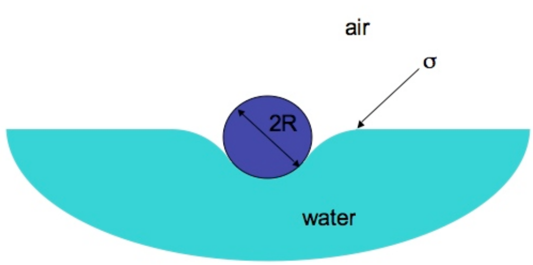
MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 2.06

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



The water strider, or pond skater, is a slender insect, about $\frac{1}{2}$ cm long, that runs or 'skates' over the surfaces of ponds and streams. It stays easily on the surface because its feet (tarsi) are equipped with numerous fine, non-wetting hairs.

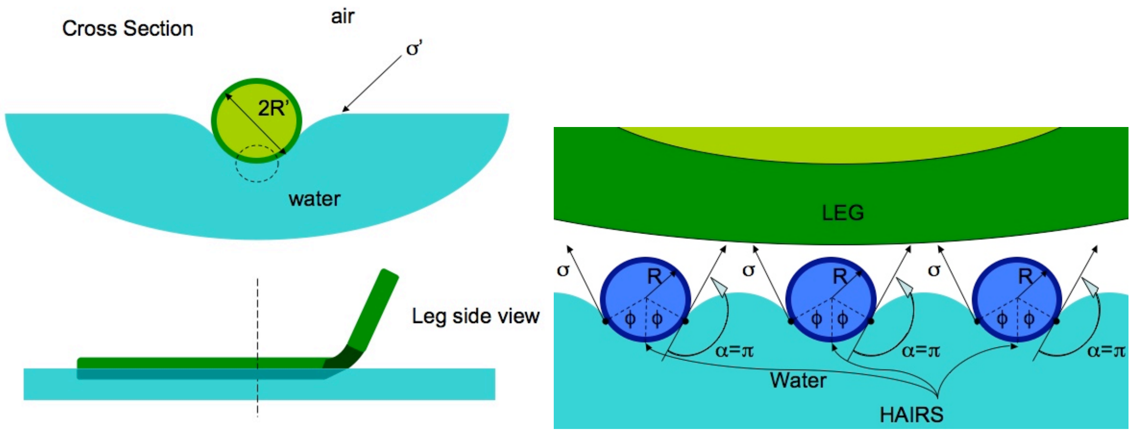
Suppose we model one of these hairs as a long cylinder of radius R made of completely non-wetting material (contact angle 180 degrees), and assume that it is set down on the water with its axis parallel to the surface, as sketched. The surface tension is $0.07[N/m]$.

- (a) Show that, as the cylinder, or hair, is brought into contact with the water and then depressed into it, the lift force exerted on it by surface tension first increases, then reaches a maximum at a certain depression and finally decreases as the cylinder is depressed further. What is the maximum value of the surface-tension-induced lift force per unit cylinder length?
- (b) What is the criterion for the gravitational effects to have a negligible effect on the (maximum) total lift force? Is it likely that this criterion is satisfied for the pond skater's tarsi?
- (c) If a pond skater weights 0.05 grams (note that this is only a guess, not a figure based on observations of the real insect) [1] , what minimum local length of hair must it have on its feet to keep it on top of the water?
- (d) What is the shape of the water surface around the leg when the contact angle is 180 degrees between the water and the leg? [2] . Idealize the leg as a perfect cylinder (no hairs) and as before assume maximum surface tension force.

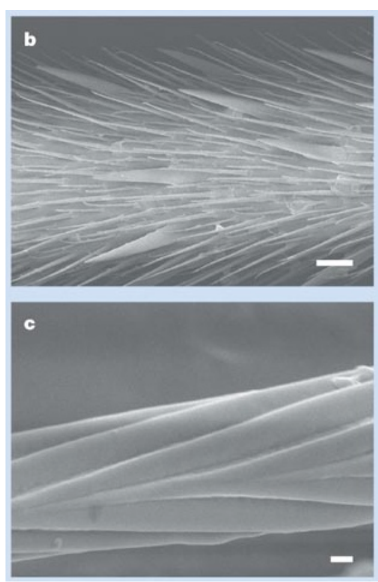
[1] Actually, accordingly to the Wikipedia the weight of these insects is 0.01 grams, not a bad guess.

[2] Accordingly to the Wikipedia the effective angle for the legs (due to the properties of the hairs) is roughly 170 degrees, superhydrophobic.

Preamble:



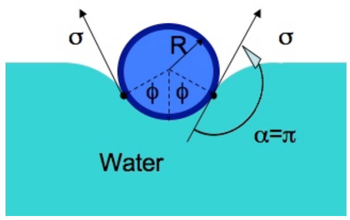
If we make a diagram of the leg, we realize that this is a highly idealized problem; the position of the hairs relative to the leg and the presence of hairs next to each other will change the force that different hairs provide to the leg. Also, if we look at an electron microscope photograph of the hairs, we realize that their shape is not circular and not constant. Nevertheless,we are not interested in an exact solution, but we want an estimate of the order of magnitude of the forces involved, and the geometric restrictions. Then, we'll consider a single cylindrical hair isolated on the water surface as shown in the problem statement, let's just keep in mind that we are just calculating approximate values for something as complex as the leg structure shown in the image.



Scanning electron microscope images of a leg showing numerous oriented spindly microsetae (b) and the fine nanoscale grooved structures on a seta (c). Scale bars: b, 20 μm; c, 200 nm. Taken from 'Water-repellent legs of water striders' by Xuefeng Gao, and Lei Jiang, Nature 432, 36 (4 November 2004).

Solution:

- (a)



If we use the diagram provided in the exercise, we notice that the force per unit length going upwards due to surface tension is

$$F'_\sigma \underline{\hat{y}} = 2\sigma \sin\phi \underline{\hat{y}}, \tag{2.06a}$$

which is maximum for $\phi = \frac{\pi}{2}$. Then, the upward surface tension force increases till the angle ϕ reaches $\frac{\pi}{2}$ and then decreases as the angle increases.

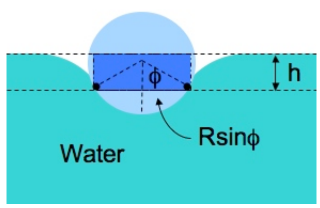
$$\boxed{F'_{\sigma\ max} = 2\sigma.} \tag{2.06b}$$

Notice that the diagram shows the force on the hair caused by surface tension, and not the force on the water surface.

- (b) The total upwards vertical force is the sum of the force due to surface tension plus the force resulting from pressure on the bottom of the hair. The pressure force can be calculated easily if we consider it as a buoyancy force, then

$$F'_p = \rho g(h2R\sin\phi) + \rho g\left(\int_0^{R(1-\cos\phi)} r(z)dz\right), \tag{2.06c}$$

where z is the height measured from the bottom of the hair.



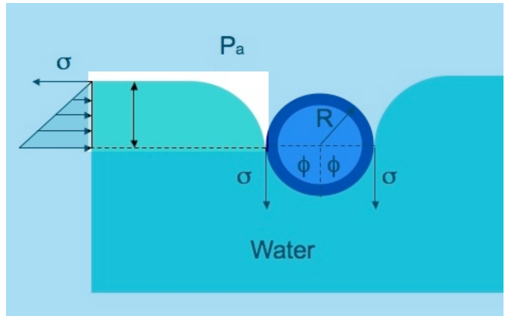
The first term represents the contribution of the water volume displaced by the rectangular area shown in the figure, and the second term represents the contribution of the area between the rectangle and the bottom of the cylinder. Then the order of magnitude of this force is

$$F'_p \sim 2\rho g h R \text{ “}+\text{” } R^2\rho g, \tag{2.06d}$$

Then, for the second term to be small,

$$\frac{R^2\rho g}{2\sigma} \sim \frac{R^2\rho g}{\sigma} \ll 1, \tag{2.06e}$$

Now, for the first term, let's estimate the height of the water. Using a control volume that encompass the liquid to the left of the hair and extending till the water surface is horizontal, let's perform a force balance in the x direction,



$$\frac{\rho g h^2}{2} - \sigma + \sigma\cos\phi = 0, \tag{2.06f}$$

simplifying,

$$h = \sqrt{\frac{2}{\rho g}\sigma(1-\cos\phi)}, \tag{2.06g}$$

then, the order of magnitude is

$$h \sim \sqrt{\frac{\sigma}{\rho g}}. \tag{2.06h}$$

Then, for the first term to be small,

$$\frac{\rho g R\sqrt{\frac{\sigma}{\rho g}}}{\sigma} \ll 1, \tag{2.06i}$$

then,

$$\left(\frac{R^2\rho g}{\sigma}\right)^{0.5} \ll 1. \tag{2.06j}$$

Then the criteria is $\left(\frac{R^2\rho g}{\sigma}\right)^{0.5} \ll 1$, or $\boxed{Bo \ll 1}$, where Bo is the Bond number.

For this geometry, $R \approx 2.5$ microns, then,

$$\frac{R^2\rho g}{\sigma} = \frac{(2.5\times10^{-6}[m])^2\left(1000\left[\frac{kg}{m^3}\right]\right)(9.8[\frac{m}{s^2}])}{0.07[\frac{N}{m}]} = 9\times10^{-9}, \tag{2.06k}$$

and can be in effect neglected. Now, if we consider the leg as a whole, the Bond number for a radius of 200 microns, is 6400 times larger, but still small.

- (c) To calculate the minimum hair length, then let's equate the weight force and the surface tension force,

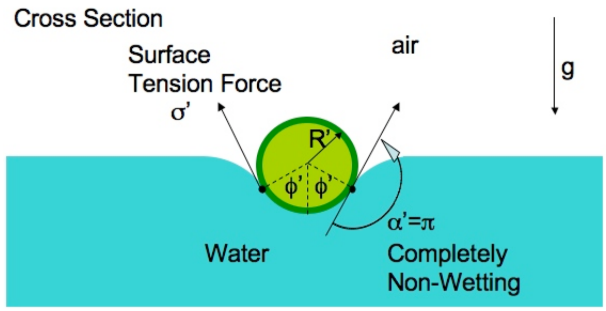
$$2\sigma l_{Tot} = m\times g,$$

then,

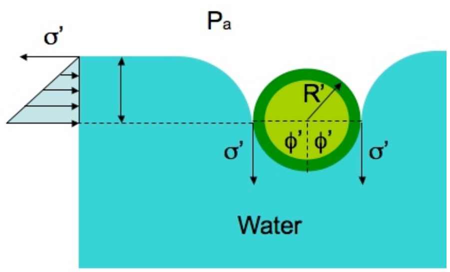
$$l_{Tot} = \frac{m\times g}{2\sigma} = \frac{(5\times10^{-5}[kg])(9.9[\frac{m}{s}])}{2(0.07[\frac{kg}{s^2}])} = 3.6\times10^{-3}[m], \tag{2.06l}$$

then this divided by 6 legs, $l^{hair}_{per\ leg} = 0.6$mm, which is more than feasible, since these creatures have many hairs per leg, and each leg is a couple of mm long. (Look again at the electron microscope figure.)

- (d) Since we are neglecting the surface irregularities of the leg, our solution will apply only at the water surface far from the leg, but will give us a good idea of the shape far away. Now, let's draw a similar diagram for the leg as the one that we made for the hair, and assume $\alpha' = \pi$.



Now, if the effective force is maximum, by analogy with the hair, $\phi' = \frac{\pi}{2}$.



Now, let's obtain an estimate for the maximum depression depth for the water surface. Making again a force balance for the liquid to the left of the leg,

$$\sigma' = \frac{\rho g h^{*2}}{2}, \tag{2.06m}$$

then,

$$h^* = \sqrt{\frac{2\sigma'}{\rho g}}, \tag{2.06n}$$

If we approach the leg from a point with null curvature, the pressure at this point inside the water is just $P(h) = P_a$. Then if we move inside the liquid, the pressure just below this point inside the liquid is $P_a + \rho g(h^* - y)$ (the origin was set at the minimum height of the surface), but this is the same pressure at any position at the same height since the liquid is in static equilibrium; then, the pressure inside the liquid is just determined by its vertical coordinate. Now, making a force balance in the y direction for a differential element with a not null curvature, [3]

[3] Figure shows vertical components of the surface tension forces only.

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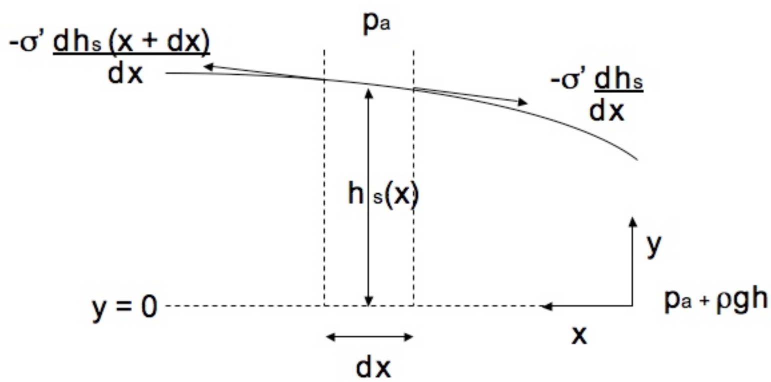


y force balance gives,

$$\sigma' \frac{d^2 h_s}{dx^2} + \rho g h^* = \rho g h_s, \tag{2.06o}$$

now, to make it an equation in terms of one variable, let's add the term $\sigma' \frac{d^2 h^*}{dx^2}$ which is equal to zero, then,

$$\sigma' \frac{d^2(h^* - h_s)}{dx^2} - \rho g (h^* - h_s) = 0,$$

or

$$L_c^2 \frac{d^2(h^* - h_s)}{dx^2} - (h^* - h_s) = 0, \tag{2.06p}$$

where $L_c = \sqrt{\frac{\sigma'}{\rho g}}$ is the capillary length. Now, let's apply the boundary conditions,

$$h^* - h_s \to 0 \qquad as \qquad x \to \infty, \tag{2.06q}$$
$$h^* - h_s = h^* \qquad when \qquad x = 0, \tag{2.06r}$$
$$\tag{2.06s}$$

which gives,

$$(h^* - h_s) = h^* e^{-\frac{x}{L_c}},$$

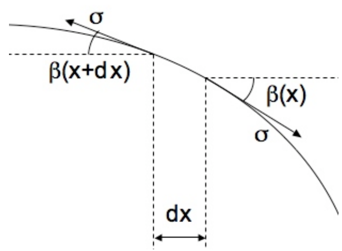
simplifying,

$$h_s = h^*(1 - e^{-\frac{x}{L_c}}), \tag{2.06t}$$

Hence the effect of surface tension extends to a distance of order L_c.

There are two problems with this solution,

- (i) The solution implicitly assumes that $\frac{dh}{dx}$ is small, which is not true next to the leg. More precisely, making a force balance in the y direction again



$$Net\ Vertical\ Force = \sigma \frac{d}{dx}(\sin\beta), \tag{2.06u}$$

but,

$$\sin\beta = \frac{h'}{(1+h'^2)^{\frac{1}{2}}}, \tag{2.06v}$$

and

$$\tfrac{d}{dx}\sin\beta = \tfrac{h''}{(1+h'^2)^{\frac{1}{2}}} - \tfrac{h'}{(1+h'^2)^{\frac{3}{2}}}h'h'',$$

simplifying,

$$\tfrac{d}{dx}\sin\beta = \tfrac{h''(1+h'^2)}{(1+h'^2)^{\frac{3}{2}}} - \tfrac{h'^2h''}{(1+h'^2)^{\frac{3}{2}}},$$

then,

$$\frac{d}{dx}\sin\beta = \frac{h''}{(1+h'^2)^{\frac{3}{2}}}, \tag{2.06w}$$

– (ii) The solution does not satisfy the contact angle condition, $\frac{dh_s}{dx} \to \infty$ next to the leg.

Exact Solution:
define $\xi = \frac{x}{L_c}$, $z = \frac{h_s}{L_c}$, $' = \frac{d}{d\xi}$, then,

$$\frac{-z''}{(1+(z')^2)^{\frac{3}{2}}} + z = z_\infty, \tag{2.06x}$$

where $z_\infty = \sqrt{2}$. Now, let's multiply by z' and integrate (after playing with the equation, you'll notice this is the integration factor required),

$$\frac{1}{(1+(z')^2)^{\frac{1}{2}}} + \frac{z^2}{2} - z_\infty z = Const, \tag{2.06y}$$

where $Const = 0$, since

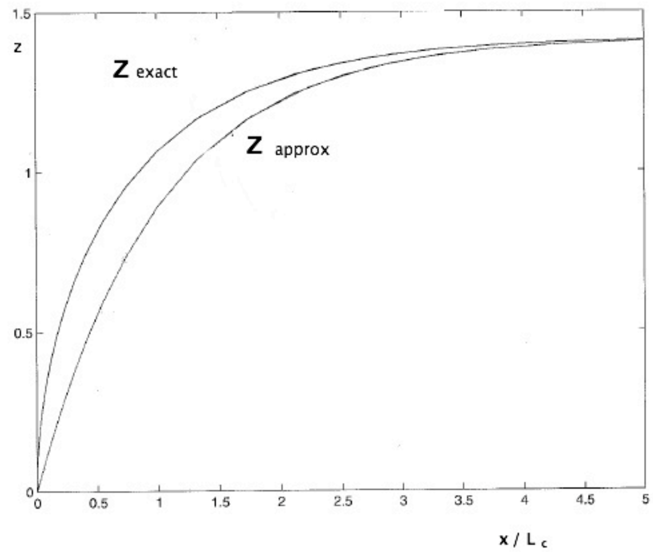
$$as\ \ \xi \to \infty,\ \ z' \to 0,\ \ and\ \ z \to z_\infty,$$

$$as\ \ \xi \to 0,\ \ z' \to \infty,\ \ and\ \ z \to 0,$$

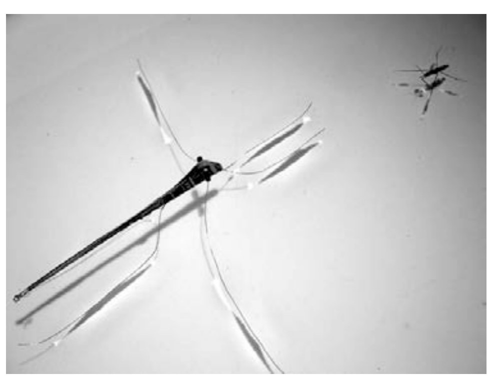
and thus,

$$z' = \left(\frac{1}{(1-(1-\frac{z}{\sqrt{2}})^2)^2} - 1\right)^{\frac{1}{2}}, \tag{2.06z}$$

where $z(0) = 0$. We can integrate numerically to obtain "z_{exact}". Note that for $\xi = 0$, z' is infinite, but $z \to 2^{\frac{1}{4}}\sqrt{\xi}$ as $\xi \to 0$, and hence we can start the integration at ξ_{min} with initial condition $z(\xi_{min}) = 2^{\frac{1}{4}}\sqrt{\xi_{min}}$. In the next figure, we plot z_{exact} and $z_{approx} = \sqrt{2}(1-e^{-\xi})$.

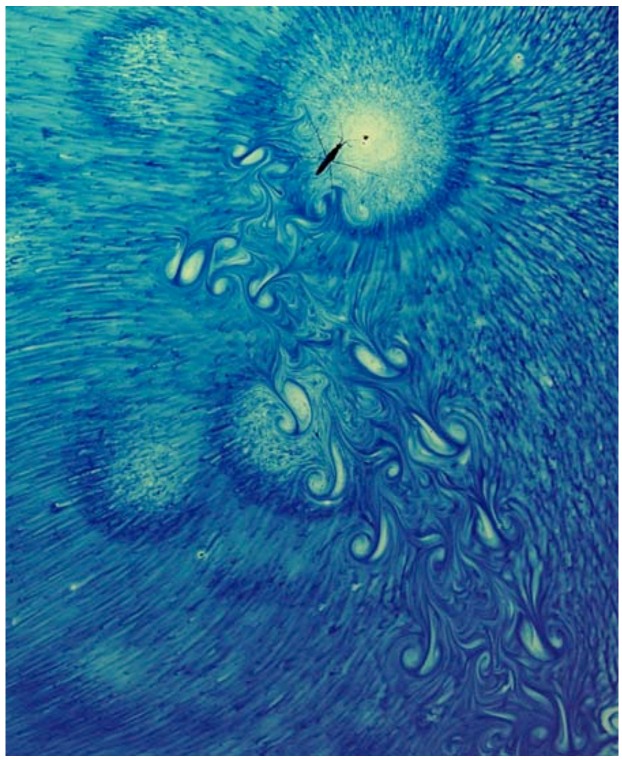


Final Comment: As the micro technologies progress (microfluidics, microfabrication and micromechanics), the role of surface tension becomes more and more important. Each day new inventions use surface tension. Below, two of such inventions are shown, the first Mechanical Water Strider, made at MIT, and the Robotic Water Strider made in Carnegie Mellon. The mechanical striders are capable of moving slowly on the surface of water using its many hydrophobic-coated legs, but since they do not have leg hairs, their lift capacity is smaller.



Water Stride (Top) and First Mechanical Water Strider (Bottom). Courtesy of John Bush, MIT, and Brian Chan, MIT.

Robotic Water Strider made in Carnegie Mellon.



Water Stride Vortices. As the water strider moves it leaves a trail of vortices that help it move. Before this observation, the method of propulsion of these insects was a mystery. This particular photo made the Nature cover some time ago, and even if it is not directly related with the problem, I couldn't resist to incorporate it to the problem. http://www-math.mit.edu/~dhu/Striderweb/striderweb.html. Courtesy of John Bush MIT.

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