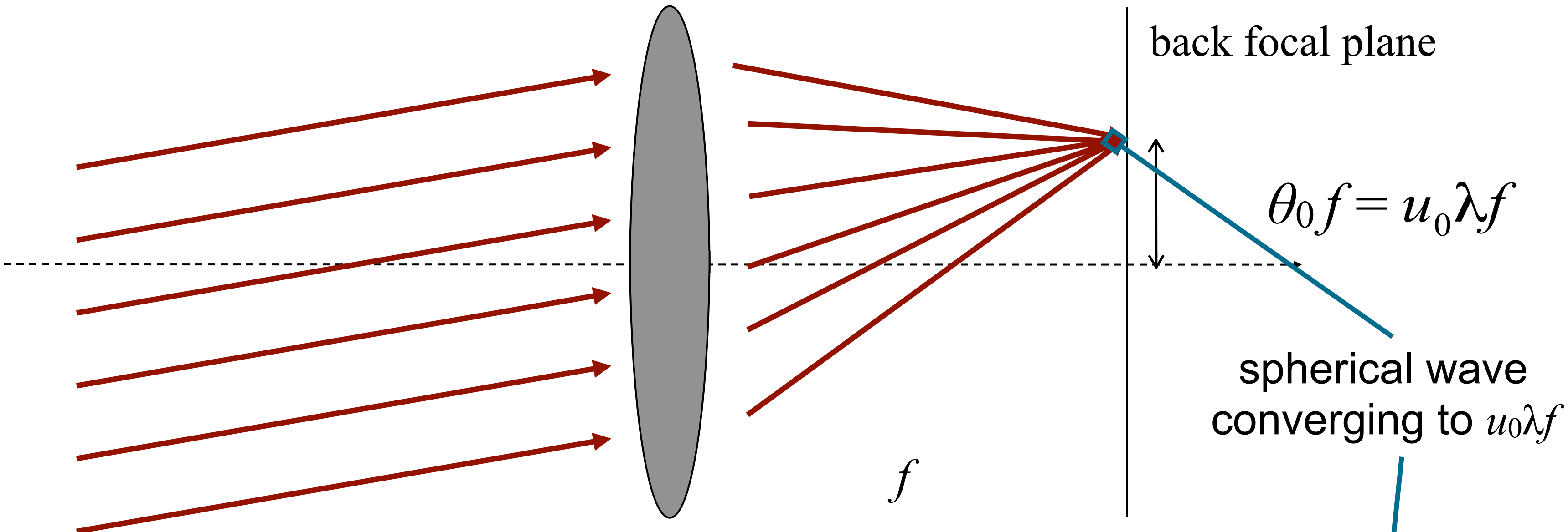
$$a_{\text{lens}}(x,y)\approx\exp\left\{i\frac{2\pi n}{\lambda}\Delta_0\right\}\exp\left\{-i\frac{2\pi}{\lambda}(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)\frac{x^2+y^2}{2}\right\}$$

this constant-phase term can be omitted

$$a_{\text{lens}}(x,y)\approx\boxed{\exp\left\{i\frac{2\pi n}{\lambda}\Delta_0\right\}}\exp\left\{-i\pi\frac{x^2+y^2}{\lambda f}\right\}$$

where $\frac{1}{f}\equiv(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)$ is the focal length

Plane wave incident on a lens



back focal plane

$\theta_0 f = u_0\lambda f$

spherical wave converging to $u_0\lambda f$

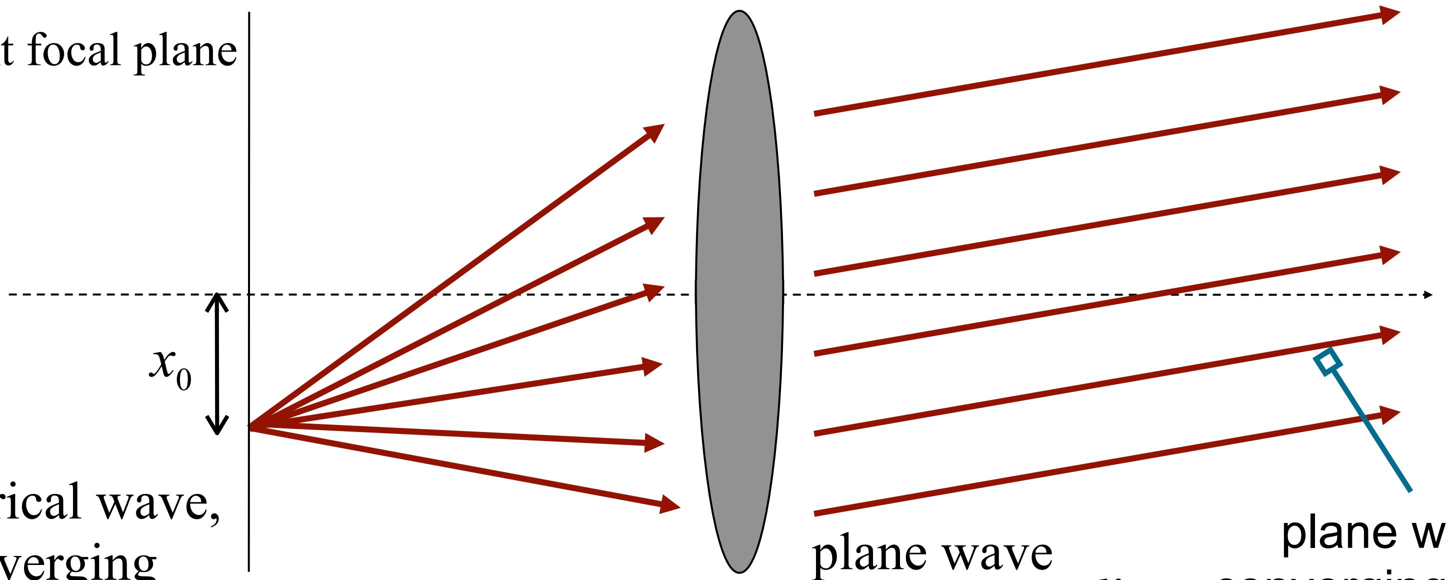
plane wave: $\exp\{i2\pi u_0 x\}$
angle θ_0, sp. freq. $u_0 \approx \theta_0/\lambda$

lens, $a_{\text{lens}}(x,y)=\exp\left\{-i\pi\frac{x^2+y^2}{\lambda f}\right\}$

$$a_+(x,y)=\exp\{i2\pi u_0 x\}\exp\left\{-i\pi\frac{x^2+y^2}{\lambda f}\right\}$$

$$a_+(x,y)=\underbrace{\exp\{i\pi u_0^2\lambda f\}}_{\text{ignore}}\exp\left\{-i\pi\frac{(x-u_0\lambda f)^2+y^2}{\lambda f}\right\}$$

Spherical wave incident on a lens



spherical wave (has propagated distance f) :

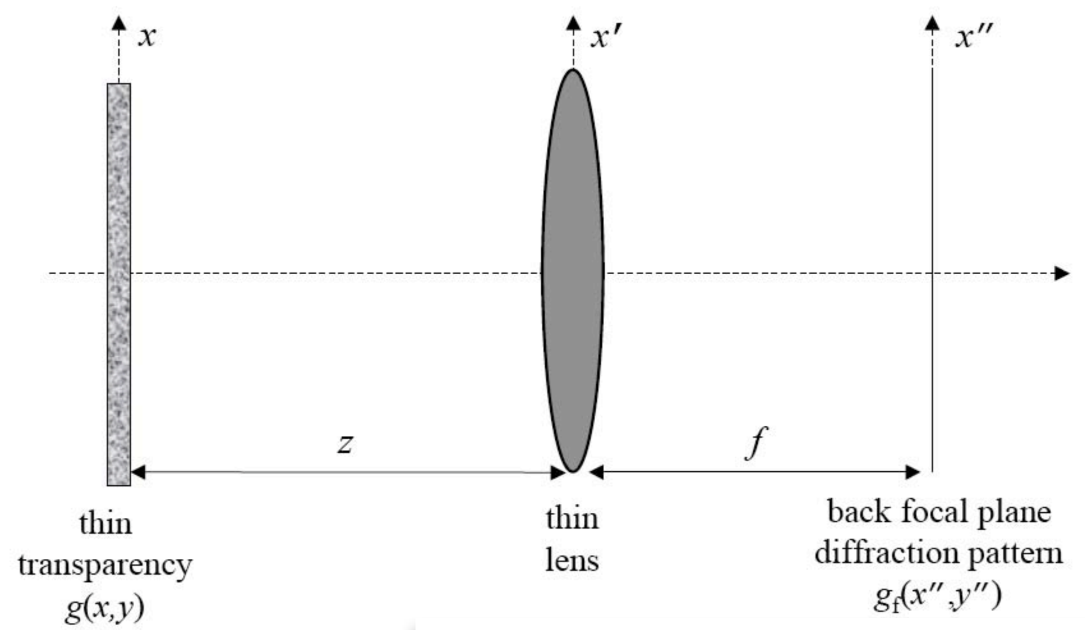
$$a_-(x,y) = \exp\left\{i2\pi\frac{f}{\lambda}\right\}\exp\left\{i\pi\frac{(x+x_0)^2+y^2}{\lambda f}\right\}$$

lens transmission function :

$$a_{\text{lens}}(x,y) = \exp\{i2\pi n\Delta_0\}\exp\left\{-i\pi\frac{x^2+y^2}{\lambda f}\right\}$$

$$a_+(x,y) = a_-(x,y)\times a_{\text{lens}}(x,y) = \exp\left\{\underset{\uparrow\ \text{ignore}}{\boxed{i2\pi\left(n\Delta_0+\frac{f}{\lambda}\right)+i\pi\frac{x_0^2}{\lambda f}}}+i2\pi\frac{x_0 x}{\lambda f}\right\}$$

Fourier transforming property of lenses



Field before lens $g_{\text{lens}-}(x') = \int g(x)\exp\left\{i\pi\frac{(x'-x)^2}{\lambda z}\right\}\mathrm{d}x$

Field after lens $g_{\text{lens}+}(x') = g_{\text{lens}-}(x')\exp\left\{-i\pi\frac{x'^2}{\lambda f}\right\}$

Field at back f.p. $g_{\mathrm{f}}(x'') = \int g_{\text{lens}+}(x')\exp\left\{i\pi\frac{(x''-x')^2}{\lambda f}\right\}\mathrm{d}x'$

1D calculation

$$g_{\mathrm{f}}(x'') = \exp\left\{i\pi\frac{x''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\}\int g(x)\exp\left\{-i2\pi\frac{xx''}{\lambda f}\right\}\mathrm{d}x$$

2D version

$$g_{\mathrm{f}}(x'',y'') = \exp\left\{i\pi\frac{x''^2+y''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\}\iint g(x,y)\exp\left\{-i2\pi\frac{xx''+yy''}{\lambda f}\right\}\mathrm{d}x\,\mathrm{d}y$$

$$g_{\mathrm{f}}(x'',y'') = \exp\left\{i\pi\frac{x''^2+y''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\}\iint g(x,y)\exp\left\{-i2\pi\frac{xx''+yy''}{\lambda f}\right\}\mathrm{d}x\,\mathrm{d}y$$

$$\therefore\; g_{\mathrm{f}}(x'',y'') = \boxed{\exp\left\{i\pi\frac{x''^2+y''^2}{\lambda f}\left(1-\frac{z}{f}\right)\right\}}\;\underbrace{G\left(\frac{x''}{\lambda f},\frac{y''}{\lambda f}\right)}_{\text{Fourier transform of } g(x,y)}$$

spherical wave-front

2.71 / 2.710 Optics
Spring 2009