Eigenproblems

2×2 Case:
Linear Oscillator

simple oscillator

$$m\ddot{u} + c\dot{u} + ku = 0$$

$$u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0$$

$$\Downarrow \qquad w_1 = u, \; w_2 = \dot{u}$$

$$\begin{pmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \quad \text{or} \quad \underbrace{\begin{pmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \end{pmatrix}}_{\frac{dw}{dt}} = \underbrace{\begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} w_1 \\ w_2 \end{pmatrix}}_{w}$$

$$\omega_n^2 = \frac{k}{m}, \quad \zeta = \frac{c}{2m\omega_n} \quad (\zeta < 1: \text{underdamped})$$

Assume solution(s) of form

$$\boxed{\underset{2\times 1}{w} = \underset{2\times 1}{\chi} \cdot \underset{1\times 1}{e^{\lambda t}} \qquad \begin{array}{l} \chi: \text{eigenvector (or mode)} \\ \lambda: \text{eigenvalue} \end{array}}$$

$$= \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} e^{\lambda t} = \begin{pmatrix} \chi_1 e^{\lambda t} \\ \chi_2 e^{\lambda t} \end{pmatrix} \begin{array}{l} \leftarrow w_1(t) \\ \leftarrow w_2(t) \end{array}$$

Then

$$\frac{dw}{dt} = \begin{pmatrix} \frac{dw_1}{dt} \\ \frac{dw_2}{dt} \end{pmatrix} = \begin{pmatrix} \chi_1 \lambda e^{\lambda t} \\ \chi_2 \lambda e^{\lambda t} \end{pmatrix} = \lambda \begin{pmatrix} \chi_1 e^{\lambda t} \\ \chi_2 e^{\lambda t} \end{pmatrix} = \lambda \chi e^{\lambda t},$$

and

$$Aw = A\chi e^{\lambda t}.$$

Hence

$$dw/dt = Aw$$

$$\Downarrow$$

$$I \curvearrowright \quad \chi \lambda e^{\lambda t} = A\chi e^{\lambda t}, \text{ or}$$

$$(A\chi - \lambda I \chi)e^{\lambda t} = 0, \text{ or}$$

$$(A - \lambda I)\chi = 0$$

$$\Rightarrow \chi = 0 \text{ or } A - \lambda I \text{ is singular } (\Rightarrow \chi \neq 0)$$

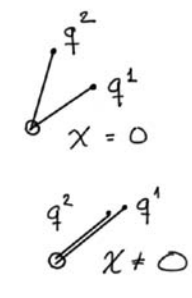
λ such that

For our oscillator, $\chi \neq 0$

$$\left(\begin{pmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}\right)\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{pmatrix} -\lambda & 1 \\ -\omega_n^2 & -2\zeta\omega_n - \lambda \end{pmatrix}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}$$

$$\Rightarrow \chi_1 \underbrace{\begin{pmatrix} -\lambda \\ -\omega_n^2 \end{pmatrix}}_{q^1} + \chi_2 \underbrace{\begin{pmatrix} 1 \\ -2\zeta\omega_n - \lambda \end{pmatrix}}_{q^2} = 0$$



Hence to obtain non-trivial χ,

$\det(A - \lambda I) = p(\lambda)$
easy way

$$q^2 = \alpha q^1, \text{ or}$$

$$\begin{pmatrix} 1 \\ -2\zeta\omega_n - \lambda \end{pmatrix} = \alpha \begin{pmatrix} -\lambda \\ -\omega_n^2 \end{pmatrix} = \begin{pmatrix} -\alpha\lambda \\ -\alpha\omega_n^2 \end{pmatrix}, \text{ or}$$

$$1 = -\alpha\lambda \Rightarrow \alpha = -1/\lambda$$

$$-2\zeta\omega_n - \lambda = -\alpha\omega_n^2 \longleftarrow 1/\lambda\, \omega_n^2$$

$$\chi = \begin{pmatrix} -\alpha \\ 1 \end{pmatrix} \cdot \text{const}$$

"one-handed"

$$\Rightarrow \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0 \qquad p(\lambda) = 0$$

$p(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2$ is characteristic polynomial

$\Rightarrow$ roots $\lambda = \lambda_1$, $\lambda = \lambda_2$ eigenvalues

$\Rightarrow (A - \lambda_1)\chi^1 = 0$. $(A - \lambda_2)\chi^2 = 0$ eigenvector. pairs

For our oscillator, $\zeta < 1$

$$p(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\Downarrow$$

$$\lambda = \frac{-2\zeta\omega_n \pm \sqrt{4\zeta^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$= \frac{-2\zeta\omega_n \pm 2i\omega_n\sqrt{1-\zeta^2}}{2}$$

$$\Downarrow$$

$$\lambda_1 = -\zeta\omega_n + i\omega_n\sqrt{1-\zeta^2} \quad \Rightarrow \chi^1$$

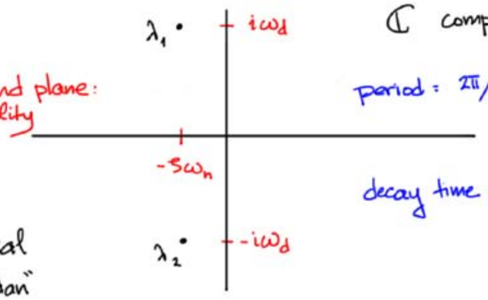
$$\lambda_2 = -\zeta\omega_n - i\omega_n\sqrt{1-\zeta^2} \quad \Rightarrow \chi^2.$$

Thus $\qquad$ initial conditions $\Rightarrow c_1, c_2$

$$u(t) = c_1 \chi^1 e^{\lambda_1 t} + c_2 \chi^2 e^{\lambda_2 t},$$

which is a decaying sinusoid ($\zeta < 1$: underdamped):

$$\lambda_{1,2} = -\zeta\omega_n \pm i\underbrace{\omega_n\sqrt{1-\zeta^2}}_{\omega_d} \; : \; e^{\lambda_{1,2} t} = \overbrace{e^{-\zeta\omega_n t}}^{\mathrm{Re}\,\lambda} \; \overbrace{e^{i\omega_d t}}^{\mathrm{Im}\,\lambda}.$$



$\mathbb{C}$ complex plane

left-hand plane: stability

period: $2\pi/\omega_d$

decay time scale: $1/\zeta\omega_n$

$\zeta > 1$: λ real

$\zeta = 1$: "Jordan"

$n \times n$ Case:

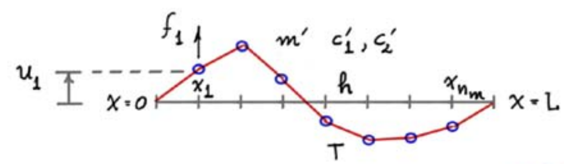
String in Tension

lambdas = eig(A) $\qquad$ eig(A, eye(n,n))

lambdas = eig($\bar{A}$, M)

(eigs)

Initial Value Problem



$h = L/(n_m+1)$

zero for $i = 1$

zero for $i = n_m$

$$\begin{cases} m'\ddot{u}_i + c_1'\dot{u}_i + c_2' T\left(\dfrac{-\dot{u}_{i-1} + 2\dot{u}_{i+1} - \dot{u}_{i+1}}{h^2}\right) \\ \qquad + T\left(\dfrac{-u_{i-1} + 2u_i - u_{i+1}}{h^2}\right) = f_i, \\ u_i(0) = (u_0)_i, \quad \dot{u}_i(0) = (\dot{u}_0)_i \qquad 1 \le i \le n_m. \end{cases}$$

State variables:

$$w_1 = u_1, \; w_2 = \dot{u}_1, \; w_3 = u_2, \; w_4 = \dot{u}_2, \; \ldots, \; w_n \text{ for } n = 2n_m.$$

State equations:

$$m'\dot{w}_1 = m'w_2$$
$$m'\dot{w}_2 = -c_1' w_2 - c_2'\frac{T}{h^2}(2w_2 - w_4) - \frac{T}{h^2}(2w_1 - w_3) + f'$$
$$m'\dot{w}_3 = m'w_4$$
$$m'\dot{w}_4 = -c_1' w_4 - \cdots$$
$$\vdots$$

$$\Rightarrow \begin{cases} M\dfrac{dw}{dt} = \overset{n\times n}{\bar{A}}\,\overset{n\times 1}{w} + \bar{F}, \quad 0 < t \le t_f \\ w(0) = w_0 \end{cases}.$$

Eigenproblem

Insert $w(t) = \chi e^{\lambda t}$ into

$$M\dot{w} = \bar{A}w \quad \text{to obtain}$$

$$M\lambda\chi e^{\lambda t} = \bar{A}\chi e^{\lambda t}$$

or

$$\boxed{\bar{A}\chi = \lambda M \chi}$$

"generalized"

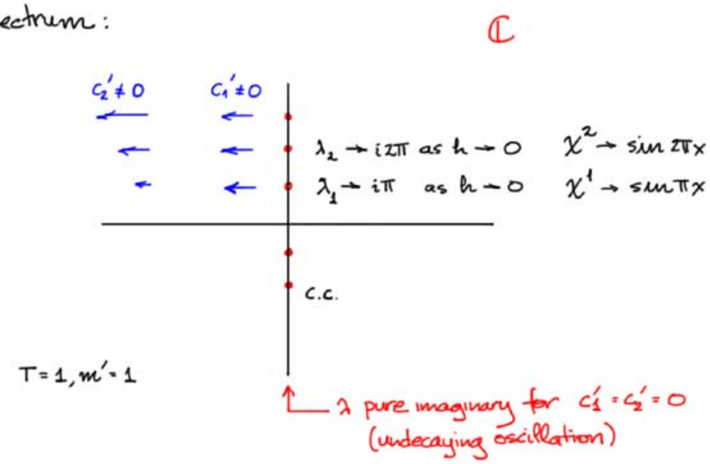
$\det(A - \lambda M) = 0$, OR columns of $A - \lambda M$ linearly dependent

$\Downarrow$

(λ_i, χ^i) pairs, $1 \le i \le n$.

Note: χ^i may not be linearly independent, but "typically"...

Spectrum:



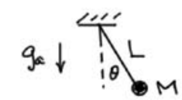
$$e^{\lambda t} = e^{\mathrm{Re}(\lambda)t} e^{i\,\mathrm{Im}(\lambda)t} \quad \begin{cases} \text{decay time scale } 1/|\mathrm{Re}(\lambda)| \\ \text{period } 2\pi/\mathrm{Im}(\lambda) \end{cases}$$

Applications of Eigenproblems:

- Linear Stability Analysis ← Pendulum
- Resonance (Avoidance)
- Modal Analysis
- ⋮

(to dynamics...)

0) formulate problem



$$\begin{cases} \dfrac{dw_1}{dt} = w_2 & w_1(0) = \theta_0 \\[2ex] \dfrac{dw_2}{dt} = -d_1 w_2 - d_2 |w_2| w_2 - \dfrac{g_a}{L}\sin(w_1) & w_2(0) = \dot{\theta}_0 \end{cases}$$

1) find equilibria: $d\bar{w}/dt = 0$

$$\underbrace{\begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\text{analyze here}}, \quad \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix}, \quad \ldots$$

2) linearize about (each) equilibrium:

Insert

$$w_1 = 0 + w_1' \qquad w_2 = 0 + w_2'$$

into state equations to obtain

$$\frac{d(0+w_1')}{dt} = (0 + w_2') \quad , \quad w_1(0) = \theta_0'$$

$$\frac{d(0+w_2')}{dt} = -d_1 \cdot (0+w_2') - d_2 \cdot |0 + w_2'|(0+w_2') - g_a/L \sin(0+w_1'), \quad w_2(0) = \dot{\theta}_0' \;;$$

assume $|w_1'|\ (|\theta_0'|), |w_2'|\ (|\dot{\theta}_0'|) \ll 1$

$$\Rightarrow \left.\begin{array}{l} \big||w_2'|\, w_2'\big| \ll |w_2'|, \\ \sin(w_1') = w_1' + O(w_1'^3) \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{l} \dfrac{dw_1'}{dt} = w_2' \\ \dfrac{dw_2'}{dt} = -\dfrac{g_a}{L} w_1' - d_1 w_2' \end{array}\right. , \text{ or}$$

$$\mathrm{Id}\, \frac{dw'}{dt} = \underbrace{\begin{pmatrix} 0 & -1 \\ -\frac{g_a}{L} & -d_1 \end{pmatrix}}_{A} \begin{pmatrix} w_1' \\ w_2' \end{pmatrix} .$$

3) pose eigenproblem: $w' = \chi e^{\lambda t}$

$$A\chi = \lambda \chi \qquad \lambda\, \mathrm{Id}\, \chi$$

4) find eigenvalues:

lambdas = eig(A)   eig(A, eye(2,2))

[generalized: lambdas = eig($\bar{A}$, M)]

here $\det(A - \lambda I) \Rightarrow$ $\zeta < 1$, but...

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d$$

$$\omega_n = \sqrt{\frac{g_a}{L}}, \quad \zeta = \frac{d}{2}\sqrt{\frac{L}{g_a}}$$

5) deduce stability   linear stability

$$u(t) = \sum_{j=1}^{n} c_j \chi^j e^{\mathrm{Re}(\lambda_j)t} e^{i\,\mathrm{Im}(\lambda_j)t}$$

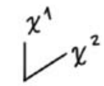
$\max_{1 \le j \le n} \mathrm{Re}(\lambda_j) > 0$ UNSTABLE

$\max_{1 \le j \le n} \mathrm{Re}(\lambda_j) < 0$ STABLE   (linearly)

$\max_{1 \le j \le n} \mathrm{Re}(\lambda_j) = 0$ MARGINAL, NEUTRAL (further analysis required)

Diagonalization

2×2 case



$$A\chi = \lambda\chi$$

(λ_1, χ^1), (λ_2, χ^2) $\quad \chi^1, \chi^2$ linearly independent

Let

$$\overset{2\times 2}{S} = \begin{array}{cc} & \\ \Big(\chi^1 & \chi^2 \Big) \\ \text{col 1} & \text{col 2} \end{array} \qquad S^{-1} \text{ exists} \; ;$$

then

$$AS = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \chi_1^1 & \chi_1^2 \\ \chi_2^1 & \chi_2^2 \end{pmatrix} = \begin{pmatrix} A_{11}\chi_1^1 + A_{12}\chi_2^1 & A_{11}\chi_1^2 + A_{12}\chi_2^2 \\ A_{21}\chi_1^1 + A_{22}\chi_2^1 & A_{21}\chi_1^2 + A_{22}\chi_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} A\chi^1 & A\chi^2 \end{pmatrix} = \begin{pmatrix} \lambda_1\chi^1 & \lambda_2\chi^2 \end{pmatrix} = \begin{pmatrix} \lambda_1\chi_1^1 & \lambda_2\chi_1^2 \\ \lambda_1\chi_2^1 & \lambda_2\chi_2^2 \end{pmatrix}$$

$$= \begin{pmatrix} \chi_1^1 & \chi_1^2 \\ \chi_2^1 & \chi_2^2 \end{pmatrix} \begin{pmatrix} \lambda_1 & \\ 0 & \lambda_2 \end{pmatrix}$$

$$= S \quad \Lambda \qquad \overset{2\times 2}{\Lambda} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} .$$

Hence

$$AS = S\Lambda \Rightarrow A = S\Lambda S^{-1}$$

or

$$\Lambda = S^{-1}AS .$$

diagonalization

n×n case

$$A\chi = \lambda\chi$$

(λ_j, χ^j), $1 \le j \le n$ $\quad \chi_j$ linearly independent (assume)

Let

$$\overset{n\times n}{S} = \begin{pmatrix} \chi^1 & \chi^2 & \cdots & \chi^n \end{pmatrix} , \quad \Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix} ;$$

then

$$A = S\Lambda S^{-1} , \quad \Lambda = S^{-1}AS .$$

[S,Λ] = eig(A)

modal interpretation of ODE IVP

Proceed from

$$\frac{dw}{dt} = Aw + F$$

$$\hookrightarrow \quad \frac{dw}{dt} = S\Lambda S^{-1}w + F$$

$$\hookrightarrow \quad \frac{d}{dt}S^{-1}w = \Lambda S^{-1}w + S^{-1}F$$

$$\hookrightarrow \quad \frac{dz}{dt} = \Lambda z + G \text{, or}$$

$$\left\{ \begin{array}{l} S^{-1}w = z \quad \text{new "state" variables} \\ S^{-1}F = G \end{array} \right.$$

$$\frac{dz}{dt} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{pmatrix} \text{, or}$$

$$\frac{dz}{dt} = \begin{pmatrix} \lambda_1 z_1 \\ \lambda_2 z_2 \\ \vdots \\ \lambda_n z_n \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \\ \vdots \\ G_n \end{pmatrix}$$

$$\boxed{\frac{dz_j}{dt} = \lambda_j z_j + G_j, \quad z_j(0) = (S^{-1}w_0)_j \quad 1 \le j \le n}$$

System of n decoupled ODEs.

Note: for $G_j = 0$,

$$z_j(t) = z_j(0)\, e^{Re(\lambda_j)t}\, e^{i\,Im(\lambda_j)t}, \quad 1 \le j \le n.$$

Many applications of "modal" decomposition...

Computation of Extreme Eigenvalues

simple power iterations

SPD matrices $B \in \mathbb{R}^{n \times n}$ $\qquad Bx = \lambda x$

definition (conditions):

1) B is symmetric $\qquad$ Symmetric
   ($\Rightarrow \lambda_j$ real, x^j orthogonal)

2) λ_j positive $\qquad$ Positive Definite
   (other equivalent criteria: $v^T B v > 0$, $v \neq 0$; ...)

an example: undamped string in tension

Recall ($c_1' = c_2' = 0$; unforced)

$$m'\ddot{u}_i + \frac{T}{h^2}(\underbrace{-u_{i-1}}_{\text{zero for } i=1} + 2u_i - \underbrace{u_{i+1}}_{\text{zero for } i=n_m}) = 0 \qquad 1 \le i \le n_m$$

or

$$\underbrace{\frac{T}{h^2 m'}\begin{pmatrix} 2 & -1 & & 0 \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \\ 0 & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}}_{\substack{K_m \\ \text{(SPD)}}}\underbrace{\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{n_m} \end{pmatrix}}_{u} = \underbrace{-\begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_{n_m} \end{pmatrix}}_{-\ddot{u}}$$

We know (from before) $u = \sum_j c_j \Upsilon_j e^{i\omega_j t}$

$$\Rightarrow K_m \Upsilon e^{\sigma t} = \omega^2 \Upsilon e^{\sigma t};$$

hence eigenvalues of

$$K_m \Upsilon = \sigma \Upsilon \qquad \sigma > 0$$

yield frequencies $\omega = \pm\sqrt{\sigma}$.

Typically of (most) interest:

$\underbrace{\sigma_1}_{\sigma_{min}}$ (smallest eigenvalue) $\Rightarrow$ lowest frequency.

Power Iteration for λ_{max} of $B\chi = \lambda\chi$ $B \in \mathbb{R}^{n\times n}$

Recall

$$B = S\Lambda S^{-1} \qquad S = \left(\chi^1 \chi^2 \cdots \chi^n\right), \ \Lambda = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}.$$

$$0 < \underset{\lambda_{min}}{\lambda_1} \le \lambda_2 \cdots \le \underset{\lambda_{max}}{\lambda_n}$$

Thus

$$B^k = \overbrace{(S\Lambda S^{-1})(S\Lambda S^{-1})\cdots(S\Lambda S^{-1})}^{k \text{ terms}}$$

$$= S\Lambda^k S^{-1}$$

$$= S\underbrace{\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}\cdots\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_n \end{pmatrix}}_{k \text{ terms}}S^{-1}$$

$$= S\begin{pmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \ddots \\ & & & \lambda_n^k \end{pmatrix}S^{-1}$$

so

$$B^k\chi_0 = \left(\chi^1 \chi^2 \cdots \chi^n\right)\begin{pmatrix} \lambda_1^k & & \\ & \lambda_2^k & \\ & & \ddots \\ & & & \lambda_n^k \end{pmatrix}\overbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}}^{S^{-1}\chi_0}$$

$$= \left(\chi^1 \chi^2 \cdots \chi^n\right)\begin{pmatrix} \alpha_1\lambda_1^k \\ \alpha_2\lambda_2^k \\ \vdots \\ \alpha_n\lambda_n^k \end{pmatrix}$$

$$= \alpha_1\lambda_1^k\chi^1 + \alpha_2\lambda_2^k\chi^2 + \cdots + \alpha_n\lambda_n^k\chi^n$$

$$\rightarrow \alpha_n\lambda_n^k\chi^n \quad \text{as } k \rightarrow \infty.$$

algorithm

given $\chi^{(0)}$, $k = 1$:

for $k = 1 : k_{max}$

$v = B\chi^{(k-1)}$

$\chi^{(k)} = v/\|v\|$ $\quad$ avoid overflow $\left(\Rightarrow \|\chi^{(k)}\|^2 = \|v\|^2/\|v\|^2 = 1\right)$

$\lambda_{max}^{(k)} = (\chi^{(k)})^T B \chi^{(k)}$ $\quad$ $\cancel{/\underbrace{\|\chi^{(k)}\|^2}_{1}}$

end

Note:

$$(\chi^n)^T B \chi^n = (\chi^n)^T \lambda_n \chi^n = \lambda_{max}$$

if

$(\chi^n)^T(\chi^n) = \|\chi^n\|^2$ (normalized) to unity.

Inverse Power Iteration for λ_{min} of $B\chi = \lambda\chi$ $\quad B \in \mathbb{R}^{n\times n}$

Recall $B = S\Lambda S^{-1}$

$\Rightarrow B^{-1} = S\Lambda^{-1}S^{-1}$ $\quad S = \begin{pmatrix} \chi^1 \chi^2 \cdots \chi^n \end{pmatrix}$, $\Lambda^{-1} = \begin{pmatrix} \lambda_1^{-1} & & \\ & \lambda_2^{-1} & \\ & & \ddots \\ & & & \lambda_n^{-1} \end{pmatrix}$.

$0 < \lambda_1 \leq \lambda_2 \cdots \leq \lambda_n$

$\lambda_{min} \qquad \lambda_{max}$

Thus

$$(B^{-1})^k = \overbrace{(S\Lambda^{-1}S^{-1})(S\Lambda^{-1}S^{-1})\cdots(S\Lambda^{-1}S^{-1})}^{k \text{ terms}}$$

$$= S(\Lambda^{-1})^k S^{-1}$$

$$= S\begin{pmatrix} \lambda_1^{-1} & & \\ & \lambda_2^{-1} & \\ & & \ddots \\ & & & \lambda_n^{-1} \end{pmatrix}\begin{pmatrix} \lambda_1^{-1} & & \\ & \lambda_2^{-1} & \\ & & \ddots \\ & & & \lambda_n^{-1} \end{pmatrix}\cdots\begin{pmatrix} \lambda_1^{-1} & & \\ & \lambda_2^{-1} & \\ & & \ddots \\ & & & \lambda_n^{-1} \end{pmatrix}S^{-1}$$

$$= S\begin{pmatrix} \lambda_1^{-k} & & \\ & \lambda_2^{-k} & \\ & & \ddots \\ & & & \lambda_n^{-k} \end{pmatrix}S^{-1}$$

so

$$(B^{-1})^k\chi_0 = \begin{pmatrix} \chi^1 \chi^2 \cdots \chi^n \end{pmatrix}\begin{pmatrix} \lambda_1^{-k} & & \\ & \lambda_2^{-k} & \\ & & \ddots \\ & & & \lambda_n^{-k} \end{pmatrix}\overbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}}^{S^{-1}\chi_0}$$

$$= \begin{pmatrix} \chi^1 \chi^2 \cdots \chi^n \end{pmatrix}\begin{pmatrix} \alpha_1\lambda_1^{-k} \\ \alpha_2\lambda_2^{-k} \\ \vdots \\ \alpha_n\lambda_n^{-k} \end{pmatrix}$$

$$= \alpha_1\lambda_1^{-k}\chi^1 + \alpha_2\lambda_2^{-k}\chi^2 + \cdots + \alpha_n\lambda_n^{-k}\chi^n$$

$$\to \alpha_1\lambda_1^{-k}\chi_1 \quad \text{as } k \to \infty.$$

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algorithm 1

given $\chi^{(0)}$:

for $k = 1 : k_{max}$

$Bv = \chi^{(k-1)}$ $\quad (v = B^{-1}\chi^{(k-1)})$

$\chi^{(k)} = v/\|v\|$ $\quad (\Rightarrow \|\chi^{(k)}\|^2 = 1)$

$\lambda^{(k)} = \chi^{(k)T} B \chi^{(k)}$

end

Note:

$$(\chi_1)^T B \chi_1 = (\chi_1)^T \lambda_1 \chi_1 = \lambda_{min}$$

if

$(\chi_1)^T(\chi_1) = \|\chi_1\|^2$ (normalized) to unity.

algorithm 2 more efficient

given $\hat{x}$, $\hat{\lambda}$, ε:

while $\| B\hat{x} - \hat{\lambda}\hat{x} \| > \varepsilon$

$B v = \hat{x}$

$\beta = \| v \|$

$\hat{\lambda} = v^T \hat{x} / \beta^2$

$\hat{x} = v / \beta$

end

(Improvements: shift, ...)