Monte Carlo
Methods

for Integration

Random Data replaced by (Pseudo)-Random Variates

"around" the
Uniform Continuous Distribution

Review: univariate case $\quad a \le X \le b$

pdf $f_X(x) = \dfrac{1}{b-a}\,, \quad a \le x \le b$

such that $\quad a \le a' \le b' \le b$

$$P(a' \le X \le b') = \int_{a'}^{b'} f_X(x)\,dx = \frac{b'-a'}{b-a}$$

which depends only on the length of the interval $[a', b']$ relative to the length of the interval $[a, b]$.

from Uniform Continuous to Bernoulli: example

Say U is uniform over $[0,1]$, $\quad a=0, b=1$

$$f_U(u) = 1 \qquad 0 \le u \le 1 .$$

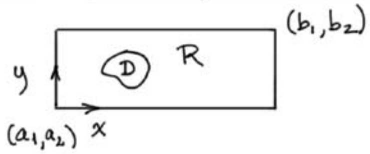
Then

$$X = g(U) = \begin{cases} 0 & 0 \le U \le 1-\theta \\ 1 & 1-\theta \le U \le 1 \end{cases}$$

is Bernoulli with parameter θ:

$$P(X=1) = P(1-\theta \le U \le 1) = \frac{1-(1-\theta)}{\underset{1 \; \; 0}{b-a}} = \theta \;\checkmark .$$

the Bivariate Uniform Distribution : random darts



$a_1 \le X \le b_1$

$a_2 \le Y \le b_2$

$$f_{X,Y}(x,y) = \frac{1}{(b_1-a_1)\cdot(b_2-a_2)} = \frac{1}{A_R} \geq 0 \qquad (x,y) \text{ in } R \; \{x_1, x_2$$

$$\boxed{P(x \le X \le x+dx,\ y \le Y \le y+dy) = f_{X,Y}(x,y)\,dx\,dy}$$

$\Rightarrow$ $P(\,(x,y) \text{ in } D\,) = \int_D f_{X,Y}\,dx\,dy = A_D / A_R$

U (OR) dxdy: mutually exclusive

$\Rightarrow$ $\boxed{P(\,(x,y) \text{ in } R\,) = 1}$

marginal pdf's:

$$f_X(x) = \int_{a_2}^{b_2} f_{X,Y}(x,y)\,dy \quad \text{any } dy \quad U\,(OR) \quad dx$$

$$= \int_{a_2}^{b_2} \frac{1}{(b_1-a_1)(b_2-a_2)}\,dy = \frac{1}{(b_1-a_1)}$$

$$f_Y(y) = \int_{a_1}^{b_1} f_{X,Y}(x,y)\,dx \quad \text{any } dx \quad U\,(OR) \quad dy$$

$$= \int_{a_1}^{b_1} \frac{1}{(b_1-a_1)(b_2-a_2)}\,dy = \frac{1}{(b_2-a_2)}$$

independence:

$$\boxed{f_X(x)\cdot f_Y(y) = \frac{1}{(b_1-a_1)}\cdot\frac{1}{(b_2-a_2)} = f_{X,Y}(x,y)}$$

$P(x \le X \le x+dx,$ AND $y \le Y \le y+dy) =$

X,Y independent

$P(x \le X \le x+dx) \cdot P(y \le Y \le y+dy)$

$\frac{dx}{b_1-a_1}$ $\qquad$ $\frac{dy}{b_2-a_2}$

(pseudo)-random variate generation

u_x = rand (1,n)

u_y = rand (1,n)

x1 xpts = $a_1 + (b_1-a_1)$*u_1 — x coordinate of dart

x2 ypts = $a_2 + (b_2-a_2)$*u_2 — y coordinate of dart

VIRTUAL DARTS

a Dart Game:

Area (or Volume, ...) Calculation

Rules

Throw darts – blindfolded – at

a rectangular dartboard R $\qquad A_R$

with a bulls-eye domain D . $\qquad A_D$

$P((X,Y)_{dart} \text{ is in } D) = A_D / A_R$

blindfolded

Bernoulli r.v. $\qquad B = g(X,Y)$

$$B = \begin{cases} 0 & (X,Y) \text{ not in } D \quad \text{probability } 1-\theta \\ 1 & (X,Y) \text{ in } D \quad \text{probability } \theta \end{cases}$$

with $\theta = P((X,Y) \text{ in } D) = A_D / A_R$.

To sample B:

throw dart $\Rightarrow (X,Y)_{dart}$;

if $(X,Y)_{dart}$ is in D, $B = 1$, otherwise $B = 0$;

note to sample B we need only determine if $(X,Y)_{dart}$ is in D.

3) Compute confidence interval for θ: $\qquad \gamma = 0.95$

$$[ci]_n = \left[\hat{\theta}_n - 1.96\sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}},\ \hat{\theta}_n + 1.96\sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}}\right]$$

confirm $n\hat{\theta}_n > 5$, $n(1-\hat{\theta}_n) > 5$.

4) Unravel to reveal A_D estimate, confidence interval:

$$\theta = A_D / A_R, \quad \hat{\theta}_n = (\hat{A}_D)_n / A_R \Rightarrow \boxed{(\hat{A}_D)_n = \hat{\theta}_n \cdot A_R}$$

$$\left[\hat{\theta}_n - 1.96\sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}} \le \theta \le \hat{\theta}_n + 1.96\sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}}\right]$$

$$\Rightarrow \boxed{[ci_{A_D}]_n = A_R\,[ci]_n}$$

Monte Carlo:

Replace real darts with pseudo-random variates:

```
u_x = rand(1,n); u_y = rand(1,n);
xpts = a1 + (b1 - a1)* u_x; ypts = a2 + (b2-a2)* u_y
theta_hat = sum ( inside_D (xpts,ypts) )/n
```

$(\hat{A}_D)_n = \ldots$

$[ci_{A_D}]_n = \ldots$

where

function [is_inside] = inside_D (x,y)

returns logical 1 if (x,y) is in D (otherwise logical 0).

Estimation of $\Theta \Rightarrow A_D$: "probability of a head" (B=1)

Now assume $\theta \equiv A_D / A_R$ is unknown.

1) Sample (throw) n darts ~ flip n coins

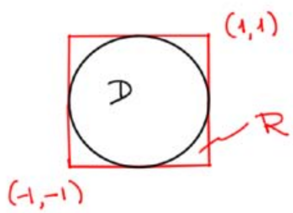
$(X,Y)_{\text{dart } 1} \to B_1 \; (\sim b_1) \cdots (X,Y)_{\text{dart}} \to B_n \; (\sim b_n)$ realization.

2) Compute sample mean estimator for Θ

$$\hat{\Theta}_n = \frac{1}{n} \sum_{i=1}^{n} B_i$$

$$\hat{\theta}_n = \frac{1}{n} \sum_{i=1}^{n} b_n \quad \text{estimate: fraction of heads}$$

example:



$A_D = \pi$

$A_R = 4$

```
function [is_inside] = inside_D(x,y)
is_inside = x.^2 + y.^2 <= 1;
end
```

DEMO

the good:

simple implementation;

good and "rigorous" error estimate;

reasonable results for small n,

convergence $\sim 1/\sqrt{n}$ independent of d(imension) volume, ...

amusing to watch;

the bad:

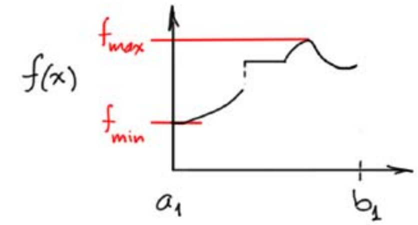
slow convergence: $\sim 1/\sqrt{n}$

but: importance sampling, ...

Monte-Carlo Integration
by
"Hit or Miss" Method

(Sample-Mean Method)

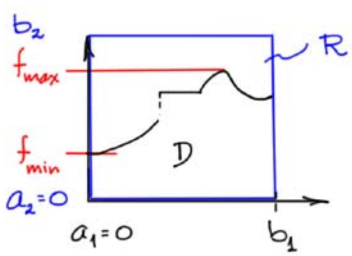
Integral...



assume $f_{min} > 0$

$$I = \int_{a_1}^{b_1} f(x)\,dx$$

... to Area



$b_2 \geq f_{max}$

... to Game of Darts