NETWORK MODELS OF TRANSMISSION LINES AND WAVE BEHAVIOR

MOTIVATION:

Ideal junction elements are power-continuous.

Power out = power out instantaneously

In reality, power transmission takes finite time.

Power out ≠ power in

Consider a lossless, continuous uniform beam.

Model it as a number of segments.

In the limit as the number of segments approaches infinity, the model competently describes wave behavior

(e.g. wave speed, characteristic impedance).

What if number of segments is finite?

- **How do you choose the parameters of each segment?**
- **What wave speed and characteristic impedance are predicted by this finite-segment model?**

APPROACH:

Consider the transmission line and each of its segments as 2-port elements relating 2 pairs of 2 variables.

There are 4 possible forms (choices of input and output).

Two of them are *causal*, the impedance form:

$$\begin{bmatrix} e_a \\ e_b \end{bmatrix} = \begin{bmatrix} \; Z \; \end{bmatrix} \begin{bmatrix} f_a \\ f_b \end{bmatrix}$$

|—— **Z** ——|

and the admittance form:

$$\begin{bmatrix} f_a \\ f_b \end{bmatrix} = \begin{bmatrix} \; Y \; \end{bmatrix} \begin{bmatrix} e_a \\ e_b \end{bmatrix}$$

——| **Y** |——

The remaining two forms are *a-causal*:

$$\begin{bmatrix} e_a \\ f_a \end{bmatrix} = \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} e_b \\ f_b \end{bmatrix}$$

$$\begin{bmatrix} e_b \\ f_b \end{bmatrix} = \begin{bmatrix} M \end{bmatrix} \begin{bmatrix} e_a \\ f_a \end{bmatrix}$$

M is called a transmission matrix.

—— **M** ——

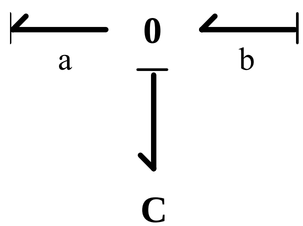
The benefit of the a-causal forms is that segments may be concatenated by matrix multiplication.

$$\text{——}\ \underset{M_1}{\overset{\mathbf{M}}{\cdot\cdot}}\ \text{——}\ \underset{M_2}{\overset{\mathbf{M}}{\cdot\cdot}}\ \text{——} \quad = \quad \text{——}\ \underset{M_1 M_2}{\overset{\mathbf{M}}{\cdot\cdot}}\ \text{——}$$

HOW IS M STRUCTURED?

Consider the elements of a linear, lossless transmission line.

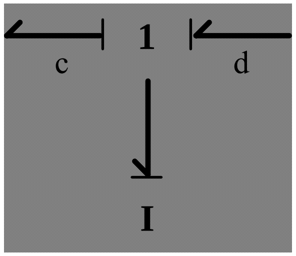
Using the Laplace variable, s, we may describe the capacitor as an impedance:



$$e_a = \frac{1}{Cs}(f_b - f_a)$$

$$\begin{bmatrix} e_a \\ e_b \end{bmatrix} = \begin{bmatrix} -1/Cs & 1/Cs \\ -1/Cs & 1/Cs \end{bmatrix} \begin{bmatrix} f_a \\ f_b \end{bmatrix}$$

We may describe the inertia as an admittance:



$$f_c = \frac{1}{Is}(e_d - e_c)$$

$$\begin{bmatrix} f_c \\ f_d \end{bmatrix} = \begin{bmatrix} -1/Is & 1/Is \\ -1/Is & 1/Is \end{bmatrix} \begin{bmatrix} e_c \\ e_d \end{bmatrix}$$

Note that there is no simple way to concatenate these.

Instead, describe them as transmission matrices.

The capacitor:

$e_a = e_b$

$f_a = f_b - Cs\, e_a$

$$\begin{bmatrix} e_a \\ f_a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -Cs & 1 \end{bmatrix} \begin{bmatrix} e_b \\ f_b \end{bmatrix}$$

The inertia:

$f_c = f_d$

$e_c = e_d - Is\, f_d$

$$\begin{bmatrix} e_c \\ f_c \end{bmatrix} = \begin{bmatrix} 1 & -Is \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_d \\ f_d \end{bmatrix}$$

These may be concatenated easily by multiplication.

NOTE:

- **The determinant of the transmission matrix, M, is unity.**
- **C (or I) can be replaced by any 1-port system.**

GENERAL FORM

To find the general form of the transmission matrix, consider the properties of an idealized lossless transmission line.

1. uniformly and infinitely decomposable.

(The line can be divided into any number of identical pieces)

Denote the transmission matrix for one-nth of the line as $M_{1/n}$.

$$— M — = — M_{1/2} — M_{1/2} — = — M_{1/3} — M_{1/3} — M_{1/3} —$$

i.e. $M = (M_{1/2})^2 = (M_{1/3})^3 = (M_{1/n})^n$

Thus M is "self-replicating".

2. transposable.

(The line should look the same from both sides)

$$\begin{bmatrix} e_a \\ f_a \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} e_b \\ f_b \end{bmatrix}$$

Determinant = 1 so the inverse is

$$\begin{bmatrix} e_b \\ f_b \end{bmatrix} = \begin{bmatrix} D & -B \\ -C & A \end{bmatrix} \begin{bmatrix} e_a \\ f_a \end{bmatrix}$$

Now change the sign so that, as before, power is positive in the direction of input to output.

$f_a = f_d$; $e_a = -e_d$; $f_c = f_b$; $e_c = -e_b$

$$\begin{bmatrix} e_c \\ f_c \end{bmatrix} = \begin{bmatrix} D & B \\ C & A \end{bmatrix} \begin{bmatrix} e_d \\ f_d \end{bmatrix}$$

Comparing:

$A = D$ and $A^2 - 1 = BC$

(determinant = 1)

Next concatenate two one-nth segments of the line.

$$(M_{1/n})(M_{1/n}) = (M_{2/n})$$

$$\begin{bmatrix} A_{1/n} & B_{1/n} \\ C_{1/n} & A_{1/n} \end{bmatrix}\begin{bmatrix} A_{1/n} & B_{1/n} \\ C_{1/n} & A_{1/n} \end{bmatrix} = \begin{bmatrix} 2A_{1/n}^2 - 1 & 2A_{1/n}B_{1/n} \\ 2A_{1/n}C_{1/n} & 2A_{1/n}^2 - 1 \end{bmatrix} = \begin{bmatrix} A_{2/n} & B_{2/n} \\ C_{2/n} & A_{2/n} \end{bmatrix}$$

write

$$A_{1/n} = \cosh \Gamma/n \qquad B_{1/n} = Z_o \sinh \Gamma/n$$

$$C_{1/n} = \frac{A_{1/n}^2 - 1}{B_{1/n}} = \frac{1}{Z_o} \sinh \Gamma/n = Y_o \sinh \Gamma/n$$

$$\cosh \Gamma = \tfrac{1}{2}(e^{\Gamma} + e^{-\Gamma}) \qquad \cosh^2 \Gamma = \tfrac{1}{2}(\cosh 2\Gamma + 1)$$

$$\sinh \Gamma = \tfrac{1}{2}(e^{\Gamma} - e^{-\Gamma}) \qquad \sinh \Gamma \cosh \Gamma = \tfrac{1}{2}(\sinh 2\Gamma)$$

Thus

$$M_{1/n} = \begin{bmatrix} \cosh \Gamma/n & Z_o \sinh \Gamma/n \\ Y_o \sinh \Gamma/n & \cosh \Gamma/n \end{bmatrix}$$

Note that $\cosh^2 \Gamma - \sinh^2 \Gamma = 1$ and $Z_o Y_o = 1$, so $\mathrm{DET}(M_{1/n}) = 1$ as required.

Check the "self-replicating" property:

$$2A_{1/n}^2 - 1 = \cosh 2\Gamma/n$$

$$2A_{1/n}B_{1/n} = Z_o \sinh 2\Gamma/n$$

$$2A_{1/n}C_{1/n} = Y_o \sinh 2\Gamma/n$$

Therefore

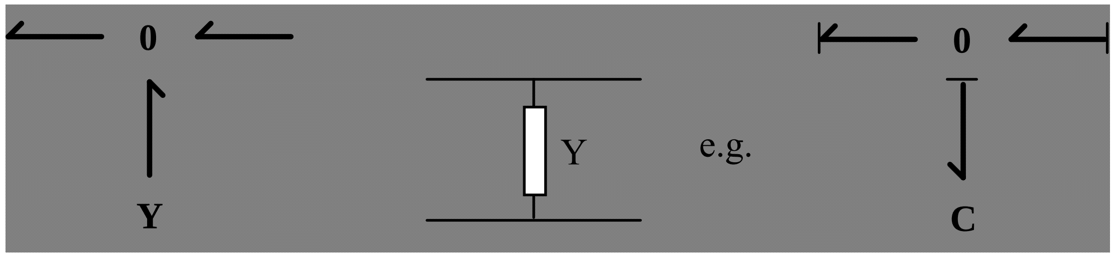
$$M = \begin{bmatrix} \cosh\Gamma & Z_o \sinh\Gamma \\ Y_o \sinh\Gamma & \cosh\Gamma \end{bmatrix}$$

Γ is a *delay parameter* inversely proportional to wave speed (see below).

Z_o is a *characteristic impedance*

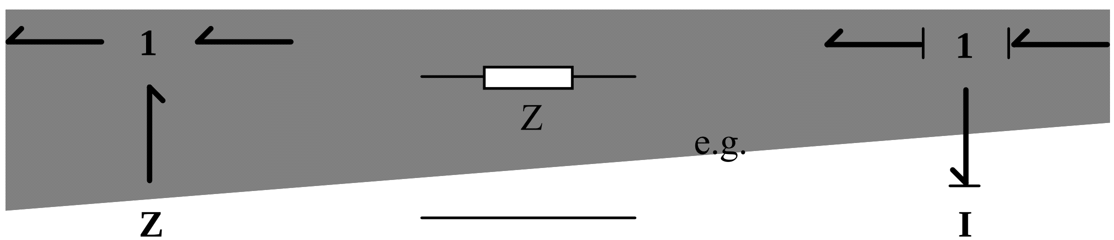
NEXT CONSIDER A LINE COMPOSED OF REPEATED SEGMENTS.

For convenience we will use the following notation:



This is a "shunt admittance".

$$M = \begin{bmatrix} 1 & 0 \\ Y & 1 \end{bmatrix}$$



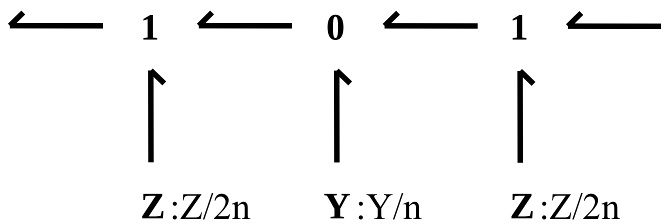
This is a "series impedance".

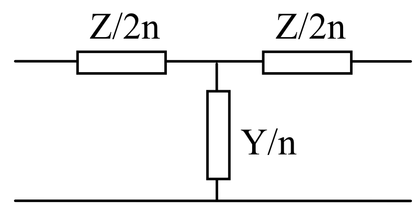
$$M = \begin{bmatrix} 1 & Z \\ 0 & 1 \end{bmatrix}$$

In the linear case Y and Z may be any functions of the Laplace variable.

Note (a) the unusual sign convention and (b) the derivative causality on the energy storage elements.

Use a symmetric primitive segment for the transmission line, the T-net:





$$\begin{bmatrix} e_a \\ f_a \end{bmatrix} = \begin{bmatrix} 1 & Z/2n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ Y/n & 1 \end{bmatrix} \begin{bmatrix} 1 & Z/2n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} e_b \\ f_b \end{bmatrix} =$$

$$\begin{bmatrix} 1 + ZY/2n^2 & Z/n + Z^2Y/4n^3 \\ Y/n & 1 + ZY/2n^2 \end{bmatrix} \begin{bmatrix} e_b \\ f_b \end{bmatrix}$$

Equate this to $M_{1/n}$

$$\cosh \Gamma/n = 1 + ZY/2n^2$$

$$\frac{Z/n + Z^2Y/4n^3}{Y/n} = Z/Y + Z^2/4n^2 = Z_o^2$$

$$\Gamma = n \cosh^{-1}(1 + ZY/2n^2)$$

$$Z_o = \sqrt{Z/Y + Z^2/4n^2}$$

These formulae permit a model with a *finite* number of segments to reproduce the wave speed and characteristic impedance of a continuous, linear lossless transmission line *exactly*.

- **Note that (for a uniform transmission line) Z_o is the same for all segments, independent of line length.**
- **The delay parameter Γ is proportional to the number of segments, i.e. proportional to line length.**

Consider the limit as the number of segments approaches infinity.

From the series expansion of cosh $\Gamma/n = 1 + \frac{1}{2}(\Gamma/n)^2 + \ldots$

$$\lim_{n\to\infty} \Gamma = \sqrt{ZY}$$

and

$$\lim_{n\to\infty} Z_o = \sqrt{Z/Y}$$

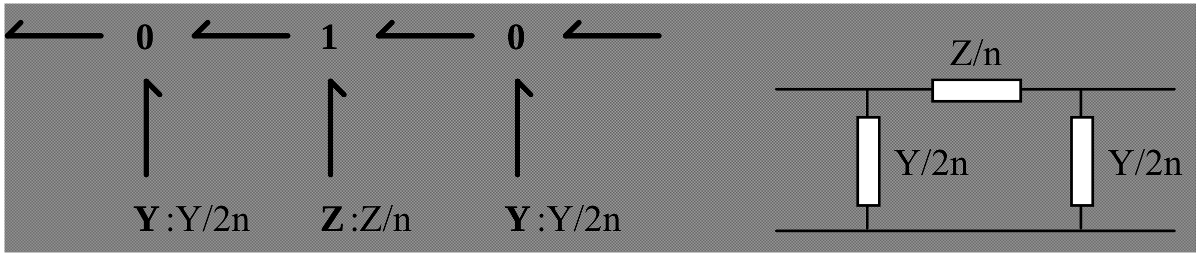
Note also that

$$\lim_{n\to\infty} M_{1/n} = 1$$

That is, as the number of segments in a approaches infinity, the transmission matrix for each approaches that of an power-continuous 0 or 1 junction

— as it should.

A comparable derivation may be performed for the other possible symmetric primitive segment, the π-net:

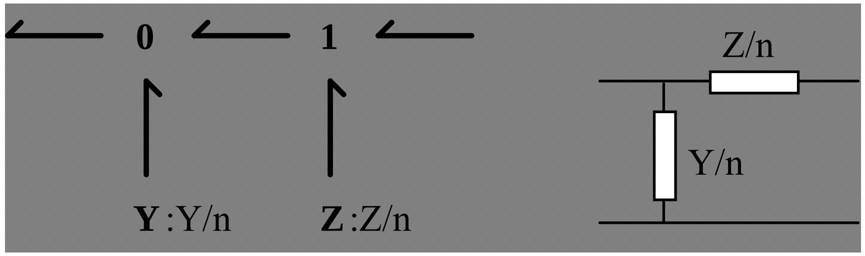


The formulae are:

$$\Gamma = n \cosh^{-1}(1 + ZY/2n^2)$$

$$Y_o = \sqrt{Y/Z + Y^2/4n^2}$$

Aside: Note that if the primitive segment is asymmetric, e.g. as follows



The transmission matrix is

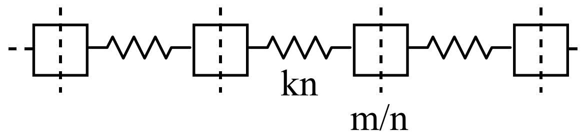
$$\begin{bmatrix} e_a \\ f_a \end{bmatrix} = \begin{bmatrix} 1 & Z/n \\ Y/n & YZ/n^2 + 1 \end{bmatrix} \begin{bmatrix} e_b \\ f_b \end{bmatrix}$$

This can only be compared to the continuous transmission line matrix in the limiting case, i.e. when $YZ/n^2 << 1$.

Thus a finite number of these segments cannot exactly reproduce the wave speed and characteristic impedance of a uniform, continuous transmission line.

EXAMPLE:

Continuous, lossless, linear beam model

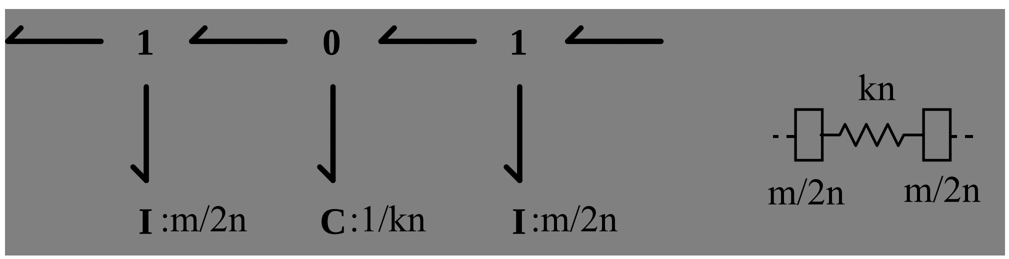


Use a T-net model for each segment:

In the Laplace domain:

mass element: F = msv Z = ms

stiffness element F = kx v = Fs/k Y = s/k



As the number of segments approaches infinity:

Delay parameter

$$\Gamma = \sqrt{ms\, s/k} = s\sqrt{m/k} = s/\omega_n$$

where

m = net beam mass,

k = net beam stiffness

ω_n = undamped natural frequency

Characteristic impedance

$$Z_o = \sqrt{\frac{ms}{s/k}} = \sqrt{mk}$$

Note that Z_o has the units of resistance (damping)

EXAMPLE:

Continuous, lossless, linear beam terminated by a lumped impedance.

$$— \mathbf{M} — \mathbf{Z}:Z_t$$

Write out the transmission matrix equations:

$$e_a = \cosh \Gamma \, e_b + Z_o \sinh \Gamma \, f_b$$

$$f_a = Y_o \sinh \Gamma \, e_b + \cosh \Gamma \, f_b$$

and the terminal impedance equation

$$e_b = Z_t \, f_b$$

substitute

$$e_a = (\cosh\Gamma\, Z_t + Z_o \sinh\Gamma)\, f_b$$

$$f_a = (Y_o \sinh\Gamma\, Z_t + \cosh\Gamma)\, f_b$$

whence

$$e_a = \frac{(\cosh\Gamma\, Z_t + Z_o \sinh\Gamma)}{(Y_o \sinh\Gamma\, Z_t + \cosh\Gamma)}\; f_a$$

Note that if $Z_t = Z_o = Y_o^{-1}$

$$e_a = Z_o \frac{(\cosh\Gamma + \sinh\Gamma)}{(\sinh\Gamma + \cosh\Gamma)}\; f_a = Z_o\, f_a$$

If the terminal impedance is matched to the characteristic impedance, the line appears identical to a lumped impedance Z_o.

Note that this result is independent of the length of the transmission line.

Because Z_o has the units of resistance, an infinitely long transmission line appears identical to a resistance.
Power may be put into the line, and though energy is conserved (the line is "lossless"), no power ever comes back out.

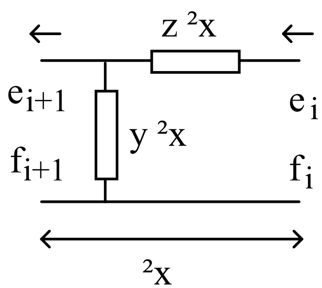
POINT:

The distinction between “lossless” and “dissipative” behavior may not be as clear as it is sometimes presented.

WAVE BEHAVIOR

Consider an asymmetric model of a section of a transmission line Δx long.

(this is OK as we will take the limit as $\Delta x \to 0$, equivalent to $n \to \infty$)



$$e_{i+1} = e_i - z\,\Delta x\, f_i$$

$$f_{i+1} = f_i - y\,\Delta x\, e_{i+1}$$

$$\Delta e = -z\,\Delta x\, f_i$$

$$\Delta f = -y\,\Delta x\, e_{i+1}$$

where z and y are impedance and admittance per unit length.

For a line of length L

$z = Z/L$

$y = Y/L$

Take the limit as Δx approaches zero

$$\lim_{\Delta x \to 0} \frac{\partial e}{\partial x} = -\, z\, f$$

$$\lim_{\Delta x \to 0} \frac{\partial f}{\partial x} = -\, y\, e$$

(partials as e and f are functions of s as well as x.)

$$\frac{\partial^2 e}{\partial x^2} = zy\, e = \gamma^2\, e$$

γ is the delay parameter per unit length.

$$\Gamma = \sqrt{ZY} = \sqrt{zLyL} = L\sqrt{zy} = \gamma L$$

Remember that the impedance per unit length is the same as the total impedance.

$$Z_o = \sqrt{Z/Y} = \sqrt{z/y}$$

A general solution to this partial differential equation is

$$e(x) = A\, e^{-\Gamma x} + B\, e^{\Gamma x}$$

where A and B are determined by the boundary conditions.

One end free: e(0) = 0 = A + B $\therefore$ B = –A

Drive the other end sinusoidally: e(L) = cos ωt

After a little algebra

(I recommend you check)

$$e(x) = \frac{1}{2\sin(\omega L/c)}\left[\sin(\omega t + \omega x/c) - \sin(\omega t - \omega x/c)\right]$$

where $c = j\omega/\gamma$ (j is the unit imaginary number)

This is a sum of a left-going wave and a right-going wave.

Identify a point of constant phase:

$d(\omega t + \omega x/c) = 0$

$dx/dt = \pm c$

Thus c is the *phase velocity*

$c = f\lambda$

where f is frequency in cycles/second (Hertz) and λ is wavelength.

$f\lambda = \omega\lambda/2\pi$

EXAMPLE:

Continuous, lossless, linear model of a uniform beam of length L, area A, density ρ, and Young's modulus E.

Net mass

$m = \rho AL$

Net stiffness

$k = EA/L$

$Z = ms = \rho ALs$

$Y = s/k = sL/EA$

Per unit length:

$z = \rho As$

$y = s/EA$

Characteristic impedance:

$$Z_o = \sqrt{z/y} = \sqrt{\rho AEA} = A\sqrt{\rho E}$$

Delay parameter:

$$\Gamma = \sqrt{ZY} = \sqrt{\rho ALs\, sL/EA} = sL\sqrt{\rho/E}$$

Per unit length:

$$\gamma = s\sqrt{\rho/E}$$

substitute s = jω and find the wave speed:

$$c = 1/\gamma = \sqrt{E/\rho}$$

— a familiar result.

WAVE SCATTERING VARIABLES

The following change of variables is extremely useful and provides considerable further insight.

Define the variables u and v as follows:

$$e = (u + v)\frac{a}{\sqrt{2}}$$

$$f = (u - v)\frac{1}{a\sqrt{2}}$$

where a is a constant parameter to be determined.

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} a/\sqrt{2} & a/\sqrt{2} \\ 1/a\sqrt{2} & -1/a\sqrt{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1/a\sqrt{2} & a/\sqrt{2} \\ 1/a\sqrt{2} & -a/\sqrt{2} \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$

Apply this change of variables to the transmission matrix.

$$M = \begin{bmatrix} \cosh\Gamma & Z_o \sinh\Gamma \\ Y_o \sinh\Gamma & \cosh\Gamma \end{bmatrix}$$

$$\begin{bmatrix} e_a \\ f_a \end{bmatrix} = \begin{bmatrix} \cosh\Gamma & Z_o \sinh\Gamma \\ Y_o \sinh\Gamma & \cosh\Gamma \end{bmatrix} \begin{bmatrix} e_b \\ f_b \end{bmatrix}$$

$$\begin{bmatrix} u_a \\ v_a \end{bmatrix} = \begin{bmatrix} 1/a\sqrt{2} & a/\sqrt{2} \\ 1/a\sqrt{2} & -a/\sqrt{2} \end{bmatrix} \begin{bmatrix} \cosh\Gamma & Z_o \sinh\Gamma \\ Y_o \sinh\Gamma & \cosh\Gamma \end{bmatrix} \begin{bmatrix} a/\sqrt{2} & a/\sqrt{2} \\ 1/a\sqrt{2} & -1/a\sqrt{2} \end{bmatrix} \begin{bmatrix} u_b \\ v_b \end{bmatrix}$$

$$\begin{bmatrix} u_a \\ v_a \end{bmatrix} = \begin{bmatrix} \cosh\Gamma + \left(\frac{a^2 Y_o}{2} + \frac{Z_o}{2a^2}\right)\sinh\Gamma & \left(\frac{a^2 Y_o}{2} - \frac{Z_o}{2a^2}\right)\sinh\Gamma \\ \left(-\frac{a^2 Y_o}{2} + \frac{Z_o}{2a^2}\right)\sinh\Gamma & \cosh\Gamma - \left(\frac{a^2 Y_o}{2} + \frac{Z_o}{2a^2}\right)\sinh\Gamma \end{bmatrix} \begin{bmatrix} u_b \\ v_b \end{bmatrix}$$

Define $\frac{a^2 Y_o}{2} = \frac{Z_o}{2a^2}$ $a^4 = Z_o^2$ $a^2 = \pm Z_o$ $a = \pm\sqrt{Z_o}$ or $a = \pm j\sqrt{Z_o}$

$$\begin{bmatrix} u_a \\ v_a \end{bmatrix} = \begin{bmatrix} \cosh\Gamma + \sinh\Gamma & 0 \\ 0 & \cosh\Gamma - \sinh\Gamma \end{bmatrix} \begin{bmatrix} u_b \\ v_b \end{bmatrix}$$

$$\begin{bmatrix} u_a \\ v_a \end{bmatrix} = \begin{bmatrix} e^{\Gamma} & 0 \\ 0 & e^{-\Gamma} \end{bmatrix} \begin{bmatrix} u_b \\ v_b \end{bmatrix}$$

Inverting the second equation

$$\begin{bmatrix} u_a \\ v_b \end{bmatrix} = \begin{bmatrix} e^{\Gamma} & 0 \\ 0 & e^{\Gamma} \end{bmatrix} \begin{bmatrix} u_b \\ v_a \end{bmatrix}$$

Now if you recall that $\Gamma = s/\omega_n$ you can see that $\exp(\Gamma) = \exp(s/\omega_n)$ is the Laplace transform of a delay time of $1/\omega_n$.

Hence the term "delay parameter" for Γ.

This change of variables has reduced the transmission line to a delay operator.

The variable u is a characteristic of the right-going or "fore wave".

The variable u_a is a delayed version of u_b.

The variable v is a characteristic of the left-going or "back wave".

The variable v_b is a delayed version of v_a.

Together, the variables u and v are known as wave scattering variables.

RELATION TO THE ONE-DIMENSIONAL WAVE EQUATION.

Rewrite the equations for an asymmetric segment in the time domain, using $z = ms/L = \rho As$ and $y = s/kL = s/EA$ and substituting d/dt for s.

$$e_{i+1}(x+\Delta x,t) = e_i(x,t) - \frac{m\Delta x}{L}\frac{d}{dt}\, f_i(x,t)$$

$$f_{i+1}(x+\Delta x,t) = f_i(x,t) - \frac{\Delta x}{kL}\frac{d}{dt}\, e_{i+1}(x+\Delta x,t)$$

Take the limit as $\Delta x \rightarrow 0$

$$\frac{\partial}{\partial x} e(x,t) = - \frac{m}{L} \frac{\partial}{\partial t} f(x,t)$$

$$\frac{\partial}{\partial x} f(x,t) = - \frac{1}{kL} \frac{\partial}{\partial t} e(x,t)$$

or

$$\frac{\partial}{\partial x} e(x,t) = - \rho A \frac{\partial}{\partial t} f(x,t)$$

$$\frac{\partial}{\partial x} f(x,t) = - \frac{1}{EA} \frac{\partial}{\partial t} e(x,t)$$

Differentiate again

$$\frac{\partial^2}{\partial x^2} e(x,t) = -\rho A \frac{\partial}{\partial t}\frac{\partial}{\partial x} f(x,t) = -\rho A \frac{\partial}{\partial t}\left(-\frac{1}{EA}\frac{\partial}{\partial t} e(x,t)\right)$$

$$\frac{\partial^2 e}{\partial x^2} = \frac{\rho}{E}\frac{\partial^2 e}{\partial t^2}$$

This is the one-dimensional wave equation

$$\frac{\partial^2 e}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 e}{\partial t^2} = 0$$

where $c = \sqrt{E/\rho} = 1/\gamma$ is the wave speed (phase velocity)

Rewrite:

$$\left(\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x}-\frac{1}{c}\frac{\partial}{\partial t}\right)e=0$$

Change variables:

$g = x + ct$

$h = x - ct$

$e(x,t) = e(g,h)$

$$\frac{\partial e}{\partial x}=\frac{\partial e}{\partial g}+\frac{\partial e}{\partial h}$$

$$\frac{1}{c}\frac{\partial e}{\partial t}=\frac{1}{c}\frac{\partial e}{\partial g}+\frac{1}{c}\frac{\partial e}{\partial h}$$

$$\left(\frac{\partial}{\partial x}+\frac{1}{c}\frac{\partial}{\partial t}\right)e = 2\frac{\partial e}{\partial g}$$

$$\left(\frac{\partial}{\partial x}-\frac{1}{c}\frac{\partial}{\partial t}\right)e = 2\frac{\partial e}{\partial h}$$

$$\therefore 4\frac{\partial^2 e}{\partial g \partial h} = 0$$

Solutions are of the form

$e(g,h) = u(g) + v(h)$

$e(x,t) = u(x + ct) + v(x - ct)$

where $u(\cdot)$ and $v(\cdot)$ are arbitrary functions (restricted only by continuity).

The functions $u(\cdot)$ and $v(\cdot)$ may be regarded as basis functions for the solution set.

Alternatively, they may be regarded as combinations of other basis functions, e.g. sinusoids.

The function u(x + ct) is a wave of shape u(x) traveling rightward (e.g. the "fore wave") at speed c.

Function v(x – ct) is a wave of shape v(x) traveling leftward (e.g. the "back wave") at speed c.

Now consider only the right going or fore wave, u(x + ct).

In general we expect power to be proportional to the square of its magnitude.

(We have assumed the medium to be linear and u(g) may be described as a linear composition of sinusoidal functions each of which contributes to power in proportion to the square of its magnitude).

Thus the power transported to the right $P_{\mathbf{u}} \propto \mathbf{u^2(g)}$**.**

By a similar argument the power transported to the left $P_{\mathbf{v}} \propto \mathbf{v^2(h)}$**.**

The net power transported to the right, $P \propto \mathbf{u^2(g) - v^2(h)}$**.**

But we may express power transport as a product $\mathcal{P}$ = e(x,t)f(x,t) so

$$\mathbf{P} \propto \left(u(g) + v(h)\right)\left(u(g) - v(h)\right)$$

Therefore if

$$e(x,t) = u(x + ct) + v(x - ct)$$

then

$$f(x,t) = u(x + ct) - v(x - ct)$$

It is important to recognize that, despite terminology, the change of variables

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} a/\sqrt{2} & a/\sqrt{2} \\ 1/a\sqrt{2} & -1/a\sqrt{2} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1/a\sqrt{2} & a/\sqrt{2} \\ 1/a\sqrt{2} & -a/\sqrt{2} \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$

may be introduced independent of any assumptions used to define wave behavior (e.g., infinite continuous line, etc.)