2.25 ADVANCED FLUID MECHANICS Fall 2013

QUIZ 1

THURSDAY, October 10th, 7:00-9:00 P.M.

OPEN QUIZ WHEN TOLD AT 7:00 PM

THERE ARE TWO PROBLEMS
OF EQUAL WEIGHT

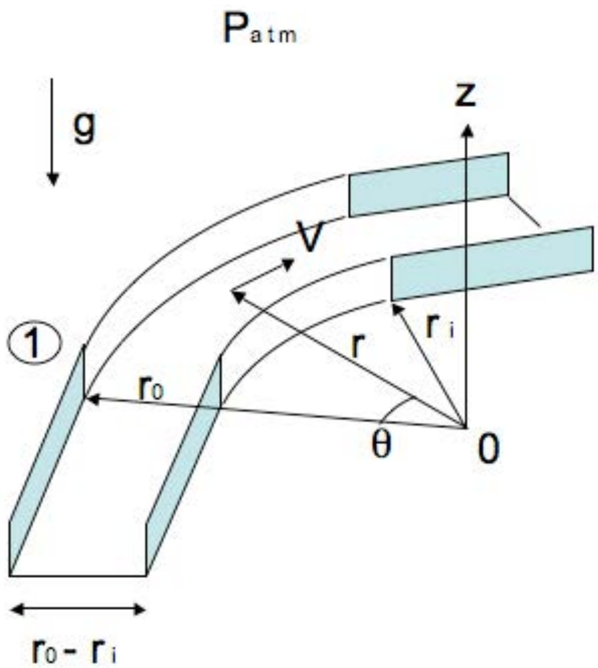
Please answer each question in DIFFERENT books

You may use the course textbook (Kundu or Panton), a binder containing your class notes, recitation problems and ONE page of handwritten notes summarizing the key equations.

Question 1: Bernoulli Equation in a Curved Channel

(a) [1] Provide a brief verbal interpretation of Bernoulli's equation (i) along a streamline, (ii) normal to a streamline.

(b) [3] Consider a 2-dimensional incompressible, inviscid, steady, *irrotational* flow. Show that in this case the Bernoulli constant $B = p/\rho + V^2/2 + gy$ is constant *everywhere throughout the flow field*, not only along a streamline. Consider a Cartesian coordinate system and let gravity act in the negative y direction.



A liquid flows steadily along an open channel of rectangular cross section having a uniform width ($r_o - r_i$). Viscous and compressibility effects are negligible. At station (1), where the flow enters a circular bend, the depth of the flow is uniformly equal to h_1 and the velocity magnitude is uniformly equal to v_1. We assume that the streamlines are straight and parallel upstream of the bend and that in the bend the streamlines are approximately circular arcs having a common center at the origin, 0. Note that the depth h in the bend will depend on r because the free surface is exposed to the uniform pressure of the atmosphere.

(c) [2] Show that the velocity in the bend is independent of both θ and z.

(d) [2] Argue that the Bernoulli constant is the same for all streamlines. Extend this result to show that the velocity in the bend is given by

$$v_\theta r = K, \tag{1}$$

where K is a constant.

(e) [2] Show that the shape of the liquid's free surface in the bend is given by

$$h(r) = h_1 + \frac{V_1^2}{2g} - \frac{K^2}{2gr^2}. \tag{2}$$

(f) [2] Show, by applying an appropriate control volume theorem, that the constant K in (1) must satisfy

$$V_1 h_1 (r_o - r_i) = K h_1 \left(1 + \frac{V_1^2}{2gh_1}\right) \ln \frac{r_o}{r_i} - \frac{K^3}{4gr_i^2}\left(1 - \frac{r_i^2}{r_o^2}\right). \tag{3}$$

Finally, consider steady rigid-body rotation of a viscous liquid at an angular velocity ω:

$$\underline{v} = \begin{bmatrix} v_r \\ v_\theta \\ v_z \end{bmatrix} = \begin{bmatrix} 0 \\ r\omega \\ 0 \end{bmatrix}. \tag{4}$$

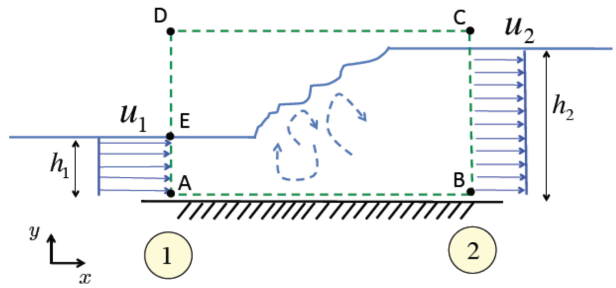
The fluid is in a circular bucket and the fluid height in the center ($r = 0$) is H. The free surface of the liquid is exposed to the uniform pressure of the atmosphere.

(g) [1] Explain why Bernoulli's equation is applicable to this case, given that the liquid has a nonzero viscosity.

(h) [2] Show that the Bernoulli constant B_{sb} for this case does not depend on z. Show that B_{sb} does depend on r and determine its value for a given r. Explain why B_{sb} varies in this case.

2. Flow over a Bump in a River

A common observation in big rivers or other fast-flowing bodies of water (e.g. during floods) is shown in the figures and sketch below. A fast moving stream of water that is steadily flowing along suddenly decelerates and the position of the free surface 'jumps' upwards. After a lot of local turbulent motion, the flow settles down again but is now steadily moving at a significantly slower speed.



We will represent the free surface height as *h(x)* and the velocity by the function *u(x)*. The fluid has constant density ρ and we will treat the problem as one-dimensional. You can assume that viscous stresses along the control surfaces of the volume shown above are negligibly small, and neglect the density of air.

PART I:

a) [2 points] consider a streamline drawn (line AB in the figure) just above the smooth flat lower surface of the channel. How is the *static pressure* in the fluid along this line related to the height of the river? How does the static pressure vary along the line DEA?

b) [4 points] Using the control volume shown in the sketch develop two expressions that relate the velocity and height of the stream at station 1 and the velocity and height of the stream at station 2. Developing a table of relevant quantities along each face of the control volume ABCDEA is highly recommended!

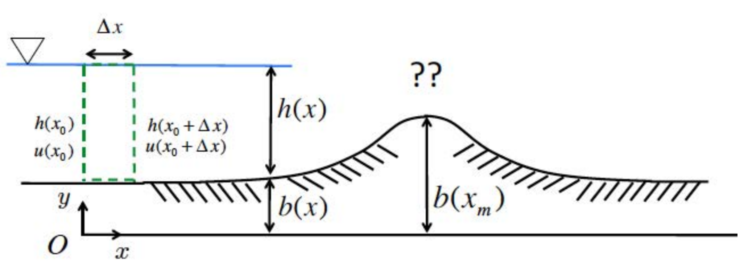
c) [2 points] Combine your expressions from (a) and (b) together to show that the speed of the river can be simply evaluated from simple measurements of the river height (e.g. using marked yardsticks attached to the channel floor):

$$u_1 = \sqrt{\frac{gh_2}{2h_1}(h_1 + h_2)} \qquad (1)$$

Part II: A deeper question to answer is *why is the water moving so fast locally to begin with*. To answer this we must consider the topography of the river bed that is upstream of station 1, as shown in the drawing below. We denote the height of the fluid stream above the river bed as *h(x)* and the height of the riverbed by *b(x)*:

d) [1 point] Consider a slice of river *dx* and show that conservation of mass can be written in the form:

$$u(x)\frac{dh(x)}{dx} + h(x)\frac{du(x)}{dx} = 0 \qquad (2)$$



e) [2 points] As the riverbed elevation rises (or the flow goes over a bump such as a submerged tree, (or house!) etc.) so that *b(x)* increases in the flow direction as shown what should happen to the height of the free surface *h(x)?* If viscous effects are negligible in this upstream region, use the Bernoulli equation along a streamline at the free surface to obtain an additional differential expression involving *U(x)*, *h(x)* and *b(x)*.

f) [2 points] Combine your equations from (d) and (e) to show that the velocity of the river, the height of the surface, and the underlying topography are all inter-related by the differential equation:

$$\frac{1}{u(x)}\frac{du(x)}{dx}\left(u(x)^2 - gh(x)\right) + g\frac{db(x)}{dx} = 0 \qquad (3)$$

g) [2 points] Finally, consider the specific point along the flow direction x_m at which the riverbed height $b(x_m)$ locally reaches a maximum, so that $db/dx|_{x_m} = 0$ as shown in the figure above. Describe very briefly in words happens to the velocity (and sketch the free surface) *just beyond* this bump in the river for the two different cases given by:

i) $u(x_m)^2 - gh(x_m) < 0$?

and

ii) $u(x_m)^2 - gh(x_m) > 0$?