13.42 Design Principles for Ocean Vehicles

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1. Fourier Series

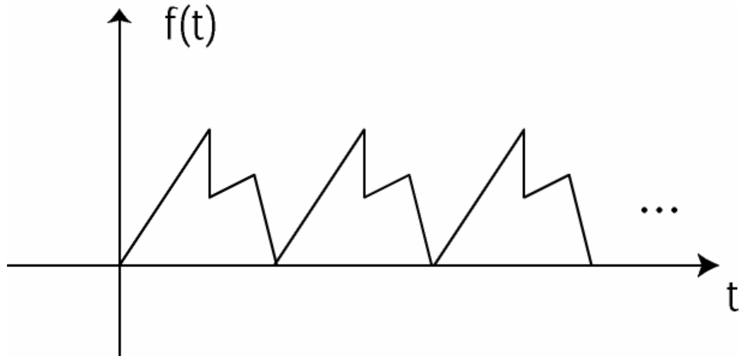

Figure 1. Periodic Signal

Fourier series are very useful in analyzing complex systems with periodic inputs as they can be used to represent a periodic signal as a summation of scaled sines and cosines:

$$f(t) = A_o + \sum_{n=1}^{\infty} \{ A_n \cos(n w_o t) + B_n \sin(n w_o t) \} \tag{1}$$

where $w_o = 2p/T$ is considered the fundamental frequency and the coefficients are written as

$$A_o = \frac{1}{T} \int_0^T f(t) dt \tag{2}$$

$$A_n = \frac{2}{T} \int_0^T f(t) \cos(n w_o t) dt \tag{3}$$

$$B_n = \frac{2}{T} \int_0^T f(t) \sin(n w_o t) dt \tag{4}$$

The Fourier series can be written more compactly using complex notation ($e^{iwt} = \cos wt + i \sin wt$).

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{inw_o t} \quad (5)$$

$$\tilde{C}_n = \frac{1}{T}\int_0^T f(t) e^{-inw_o t}\, dt \quad (6)$$

We can use Fourier series to represent a periodic, absolutely integrable function, $f(t)$.

> **N.B.** An absolutely integrable function is one whose integral converges when between minus and plus infinity or which has a finite number of discontinuities that can be integrated around:

$$\int_{-\infty}^{\infty} | f(t) |\, dt < \infty \quad (7)$$

2. Fourier Transform

The Fourier transform (FT) converts a function of time into a function of frequency. The inverse Fourier transform (IFT) reverts the function in the frequency domain back to the time domain. We will assume that $f(t)$ is absolutely integrable.

The Fourier Transform of $f(t)$ is $\tilde{f}(w)$ such that

$$\tilde{f}(w) = \int_{-\infty}^{+\infty} f(t) e^{-iwt}\, dt \quad (8)$$

The inverse Fourier Transform of $\tilde{f}(w)$ is $f(t)$

$$f(t) = \frac{1}{2p}\int_{-\infty}^{+\infty} \tilde{f}(w) e^{iwt}\, dw \quad (9)$$

Example 1: Let

$$x(t) = \begin{cases} 1; & |t| \le T_1 \\ 0; & |t| > T_1 \end{cases} \tag{10}$$

Take Fourier transform of $x(t)$:

$$\tilde{x}(w) = \int_{-\infty}^{+\infty} x(t) e^{-iwt} dt = \int_{-T_1}^{T_1} (1) e^{-iwt} dt = \frac{2 \sin wT_1}{w} \tag{11}$$

$$\tilde{x}(w) = \frac{2 \sin wT_1}{w}, \; -\infty < w < +\infty \tag{12}$$

Next we take the inverse Fourier transform of $\tilde{x}(w)$:

$$x(t) = \frac{1}{2p} \int_{-\infty}^{+\infty} \tilde{x}(w) e^{iwt} dw \tag{13}$$

$$x(t) = \frac{1}{2p} \int_{-\infty}^{+\infty} \frac{2 \sin wT_1}{w} e^{iwt} dw \tag{14}$$

So we arrive back at the original function $x(t)$

$$x(t) = \begin{cases} 1; & |t| \le T_1 \\ 0; & |t| > T_1 \end{cases} \tag{15}$$

Example 2: Given some function in frequency space, $\tilde{x}(w)$, such that

$$\tilde{x}(w) = \begin{cases} 1; & |w| \le W_1 \\ 0; & |w| > W_1 \end{cases} \tag{16}$$

We can take the inverse Fourier transform of this function

$$x(t) = \frac{1}{2p} \int_{-\infty}^{+\infty} \tilde{x}(w) e^{iwt} dw = \frac{1}{2p} \int_{-W_1}^{+W_1} (1) e^{iwt} dw = \frac{\sin W_1 t}{pt} \tag{17}$$

$$x(t) = \frac{\sin W_1 t}{p\,t}, \; -\infty < t < +\infty \tag{18}$$

Notice the similarity between the two functions in examples 1 and 2 – specifically equations (10) and (16), and also equations (12) and (18). Parseval's theorem explains that there exists a dual pair of functions with time and frequency interchanged – i.e. a symmetric pair of functions.

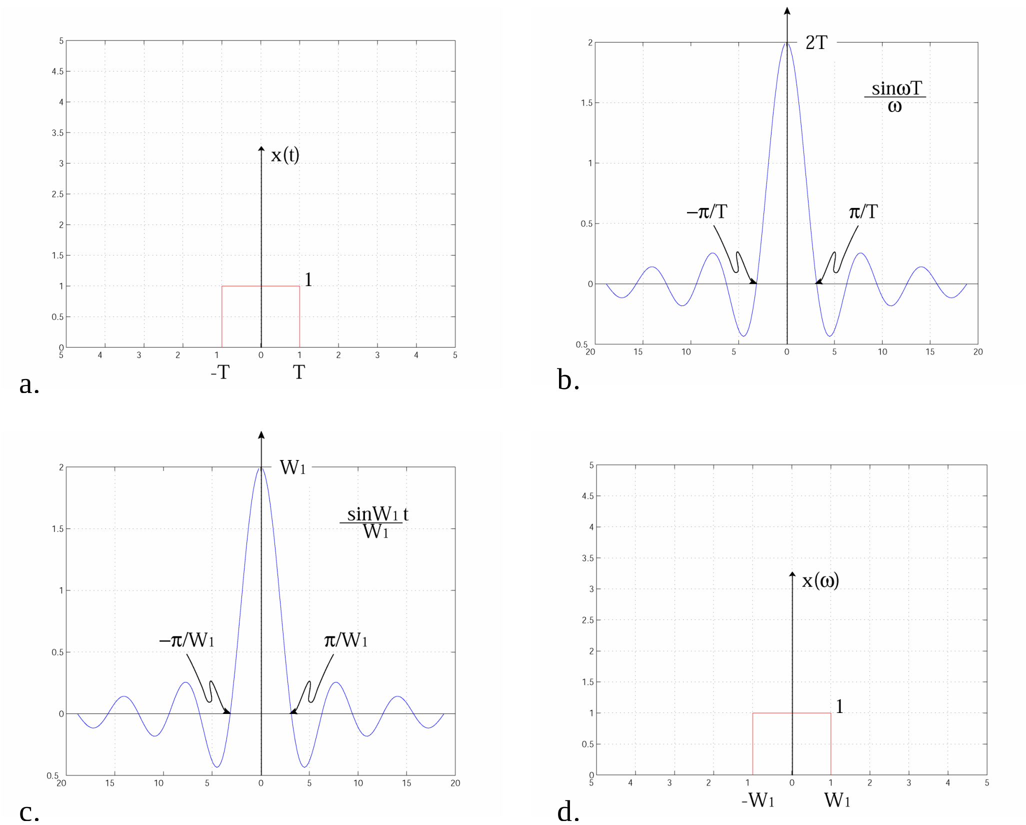

Figure 2. Symmetric Functions: (a.) Function of time: x(t); (b.) Fourier transform of x(t) in the frequency domain. (c.) Function of frequency: $\tilde{x}(w)$; (d.) Inverse Fourier transform back to the time domain.

3. Convolution and the Fourier Transform

For LTI systems, the Fourier transform turns the convolution integral into simple multiplication. Given a continuous time LTI system with impulse response, $h(t)$, and system input, $x(t)$, such that the output of the system is

$$y(t) = \int_{-\infty}^{+\infty} x(t)h(t-t)\,dt\,, \tag{19}$$

the Fourier transform of the output is written as

$$\tilde{y}(w) = FT\left\{ \int_{-\infty}^{+\infty} x(t)\,h(t-t)\,dt \right\} \tag{20}$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \left[x(t)h(t-t)dt \right] e^{-iwt}dt \tag{21}$$

We can then rewrite the exponential e^{-iwt} as $e^{-iw(t+t-t)} = e^{-iw(t-t)}\, e^{iwt}$ without changing our equation. Then let $t_1 = t - t$ and $dt_1 = dt$, such that equation (21) becomes

$$\tilde{y}(w) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t)h(t_1)e^{iw(t_1+t)}dt\,dt_1 = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x(t)h(t_1)e^{iwt_1}e^{iwt}\,dt\,dt_1 \tag{22}$$

Reordering the terms within the integrals we see that we have two separable integrals such that

$$\tilde{y}(w) = \underbrace{\left\{\int_{-\infty}^{+\infty} h(t_1)e^{-iwt_1}dt_1\right\}}_{\tilde{h}(w)}\underbrace{\left\{\int_{-\infty}^{+\infty} x(t)e^{-iwt}\,dt\right\}}_{\tilde{x}(w)}. \tag{23}$$

Note the form of the two separate integrals on the RHS of equation (42) – they are the Fourier transform of the input and the impulse response. Now, the Fourier transform out the system output is simply the multiplication of the Fourier transform of the input and impulse response:

$$\tilde{y}(w) = \tilde{h}(w) \cdot \tilde{x}(w) . \tag{24}$$

Where $\tilde{h}(w)$ is the Fourier transform of the impulse response and is referred to as the TRANSFER FUNCTION, commonly written as $H(w)$.

$$\tilde{y}(w) = H(w) \cdot \tilde{x}(w) \tag{25}$$

4. Recap of Fourier Transform

- Convolution: $y(t) = h(t) * x(t)$
- Multiplication: $\tilde{y}(w) = H(w) \cdot \tilde{x}(w)$
- Linearity: If $x(t) \to \tilde{x}(w)$ and $y(t) \to \tilde{y}(w)$ then $ax(t) + by(t) \to a\tilde{x}(w) + b\tilde{y}(w)$.

5. LTI Systems and Fourier Transforms

To evaluate a LTI system you can use the Fourier transform and convolution to find the output $y(t)$ given the input and the transfer function.

1. $x(t) \to \tilde{x}(w)$ take the FT of the input.
2. $\tilde{y}(w) = H(w)\tilde{x}(w)$ convolve the FT of the input and the transform function.
3. $\tilde{y}(w) \to y(t)$ take the inverse FT to find the system output.

For a given a harmonic input $u(t)$ and transfer function $H(w)$ we can easily write the output of our LTI system in terms of the amplitude and freque ncy of the input and the amplitude and argument (phase) of the transfer function.

$$x(t) = x_o \cos(w_o t + y\ ) \tag{26}$$

$$y(t) = y_o \cos(w_o t + y\ + y_1) \tag{27}$$

where $y_o = x_o |H(w_o)|$ is the amplitude of the response and $y_1 = arg\{H(w_o)\}$ is the phase shift of the response from the input.

From complex math we can write a function $H(w)$ in terms of its amplitude, $|H(w)|$, and its argument, $\angle H(w)$ as follows .

$$H(w) = |H(w)| e^{i\angle H(w)} \tag{28}$$

Now lets look at the real part of a complex function:

$$u(t) = Re\{x_o e^{i(w_o t + y)}\} = Re\{\tilde{x}_o e^{iw_o t}\} \tag{29}$$

where $\tilde{x}_o = x_o e^{iy}$. Taking the convolution $y(t) = h(t) * x(t)$ we have

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(t) Re\{\tilde{x}_o e^{iw_o(t-t)}\} dt \\ &= Re\left\{\left\{\int_{-\infty}^{+\infty} h(t) e^{-iw_o t} dt\right\} \tilde{x}_o e^{iw_o t}\right\} \\ &= Re\{H(w_o)\tilde{x}_o e^{iw_o t}\} \end{aligned}$$

where $H(w_o) = |H(w_o)| e^{i\angle H(w_o)}$.

Since we are only interested in the real part of $y(t)$ (the input was cosine) we have

$$y(t) = x_o |H(w_o)| \cos(w_o t + y + \angle\{H(w_o)\} \tag{30}$$

This process can be extended in the same fashion for an input of sine. However instead of the real component we look at the imaginary component of $e^{iwt} = \cos(wt) + i\sin(wt)$ which is $\sin(wt)$.

6. Useful References

There are several good texts on signals and systems that give a thorough discussion of Linear Time Invariant systems and their properties. A few suggestions are listed below.

- A.V. Oppenhein, A. S. Willsky, S.H. Nawab (1997) *Signals and Systems, 2nd ed.* Prentice Hall Signal Processing Series, New Jersey. (6.003 Course text book)
- Triantafyllou and Chryssostomidis, (1980) "Environment Description, Force Prediction and Statistics for Design Applications in Ocean Engineering" Course Supplement.