2.093 Term Project
Frequency Response of Trees

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1 Introduction and Motivation

Plants have always been a ubiquitous component of human experience. Plants are a permanent fixture in the natural world, imported or accounted for in the urban setting, and harvested to sustain a population. When one thinks of plants, the mind naturally wanders to the swaying of branches or waves rippling through crop fields. These images are apt, as our most common experiences with plants (the live ones, at least) involve a common component: wind. The response of plants to wind loading is an essential question. Permanent uprooting as well as mechanical failure of trunks, branches, and/or stems causes economic damage, poses a threat to human safety, and can cause the loss of significant portions of agricultural crops [1]. Perhaps surprisingly, the other major concern for modeling the interaction of wind and plants comes from the computer graphics industry, where a realistic behavior must be attained in order to fool the eye - and must be done in the computationally cheapest way possible [1].

The current study investigates the response of trees to harmonic wind loading. This response is dictated by three main components: (1) Time-varying excitation load caused by wind-induced drag; (2) The dynamic behavior of the tree, and; (3) damping processes. Wind inputs energy into the system. The dynamic behavior of the tree is dictated by the exchange between kinetic energy (expressed by portions of the tree moving with certain velocities) and elastic strain energy (expressed by the deformation of portions of the tree that have the potential to spring back into their original configuration). Damping removes energy of the system and is responsible for the decay of oscillatory amplitudes. It is caused mainly by turbulence created by the interaction between parts of the tree and air as well as (to a lesser extent) the production of heat through the internal material friction in the tree.

Obviously, the dynamic response of a tree depends heavily on its actual geometric structure as well as its physical properties. Most trees have a branched structure, expressing repeating architectures on different scales. One common mode of branching is sympodial branching, which involves a single segment growing until it branches into two lateral segments and no axial segment continuing [2]. This architecture shall be used in the model tree described in the next section.

The paper shall proceed to describe a model tree and its material properties and explain the finite element model used. It shall then provide a simplified model that was tested for agreement with a mathematical solution. The paper shall then proceed to explain results obtained from finite element modeling. Natural frequencies for the first 25 modes shall be shown. The evolution of mode shapes shall be illustrated for both increasing frequencies as well as decreasing element size. The undamped response of the model tree shall be shown for a particular harmonic frequency and then the damped response shall be shown over a range of driving frequencies. Finally, the effect of rigid end zones shall be explored in more detail, combining the approaches from the previous sections to show the importance of rigid end zones.

2 Model Tree

The physical modeling of trees, and in particular their dynamic behavior, is often accomplished in one of two ways. A particular representative tree can be selected, its exact geometry and material properties measured, and calculations and experiments can be performed on that particular arrangement. This method allows for experimental validation as well for specific parameters to be used. The disadvantage of this approach, however, lies in applying results to a broader population. What portions of the results are specific to the model tree? Is the model tree representative of trees in general, trees expressing a "similar" growth pattern, trees of the same species, trees from the same forest, or simply non-representative of any other tree's dynamic behavior? The second method for modeling the dynamic behavior of trees is to construct fractal trees [2]. These are idealized structures with self-repeating structures. Particular behavior cannot be predicted, but general trends and scaling laws can be investigated. Obviously, the model lacks the more random components of actual trees, and the question of how to apply results to trees that do not exactly fit architectures investigated is a valid one.

A large amount of work has gone into studying the allometry (relationship between size and shape) of

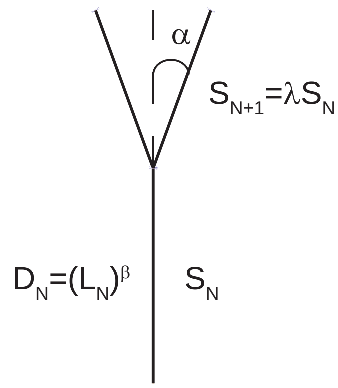


Figure 1: Geometric parameters required to describe branching between branch N and branches N+1 in a fractal representation of a sympodial tree. The length of a branch is related to its diameter through the parameter β, the area of of a branch is related to its parent's area by the parameter λ, and α defines the angle of branching.

trees [1, 3]. For the model tree developed for this paper, five numbers were needed: the base diameter, a parameter relating length of a branch to its diameter, an area reduction parameter, the angle of branching, and the number of levels of branching. The length of a branch has been found to be related to its diameter (specifically, its diameter at its base - for this model tree, branches were assumed not to taper) through a simple power law, $D = L^\beta$. The diameter of a branch can be found if one knows the diameter of its parent branch through an area reduction factor λ. For a fractal tree, an axial branch grows and then branches into two lateral segments that each branch away from the original branch's axis by an angle α [2]. These parameters are shown in Figure 1.

Also needed are the number of levels of branching of a fractal tree. A level is a group of branches that can all be traced back to the main trunk through the same number of branches. Each branch on a given level has the same diameter and same length. Figure 2 shows the four levels of branching used for this paper as well as computed diameters and lengths of each branch. The figure also shows representative points used (one for each level) for measuring displacements for particular levels.

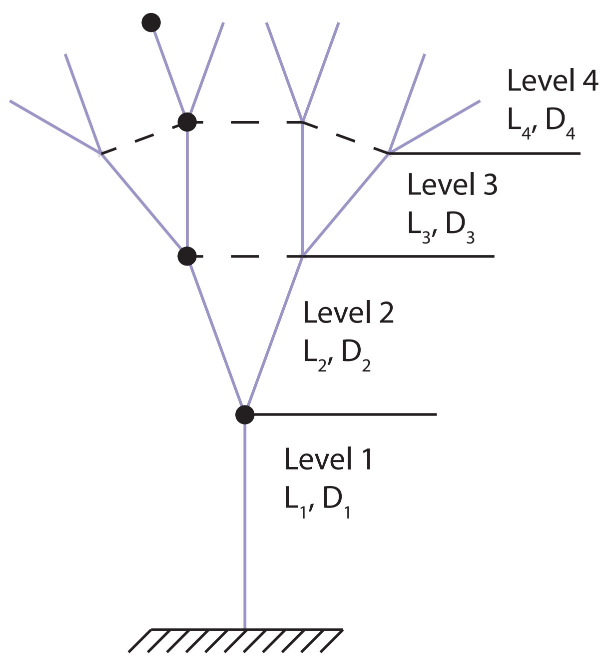
For this paper, geometric parameters that had been used in a previous study were selected [2] and other material properties were estimated assuming a walnut tree from [3]. See Table 1 for numerical values. All parameters were converted to use centimeters for length, kilograms for mass, and seconds for time in the finite element program ADINA [4].

Table 1: Geometric and mechanical properties of the model tree.

Base Diameter [cm]	α [deg]	β	λ	$\rho[kg * m^{-3}]$	E [GPa]	ν
18	20	1.5	0.5	805	11.3	0.38

A planar model was selected for ease of modal analysis. The model used 3-D beam elements with circular cross sections. Nodal translations were restricted to the y-z plane and only in-plane rotations were allowed. Three-dimensional beam elements were chosen as the model tree is composed of slender elements for which bending is key. The elements also made for quick construction. Shear deformation effects were not included and behavior was described with the cross-sectional area and the isotropic, linear elastic material model. While the model could have also been built using generic three-dimensional elements, beam elements are tailored for applications such as the problem under investigation. As will be shown, results quickly converged and had initially small errors.

Boundary conditions for the model tree are shown in Figure 3. The tree was clamped at its base, and rotation was fixed at each branching point (this boundary condition was necessary for the branching points to support moments). While not technically a boundary condition, ADINA allows for rigid end zones. This allows the



Level	Diameter [cm]	Length [m]
1	18	6.9
2	12.7	5.4
3	9	4.3
4	6.4	3.4

Figure 2: Four levels of branching were used for this paper. Each level of branching can be reached from the base in the same number of steps and each branch in a given level shares an identical length and diameter with other branches in that level. Values used for the different branch diameters and lengths are also given.

user to define finite lengths near the ends of beams as either infinitely or very (10^6 times the beam's material stiffness) rigid. In classical structural systems, this can be thought of as the connections between trusses. In trees, an increase at branching points is not uncommon, and these branching points are often modeled as rigid end zones [2]. The model tree used all three options: no rigid end zones, very rigid end zones, and infinitely rigid end zones. The effect of the length of the end zones was also investigated. It can be expected from common experience with truss structures that the length of these rigid zones will have a dramatic effect on the dynamic response of the tree (imagine using larger and large connection plates in a truss - the response becomes more and more rigid).

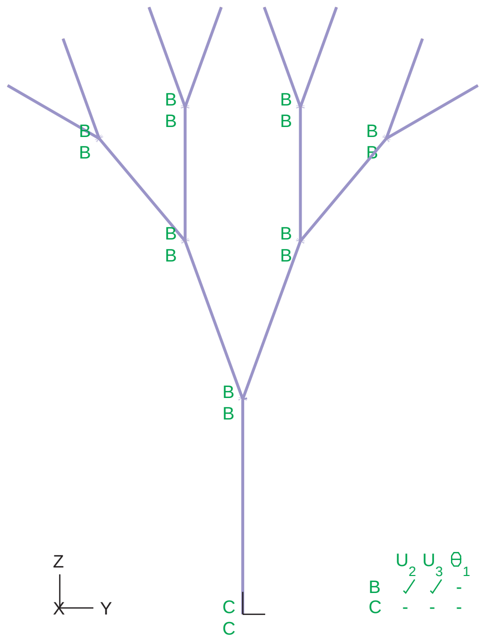


Figure 3: Boundary conditions used for the model tree. The tree is clamped at its base, and rotation is prevented at each branch point.

Finally, drag-induced loading from wind was modeled in a simplistic way. Wind load per unit exposed length of a given branch at level N, q_N, was expressed as $q_N = \rho_{air} \cdot V^2 \cdot D_N$, where the wind velocity V was taken to be five meters per second and D_N was taken to be the diameter of a branch at level N. This resulted in a line load of 21.6 kg/s^2 for the base of the tree (Level 1), 15.2 kg/s^2 for Level 2, 10.8 kg/s^2 for Level 3, and 7.68 kg/s^2 for Level 4. These loads were applied as line loads as y-tractions (the entire load was applied in a horizontal direction). No reduction in load was applied across the tree, i.e., loads were identical for every branch at a given level. This does not reflect the reality of the situation as energy is removed from the flow as it passes around the leading branches. A simple model of wind loading was desired, however, and this model was deemed sufficient as only relative comparisons between results were being made. Figure 4 shows the loading used for the model tree.

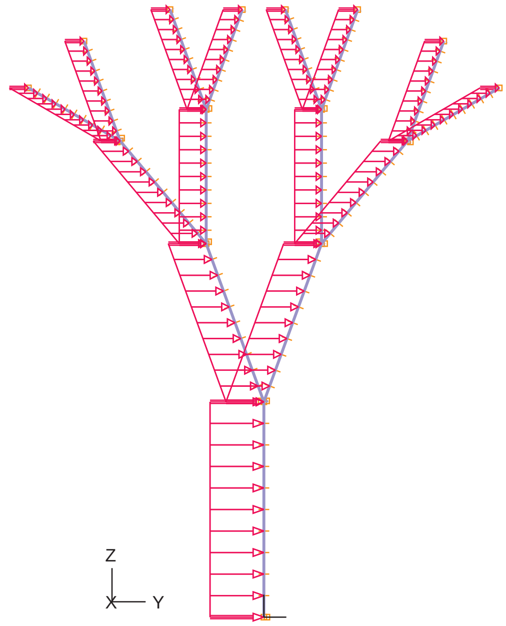


Figure 4: Loading configuration for the model tree. Line loads of 21.6 kg/s^2 (Level 1 - base), 15.2 kg/s^2 (Level 2), 10.8 kg/s^2 (Level 3), and 7.68 kg/s^2 (Level 4 - top level of branches) were used. All loads were applied as tractions in the y-direction and were identical for each branch in a given level.

3 Mathematical Model

Before any in depth calculations were performed, a simple mathematical model was established as a check for an analysis performed in ADINA. A cantilevered beam with properties equal to those of the Level 1 branch (see Figure 1 and Table 1 for values) was taken as the base case. The fundamental frequency of a beam with distributed mass can be expressed explicitly as:

$$f_1 = \frac{1}{2\pi}(3.5156)\sqrt{\frac{EI}{ML^3}}. \tag{1}$$

Values from Table 1 were plugged into Equation 3, and the fundamental mode for the simple cantilevered beam was calculated to be 1.98313 Hz. Several runs of ADINA were made, subdividing the beam into an increasing number of elements. The results converged to a value of 1.98145 after 10 subdivisions, giving and error of less than 0.1%, and are summarized in 2.

Table 2: Fundamental frequencies of a cantilevered beam matching Level 1 computed using ADINA. The analytical solution is $f_1 = 1.98Hz$.

Number of Subdivisions (N)	f_1 [Hz]
1	1.99087
2	1.98241
5	1.98148
10	1.98145
100	1.98145
450	1.98145

4 Natural Frequencies

The first 25 natural frequencies of the model tree were calculated using the modal participation factor method. For this trial, three variations of the model tree were used, varying the rigid end zones of each branch. Rigid end zones were set to be one centimeter long for every branch, and were either set to the material stiffness (no rigid end zone), very stiff (10^6 times the material stiffness), or infinitely rigid. Figure 5 shows the first 25 frequencies for the case of no rigid end zones for a varying number of subdivisions per branch. One, two, five, 10, 20, and 30 subdivisions per branch were selected. What is immediately apparent is that the case of 1 subdivision per branch shows a dramatic increase in natural frequency compared to all other solutions starting at mode 16. All runs showed a jump at the 16th mode, with relatively little variation amongst the first 15 modes. In the case of one subdivision, a relatively large jump at modes 24 and 25 was also noted, but not in the other runs. For more than 1 subdivision, there appears to be little variation in the results for each mode. Figures 6 and 7 show the case of very rigid and infinitely rigid end zones, respectively. Again, rigid end zone length was defined as 1 cm for each branch. No dramatic differences are noticed between the different figures of results, but there is a very slight increase in all natural frequencies as one goes from no end zones to very stiff end zones to rigid end zones. This result agrees with theoretical expectations: when any part of a dynamic system is stiffened, the natural frequencies of that system will increase. In all cases, the first fifteen modes are approximately lumped between 1 Hz and 5 Hz.

While Figures 5 through 7 would indicate that using any number of subdivisions per branch greater than one will give converging results, a closer look may be warranted. For this, the fundamental mode of each of these same systems was examined. Figure 8 shows the fundamental mode for the three end zone conditions (again, end zones are defined as non-rigid, very stiff, or rigid for within one centimeter of a branching point), for an increasing number of subdivisions per branch. It becomes apparent in Figure 8 that for the case of non-rigid and infinitely rigid end zones, the fundamental mode converges after two subdivisions per branch and remains essentially constant after 5 subdivisions per branch. In the case of the very stiff end zones, however, a truly convergent solution does not appear within the number of subdivisions per branch used. A minimum is reached at 10 subdivisions per branch. Most of the solutions can be rejected using a physical argument: for 1, 20, and 30 subdivisions per branch, the natural frequency shown is *higher* than that shown for infinitely rigid end zones and for ten subdivisions, the natural frequency shown is *lower* than that shown for the case of no end zones. It should be noted that for the cases of no end zones and rigid end zones, variation in the fundamental frequency is on the order of a tenth of one percent, and variation in the fundamental frequency in the very stiff end zone results is on the order of two percent. A tabulated list of values of the fundamental modes shown in Figure 8 is shown in Table 3. Further discussion of the effect of rigid end zones will take place in a later section. Unless indicated otherwise, the following analyses were completed with a rigid end zone of one centimeter. This was fixed to allow for further investigation of other parameters, but will be explored in a later section.

So what did the modes look like? Figure 9 shows the evolution of the first six mode shapes (here shown for five subdivisions per branch). In the first mode, we see most of the deformation occurring in the base of the tree, with a small amount of bending in the following levels. The second mode shows a symmetric deformation of the level two branches, with some bending in the higher levels. The third mode shows an anti-symmetric action in the third level (with some bending in the fourth level of branches). Note also that

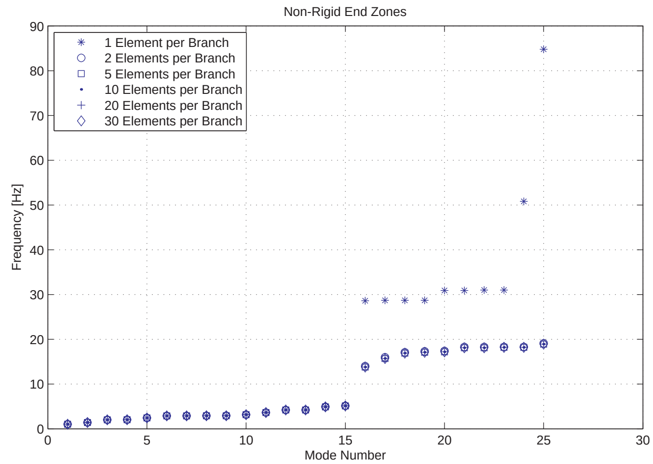


Figure 5: The first 25 natural frequencies of the model tree assuming no rigid end zones.

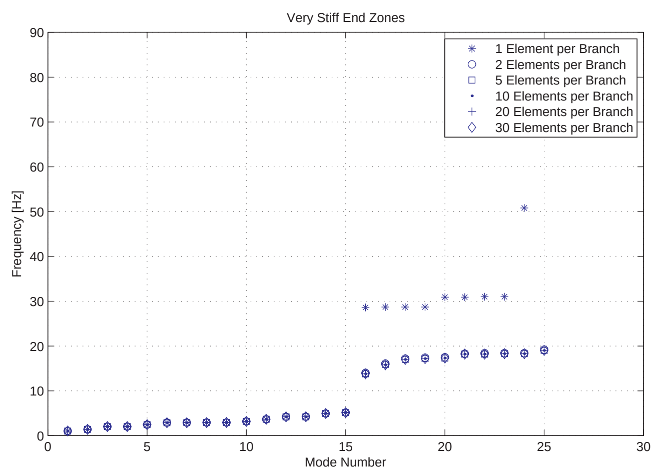


Figure 6: The first 25 natural frequencies of the model tree assuming very (10^6 times material stiffness) stiff end zones one centimeter long at each branch end.

Table 3: Fundamental frequencies (f_1) computed for different number of subdivisions per branch (N) for the three rigid end zones investigated.

N	f_n [Hz] (no end zones)	f_n [Hz] (very stiff end zones)	f_n [Hz] (rigid end zones)
1	1.01287	1.02919	1.01678
2	1.01177	1.01401	1.01567
5	1.01170	1.01501	1.01558
10	1.01170	1.01081	1.01557
20	1.01169	1.02172	1.01557
30	1.01169	1.02976	1.01557

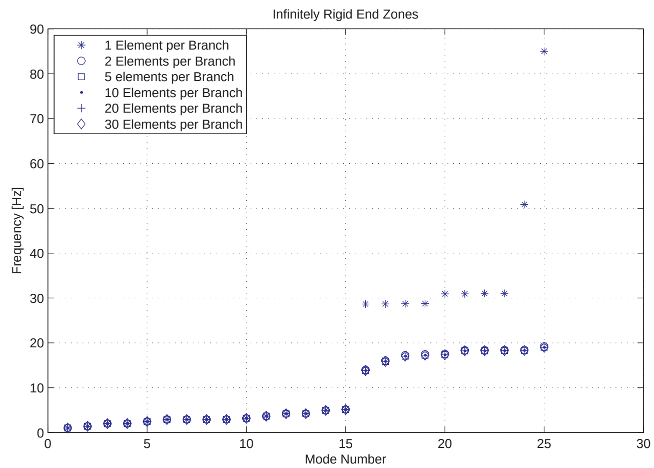


Figure 7: The first 25 natural frequencies of the model tree assuming infinitely rigid end zones one centimeter long at each branch end.

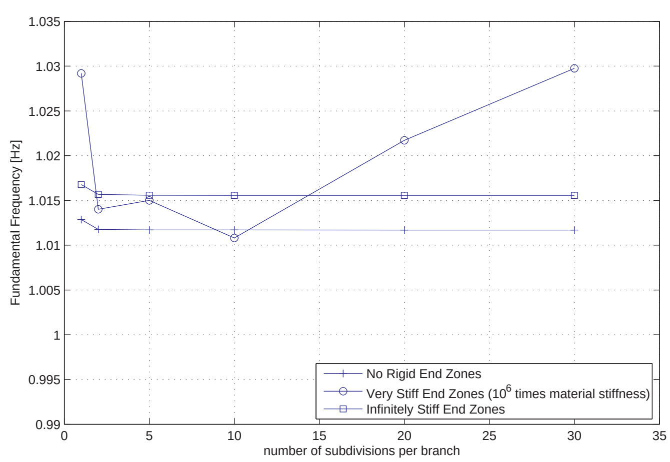


Figure 8: The fundamental mode of the model tree for three different one-centimeter-long end zone conditions for an increasing number of subdivisions.

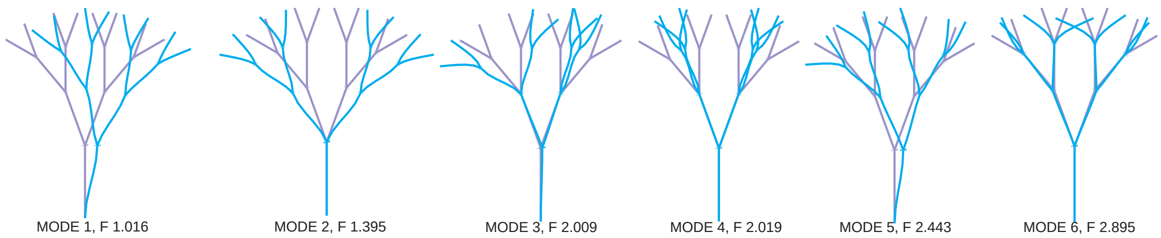


Figure 9: The first six mode shapes for five subdivisions per branch. Branches shown in purple are the original configuration while blue lines give the deformed modal shape (note that deformations are magnified extensively). One centimeter rigid end zones are used for each branch in each case.

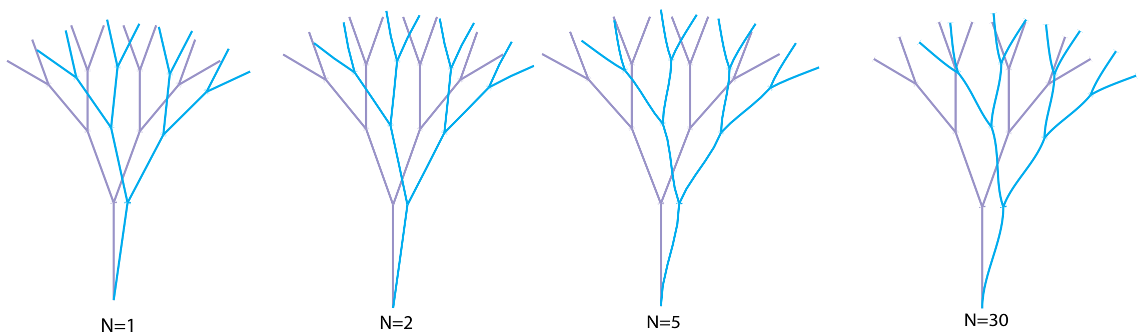


Figure 10: The evolution of the fundamental mode of the model tree for an increasing number of subdivisions per branch, N. One centimeter rigid end zones for each branch are used in all cases.

the third mode shows a very slight bending in the base of the tree, a trait common to any mode with anti-symmetric motion. This is expected from a physical need to conserve angular momentum in the system. Mode four shows a symmetric motion of level three. Mode five shows an anti-symmetric mode in level four with some bending in level three, while mode six shows a symmetric motion in level four. Ensuing modes are typified first by a continuation in the combination of motions in the fourth level, followed by complicated combinations of actions throughout multiple levels.

Figure 10 shows the evolution of the fundamental mode as the number of subdivisions increases. For one to two subdivisions per branch, deformation is linear, with almost all the deformation coming from the base of the tree, with the higher levels simply rotating. As the number of subdivisions per branch is increased to five up to thirty, we see more of a curved deformation of the base, as well as slight deformation *and* rotation of the upper levels. Very little in the way of differences can be seen between five and thirty subdivisions, which also reflects the results obtained in Figure 8, which showed essentially no change in the fundamental frequency between five and thirty subdivisions. The first eight modes for each number of subdivisions per branch is shown in Appendix A.

5 Response to Harmonic Loading (Part 1)

The previous section dealt with the natural frequencies of the trees, and we saw that the fundamental frequency in all cases was approximately 1 Hz. While the mode shapes themselves showed a change from linear to curved characteristics after two subdivisions per branch, it was important to look at the convergent nature of displacements. A series of modal superposition analyses were completed using a forcing function with amplitude as derived earlier (corresponding to a mild wind), and frequency taken to be 0.78 Hz. This frequency was selected to be below any resonant frequencies. All 25 modes were used in the formulation of a solution. While modal damping was not added, the simulations were run for 50 seconds (with a 0.05 second time step) to allow a maximum response to develop. The maximum displacement over that 50 second period was recorded for an increasing number of subdivisions. As stated before, all analyses were completed with a one centimeter rigid end zone for each branch. Table 4 tabulates the maximum displacements and Figure 11 shows the deformed shapes recorded for each envelope response.

Table 4: Maximum displacements calculated for the model tree over 50s of approximately 0.8 Hz harmonic loading for increasing number of subdivisions per branch (N).

N	Maximum Displacement [cm]
1	32.51
2	32.49
5	32.49
10	32.49
20	32.49
30	32.49

The different responses overall show a similar trend to the evolution of the first mode; that is they express bending after two subdivisions per branch. The point of maximum deflection is also always at the same branch tip in the fourth level. This is the point that was also indicated in Figure 2 as the point that would be tracked in future analyses. As can be seen in Figure 11 and Table 4, the maximum displacement converges with two subdivisions per branch, but has an error of less than 0.1% with just one subdivision per branch.

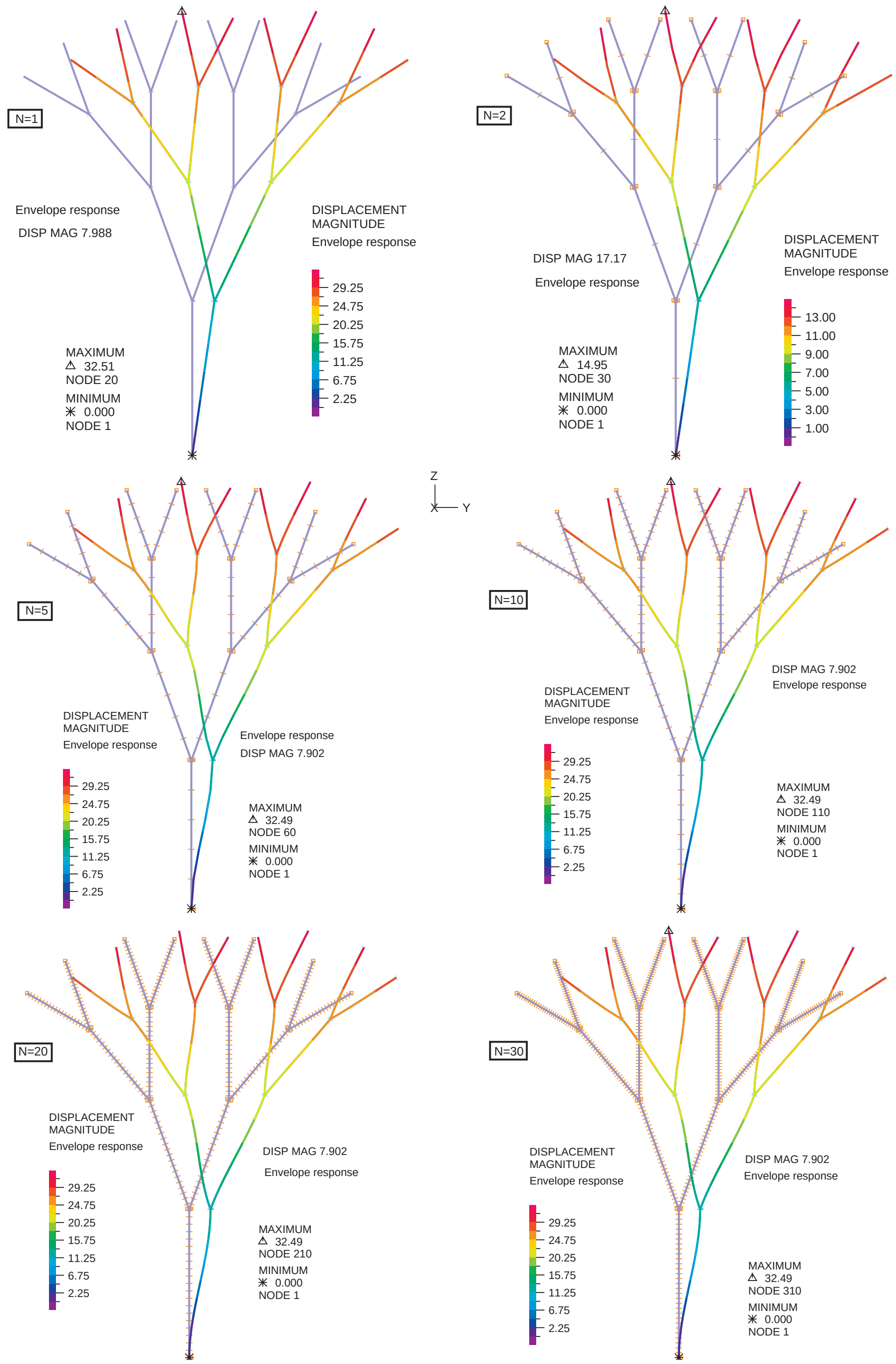


Figure 11: Envelope response of the model tree to a harmonic load forcing at approximately 0.8 Hz for different numbers of subdivisions per branch (N).

6 Response to Harmonic Loading (Part 2)

In the previous section, one particular loading frequency was selected and only used over a relatively short period of loading. No damping was included in the previous analyses, but it is clearly important in the dynamic response of a tree. For this reason, modal participation analyses were completed to allow for a frequency response to be developed for the model tree. Damping was assumed constant for every mode and was 1, 5, or 10% damping. It was expected (and confirmed in the previous section for sub-resonant loading) for maximum displacements to occur at the tip of a branch in Level 4 (the top level of the tree). Figure 12 shows the frequency response of the model tree to harmonic loading between 0 Hz (quasi-static) and 5 Hz for one, two, and five subdivisions per branch. It is interesting to note that peak displacement at resonance *increases* between one and two subdivisions per branch. This is most clearly seen by comparing peak response with only 1% damping. Peak response stays constant after five subdivisions per branch. Maximum response was noted to be between 1.0 Hz and 1.1 Hz, which agrees with the lower fundamental modes of the model tree. For light damping, other, smaller peaks are seen at approximately 2.0, 2.5, 2.9 and 3.7 Hz. Damping is seen to have a very strong effect on peak response at resonance. With 1% damping, a peak response of 3 meters is predicted for a relatively mild wind! Slight increases in damping bring this response down quickly with a smooth harmonic peak predicted with 10% damping. Damping of trees is relatively high when compared to many physical systems, especially in the presence of leaves, so damping on the order of 5-10% is reasonable.

A frequency response plot for each level of branching for a model tree with 30 subdivisions per branch can be seen in Appendix B. This run was completed to ensure that characteristics noticed at the top level were persistent throughout the branched structure.

As already discussed, the maximum response of the model tree does not vary a great deal with a differing number of subdivisions. To show this point in another way, the resonant response of the model tree (as evaluated at a top level branch) can be compared. Figure 13 shows a frequency response plot of the tip of a top level branch at 5% damping for all modes for increasing number of subdivisions per branch. It is clear that the peak response is practically identical for all model trees considered, with a *slight increase* in magnitude between one and two subdivisions per branch (as we see more clearly in the 1% damping case in Figure 12

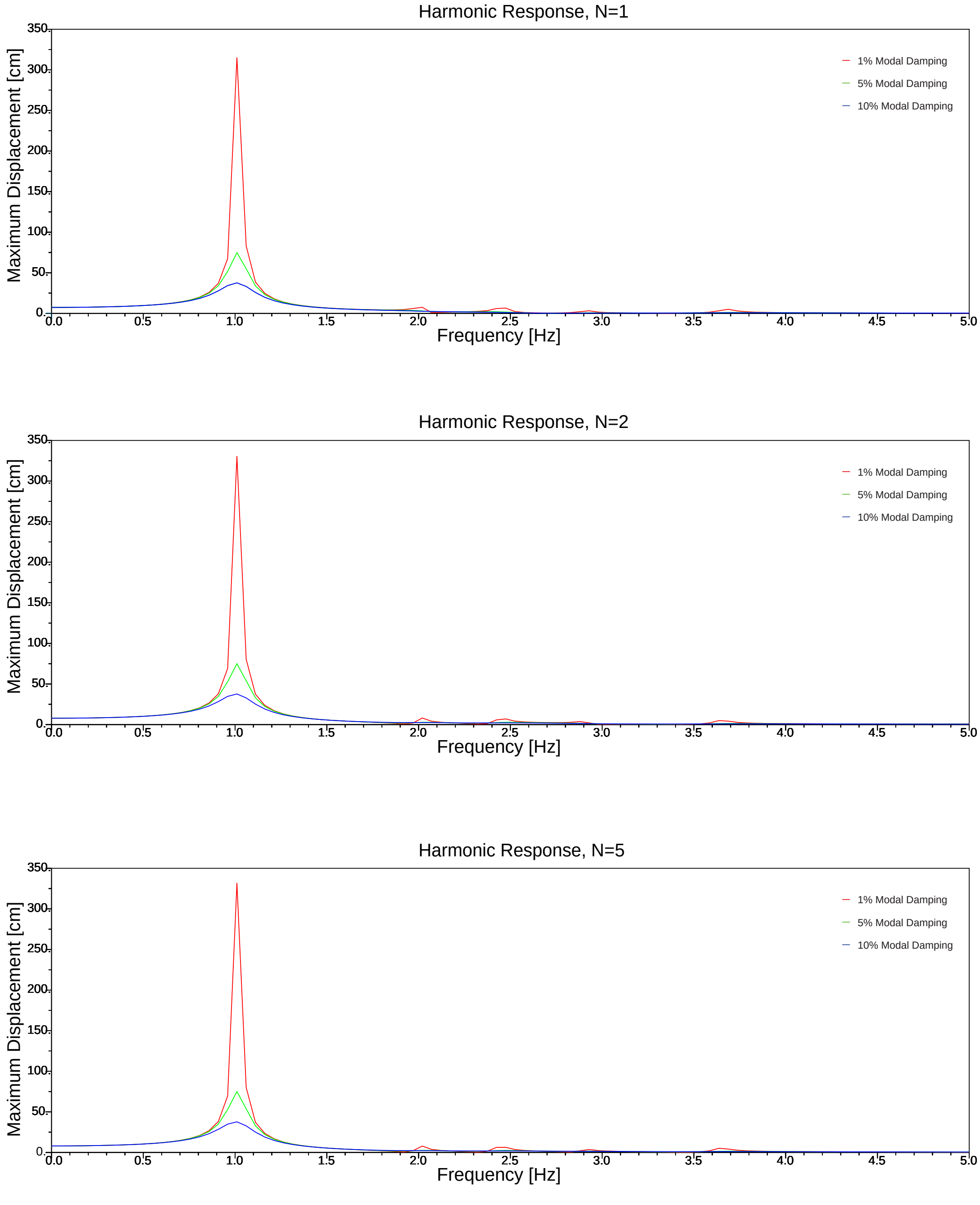


Figure 12: Frequency response of a top level (Level 4) branch of the model tree for different numbers of subdivisions per branch (N) at different levels of damping.

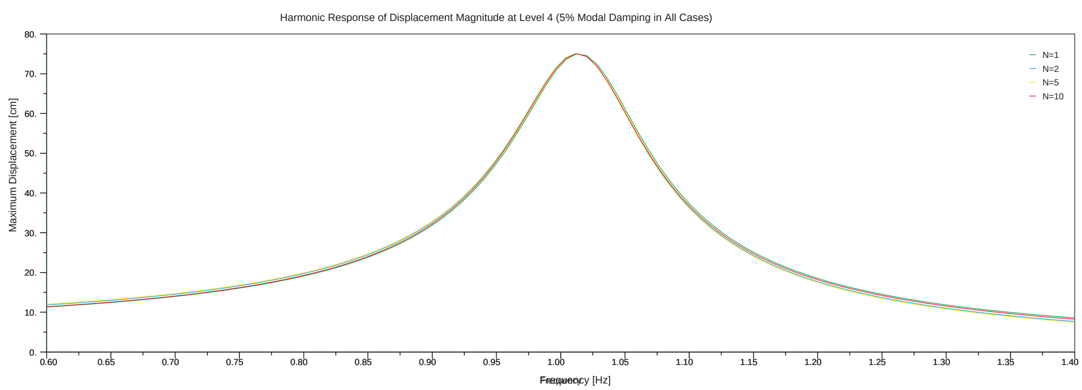


Figure 13: Frequency response of a top level (Level 4) branch of the model tree for 5% damping for every mode for an increasing number of subdivisions per branch.

7 Rigid Zone Length

In the preceding analyses, the end zones of each branch at a branching point were assumed to be rigid for one centimeter. For most practical purposes, very little difference was seen in increasing the number of subdivisions per branch, except in the shape of the response. What if we play more with these rigid end zones? Children who climb trees all have surely noticed that a tree is stiffest near its branching points. The branching points even sometimes express this stiffening through additional material near branching points in some species of trees. While there is not this clear division between branch material and stiffened material, the length scale can easily be imagined to be up to approximately 10 centimeters. As these regions grow, physical arguments would predict an *increase* in the natural frequencies of the model tree as well as a *decrease* in the maximum displacement of the tree. As noted, experience with truss structures would predicate a tangible change in dynamic properties.

Using rigid end zone lengths of one, five, and ten centimeters, the natural frequencies and maximum displacement (to a 0.78 Hz load over a time of 50 seconds) were measured for model trees with five subdivisions per branch. This number of subdivisions were chosen based on the convergence of all previous results with this many elements per branch. Table 5 summarizes these results. The mode shapes of these different model trees appeared to be consistent, however, and the first three modes of each zone can be seen in Appendix C.

Table 5: Fundamental frequency and maximum displacement of a model tree with different rigid end zone lengths. Maximum displacement was measured for a 0.8 Hz harmonic loading over 50 seconds.

Rigid End Zone Length (cm)	Fundamental Frequency [Hz]	Maximum Displacement [cm]
1	1.01558	32.49
5	1.03125	29.87
10	1.0512	26.80

We can see that changing the rigid end zone length plays a key factor in deciding the response of the tree as a structure, as previously expected. By lengthening the rigid end zone length from one centimeter to ten centimeters, the fundamental frequency of the system changed by greater than three percent and the maximum displacement by greater than 15 percent! Compared to other magnitude changes we have seen other than increases in damping, this is quite substantial.

The presence of rigid end zones plays a large role in the dynamic response of the model tree. The modeling approach used for these end zones was perhaps over-simplified, with a constant length throughout the tree (instead of following a type of allometry law). The use of infinitely stiff rigid end zones was a decision based on the non-convergence of the very stiff end zones in the previous section, but does not best reflect reality. Clearly, the use of rigid end zones is warranted by physical experience, but the modeling of these zones is not a task that can be taken too lightly when computing displacements and natural frequencies.

8 Conclusions

We have examined the dynamic response of trees in this paper. A model tree was constructed with three-dimensional beam elements with circular cross sections. Dimensions of this model tree were developed following simple allometric laws previously developed. Boundary conditions were established using simple physical reasoning and wind load magnitudes were established for a mild wind. A simplified model was verified for finding the fundamental frequency before proceeding to more complex analysis. The first 25 natural frequencies were found for model trees with 1, 2, 5, 10, 20, and 30 subdivisions per branch and with one centimeter end zones with the material stiffness, 10^6 times the material stiffness (very stiff), and infinitely stiff (rigid). It was found that starting with two subdivisions per branch, the frequencies for the first 25 modes agreed well. The fundamental mode was found to converge nicely for the model trees with end zones matching material stiffness and rigid end zones, but not for those with very stiff end zones. The modal shapes were found to evolve as involving higher levels of branching, with both symmetric and anti-symmetric motions found. In the case of symmetric motions, the base level of the tree remained stationary, while in the case of anti-symmetric motions, the base was found to sway. With more than two subdivisions per branch, the modal shapes showed a curved nature.

The response to harmonic loading was investigated in two ways. The maximum response to a sub-resonant wind excitation over a prescribed time was measured for varying numbers of subdivisions per branch (but in the absence of damping). Then the frequency response was measured for three levels of damping across all modes: 1%, 5%, and 10%. Damping, as would be expected, was found to have a tremendous effect on resonant response. In both of these cases, the number of subdivisions per branch was found to have a very small effect on the overall response, with results converging quickly and with small initial errors. This can be attributed to the model tree being an excellent application for beam elements.

Finally, the rigid end zones were investigated. By varying this parameter in a simple way, response was shown to change dramatically. This makes physical sense, but underlines the importance of understanding how to describe these rigid end zones. Extreme care is needed in selecting values for modelers to use. For a fractal tree such as the one investigated, however, the modes shapes did remain consistent despite the lengthening of the rigid end zones.

For most applications, one beam element per branch is most likely-sufficient. For applications emphasizing the actual shape of the tree (think computer graphics), it may be necessary to extend to greater than two elements per branch.

Open questions or further avenues for future research involve better characterizing the rigid end zones with a very stiff model, a comparison of three-dimensional beam elements to a model built with generic three-dimensional elements (comparing computation time as well as convergence), and the essential number of modes needed to capture the dynamic response. A better evaluation of the stresses developed within each branch could be beneficial as well.

References

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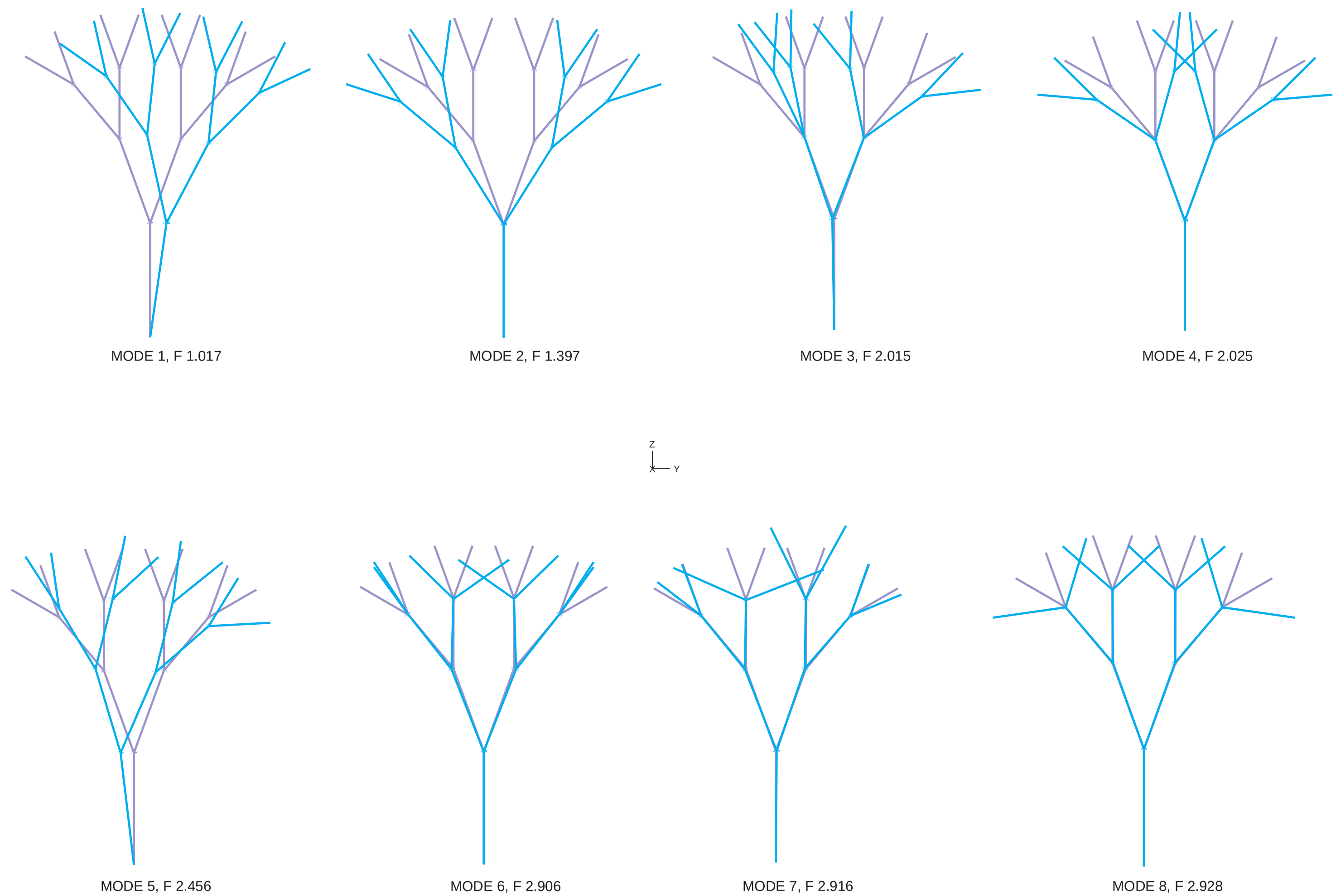
[2] M Rodrigues, E de Langre, and Bruno Moulia, A Scaling Law For The Effects Of Architecture And Allometry On Tree Vibration Modes Suggests A Biological Tuning To Modal Compartmentalization. *Amer. J. of Botany*, **95** (12), 1523-1537 (2008).

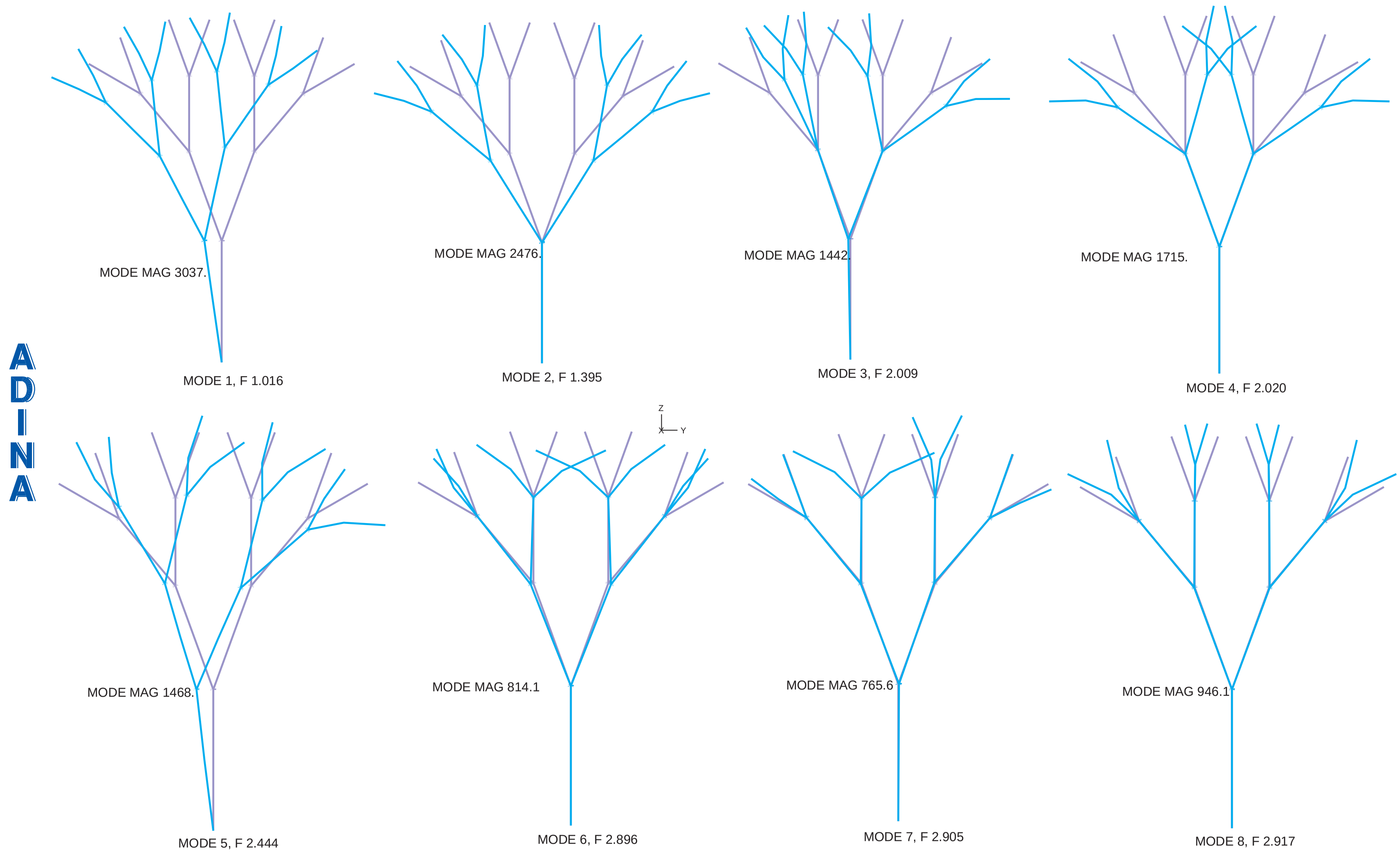
[3] M Fournier, A Stokes, C Coutand, T Fourcaud, and B Moulia, Tree Biomechanics and Growth Strategies In The Context Of Forest Functional Ecology. *In* A Herrel, T Speck, and N. Rowe [eds.], *Ecology and Biomechanics: A Biomechanical Approach Of The Ecology Of Animals And Plants*, CRC Taylor and Francis, Boca Raton, Florida, USA. 1-33 (2005).

[4] Automatic Dynamic Incremental Nonlinear Analysis. See `http://www.adina.com/index.shtml` for details.

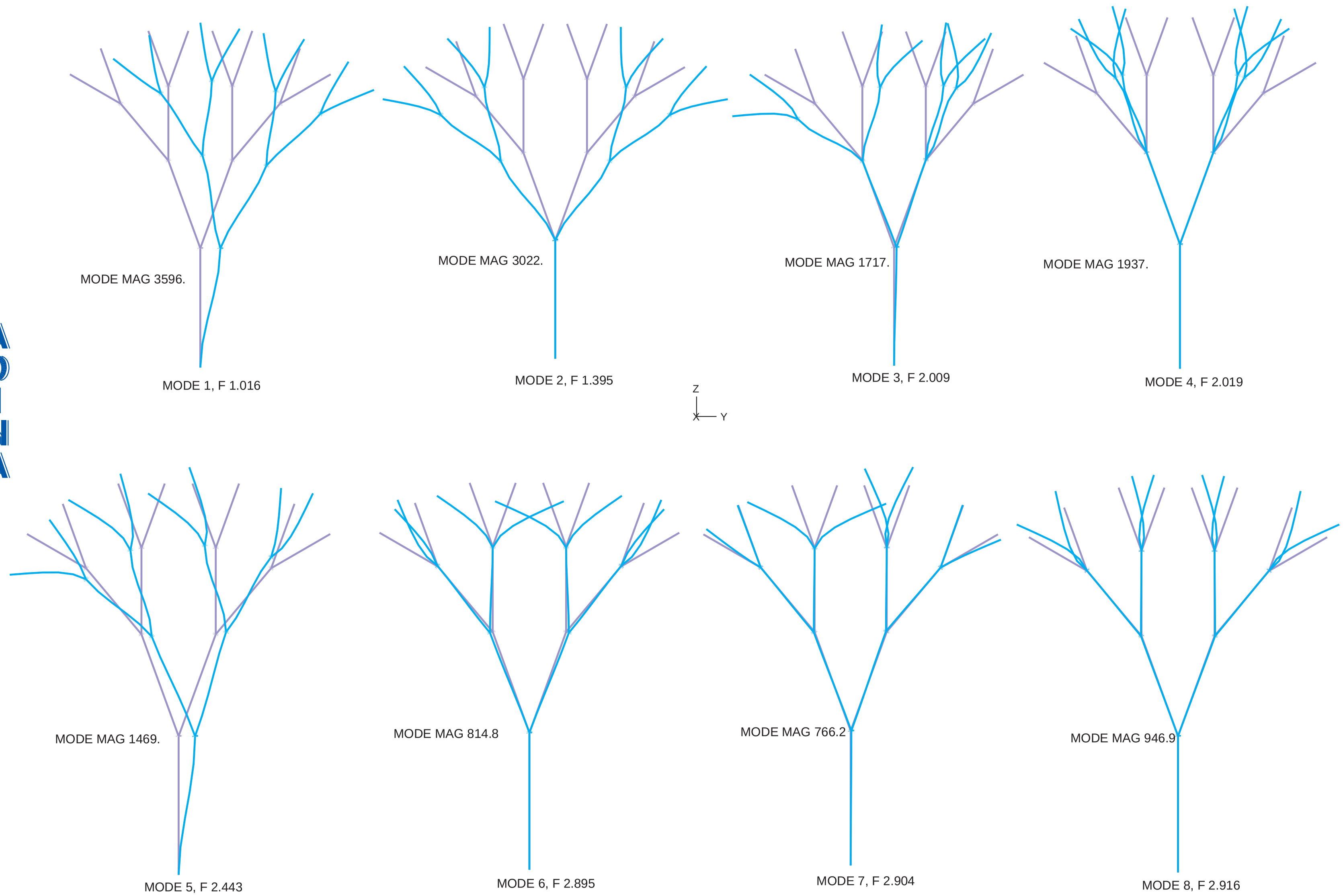
A Appendix A: Mode Shapes for Different Number of Subdivisions per Branch (N)



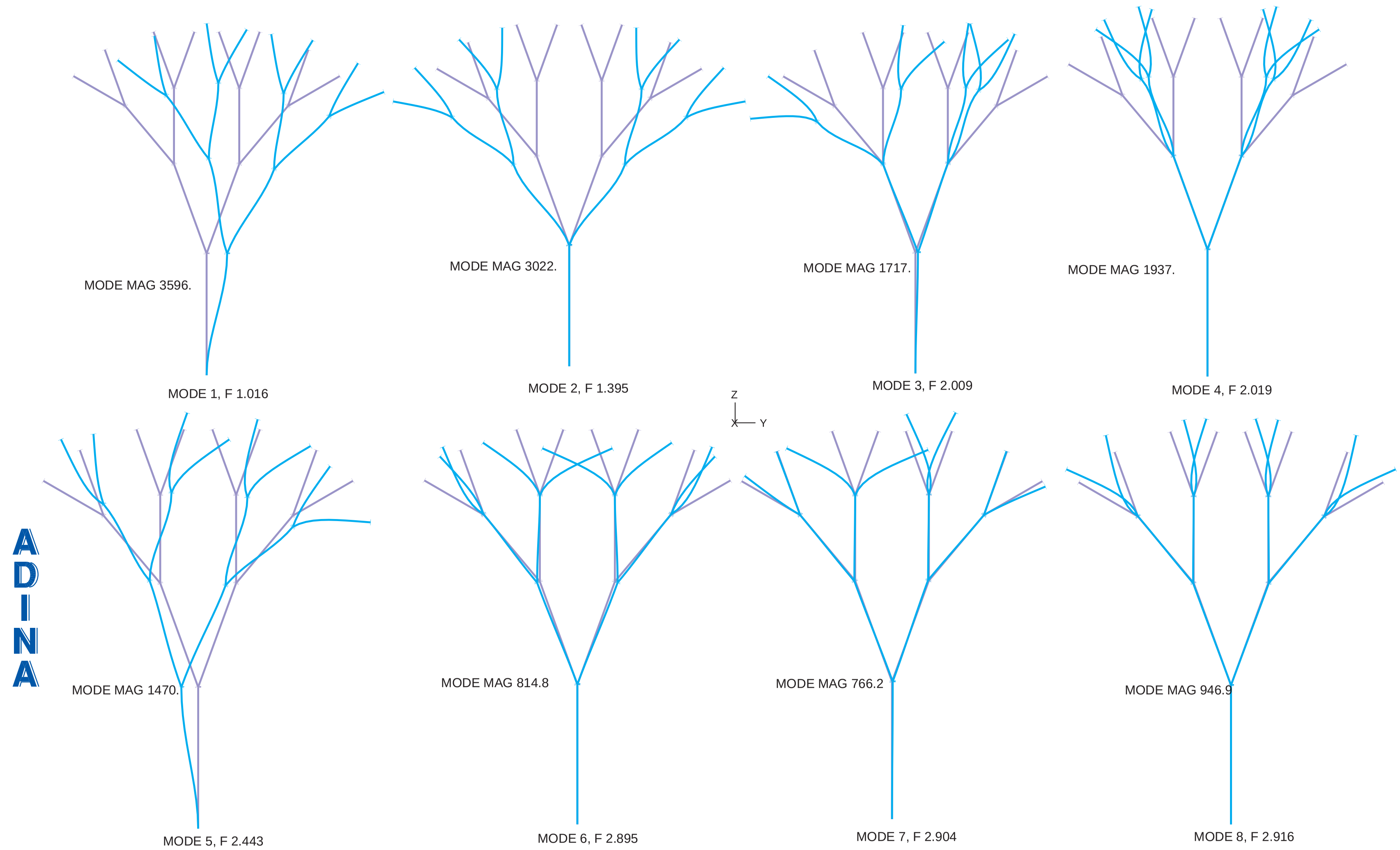
Modes 1 through 8 for N=1 (Frequencies in Hz)



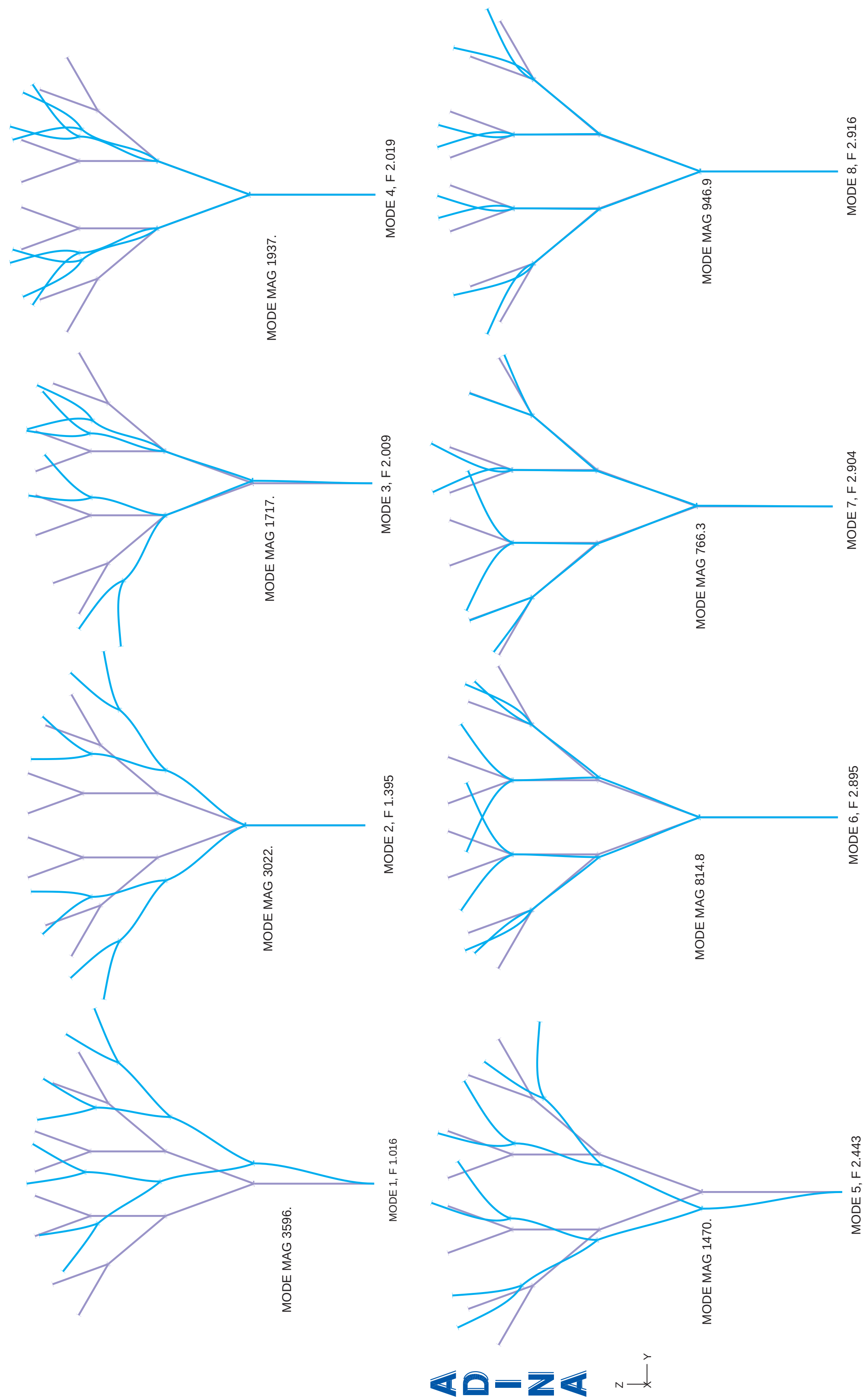
Modes 1 through 8 for N=2 (Frequencies in Hz)



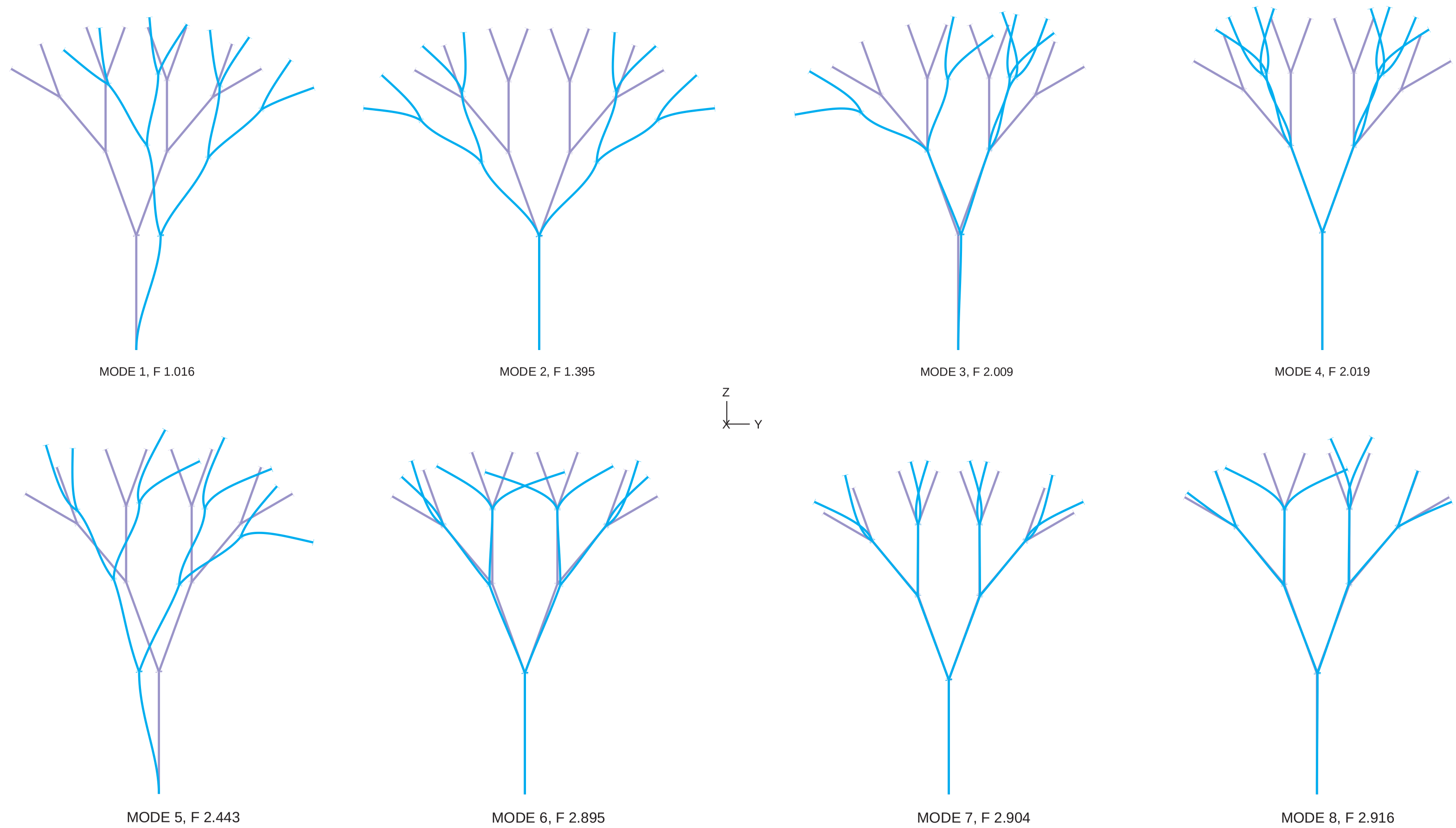
Modes 1 through 8 for N=5 (Frequencies in Hz)



Modes 1 through 8 for N=10 (Frequencies in Hz)

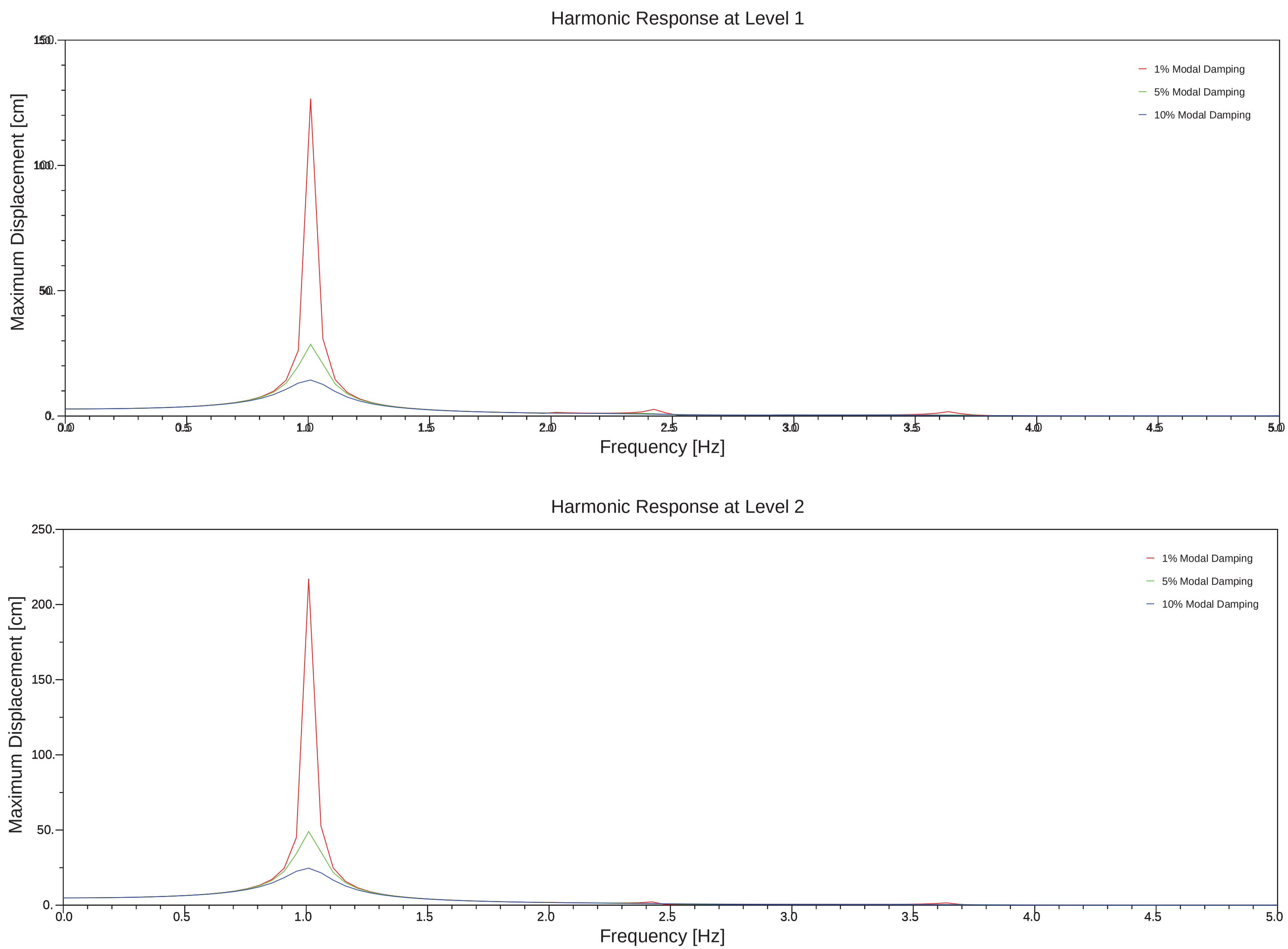


Modes 1 through 8 for N=20 (Frequencies in Hz)

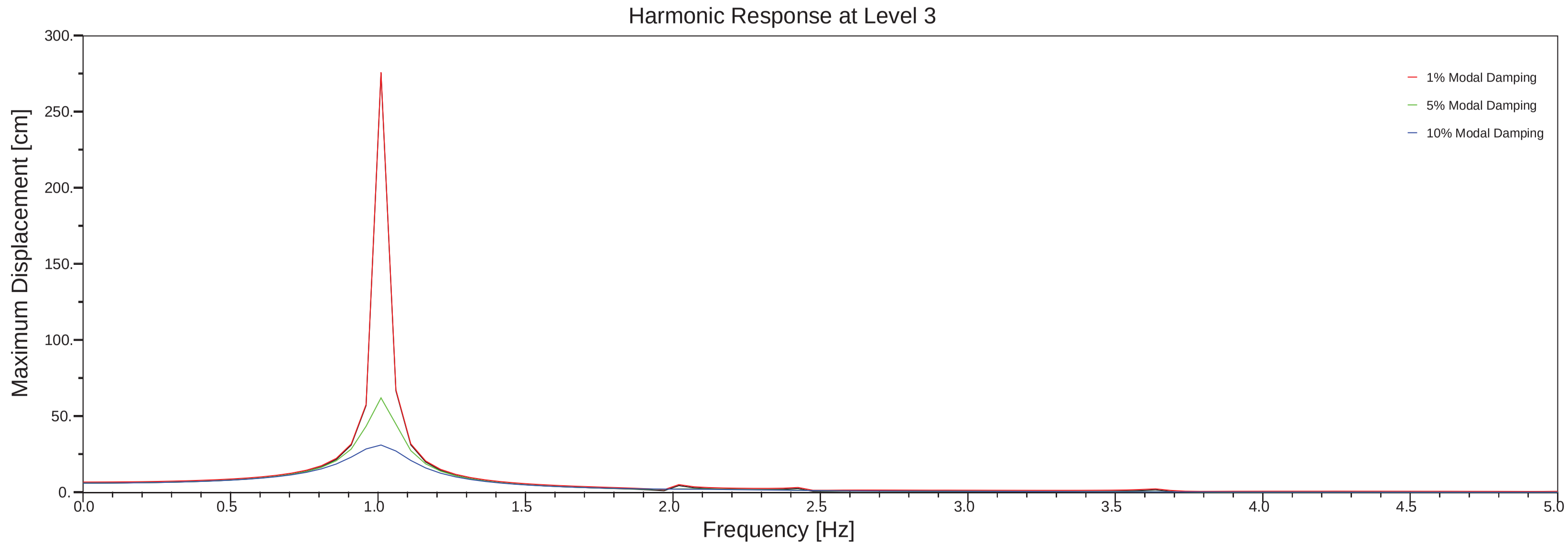


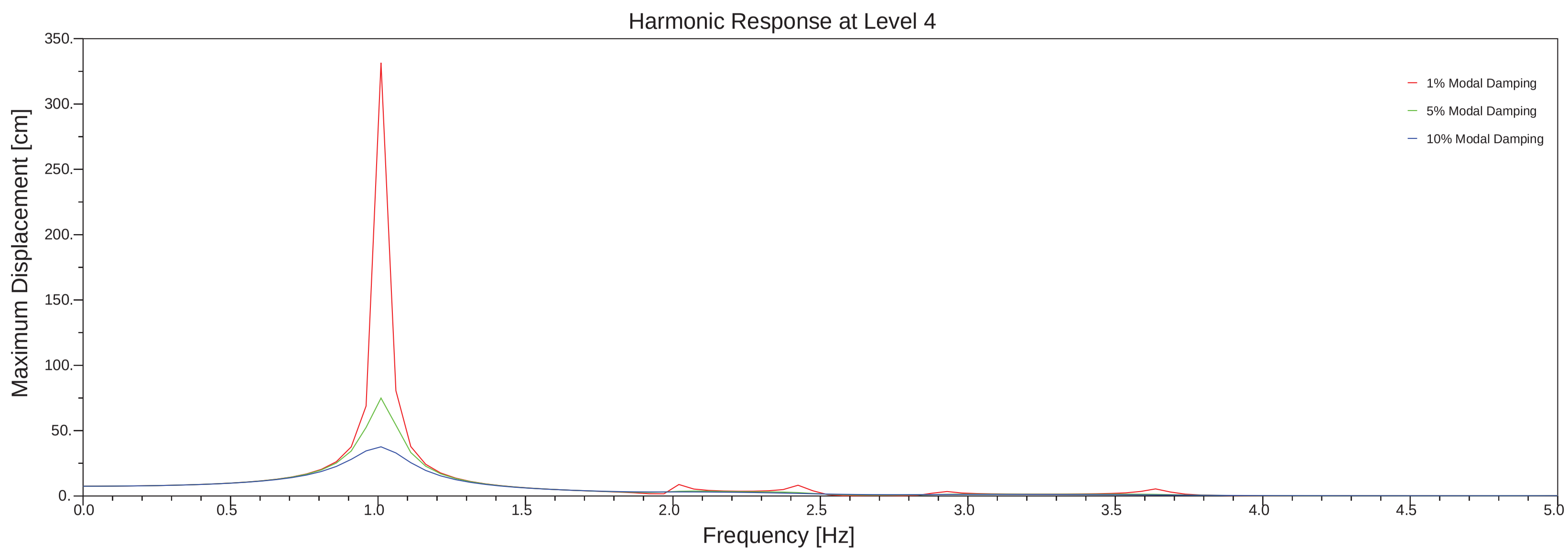
Modes 1 through 8 for N=30 (Frequencies in Hz)

B Appendix B: Frequency Response of a Model Tree with 30 Subdivisions per Branch



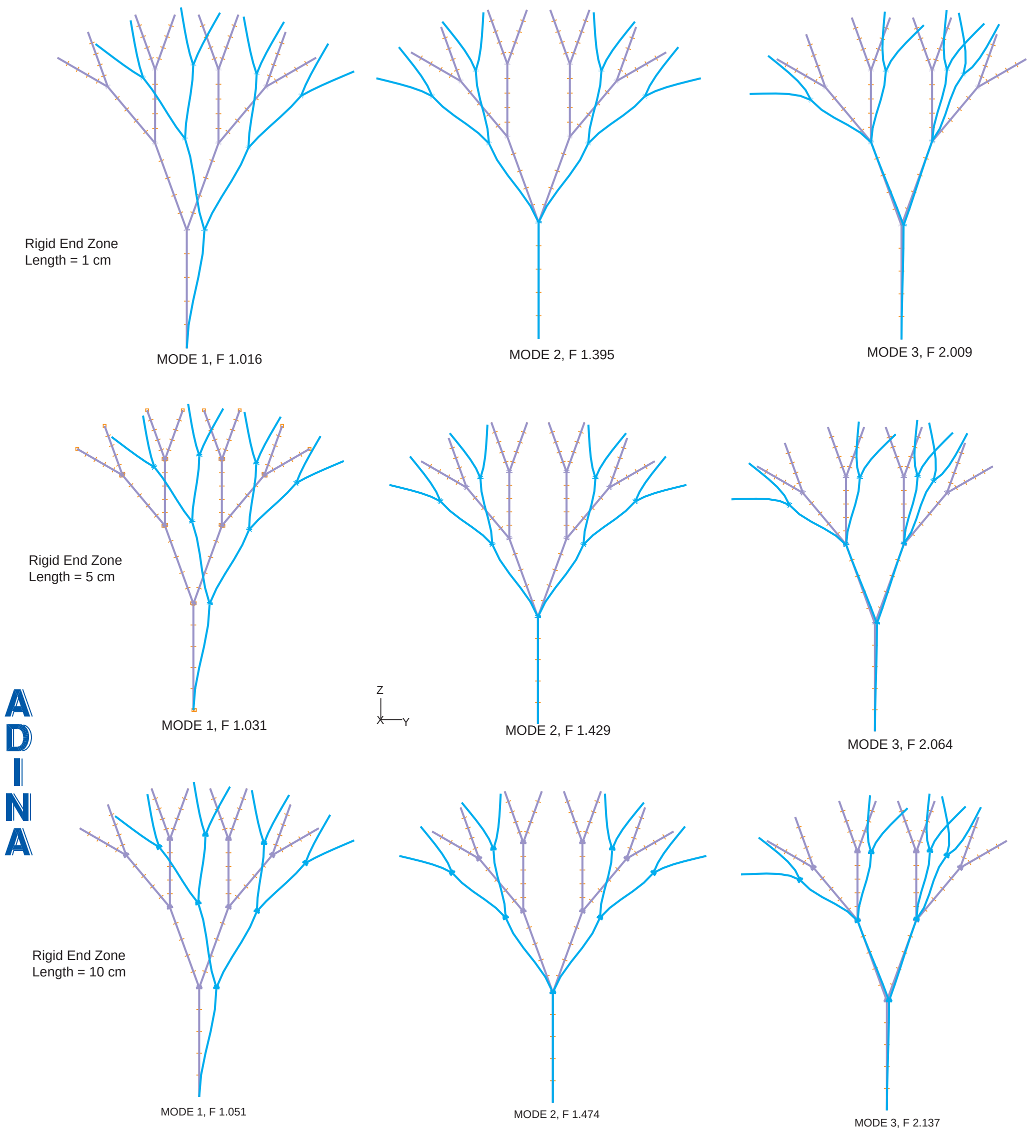
Effect of Damping on Frequency Response at Levels 1 and 2 (for N=30)





Effect of Damping on Frequency Response at Levels 3 and 4 (for N=30)

C Appendix C: Mode Shapes for Changing Rigid End Zone Length



Comparison of First Three Modes for Different Lengths of Rigid End Zones
Frequencies in Hz.