Problem Set 3 — 2.086 Spring 2012

Released: *Tuesday, 13 March 2012*
Due: *Tuesday, 3 April, at 2:30 PM by hardcopy at the beginning of class.*

Note that no MATLAB *scripts need be uploaded to Stellar for this problem set.*

Introduction

In this problem set, we will determine the friction coefficient between a robot's wheels and the ground. The friction coefficient directly affects the robot's resistance to slippage, and hence also the tasks the robot may perform. In particular, the friction coefficient can limit the load which the robot can pull or push.

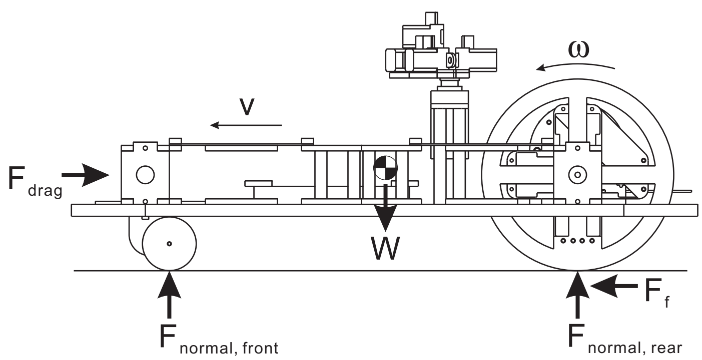


Figure 1: A mobile robot in motion.

When a mobile robot such as shown in Figure 1 is commanded to move forward, a number of forces come into play. Internally, the drive motors exert a torque (not shown in the figure) on the wheels, which is resisted by the friction force F_f between the wheels and the ground. If the magnitude of F_f dictated by the sum of the drag force F_drag (a combination of all forces resisting the robot's motion) and the product of the robot's mass and acceleration is less than the maximum static friction force $F_{\mathrm{f,\,static}}^{\mathrm{max}}$ between the wheels and the ground, the wheels will roll without slipping and the robot will move forward with velocity $v = \omega r_\mathrm{wheel}$ (here ω is the angular velocity of the wheel). If, however, the magnitude of F_f reaches $F_{\mathrm{f,\,static}}^{\mathrm{max}}$, the wheels will begin to slip and F_f will drop to a lower level $F_{\mathrm{f,\,kinetic}}$, the kinetic friction force. The wheels will continue to slip ($v < \omega r_\mathrm{wheel}$) until zero relative motion between the wheels and the ground is restored (when $v = \omega r_\mathrm{wheel}$).

The critical value defining the boundary between rolling and slipping, therefore, is the maximum static friction force. Amontons' "law" states that

$$F_{\mathrm{f,\,static}}^{\mathrm{max}} = \mu_\mathrm{s}\, F_{\mathrm{normal,\,rear}}, \tag{1}$$

where μ_s is the static coefficient of friction and $F_{\mathrm{normal,\,rear}}$ is the normal force from the ground on the rear, driving, wheels. In order to minimize the risk of slippage, robot wheels should be designed for a high value of μ_s between the wheels and the ground. This value, although difficult to predict accurately by modeling, can be determined by experiment.

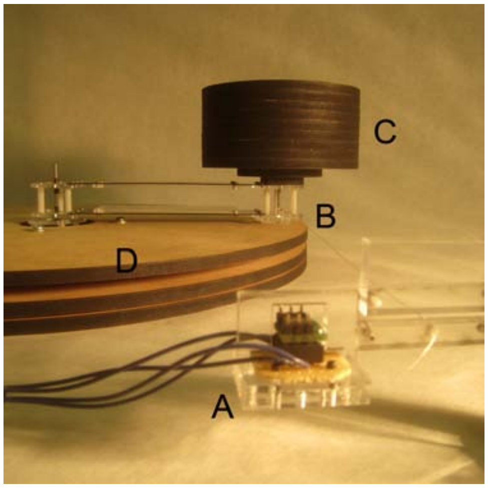
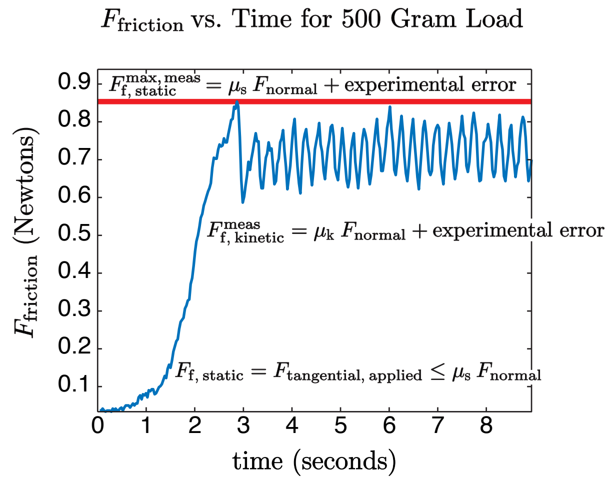


Figure 2: Experimental setup for friction measurement: Force transducer (A) is connected to contact area (B) by a thin wire. Normal force is exerted on the contact area by load stack (C). Tangential force is applied using turntable (D) via the friction between the turntable surface and the contact area. Apparatus and photograph courtesy of James Penn.



Data courtesy of James Penn.

Figure 3: Sample data for one friction measurement, yielding one data point for $F^{\text{max, meas}}_{\text{f , static}}$.

We first conduct experiments to determine the friction force $F^{\text{max}}_{\text{f, static}}$ (in Newtons) as a function of normal load $F_{\text{normal, applied}}$ (in Newtons) and (nominal) surface area of contact A_{surface} (in cm^2) with the 2.086 friction turntable apparatus depicted in Figure 2. Weights permit us to vary the normal load and "washer" inserts permit us to vary the nominal surface area of contact. We next apply a force $F_{\text{tangential, applied}}$ to the turntable which is balanced by the friction force F_{friction}; the latter is then measured by a transducer which relates voltage to deflection to force. A typical measurement point (at a particular prescribed value of $F_{\text{normal, applied}}$ and A_{surface}) yields the time trace of Figure 3: we increase $F_{\text{tangential, applied}}$ until slippage occurs; $F^{\text{max, meas}}_{\text{f, static}}$ (our measurement of $F^{\text{max}}_{\text{f, static}}$) is deduced as the maximum force achieved.

The experimental data comprises 50 measurements: 2 (distinct) measurements at each of 25 points on a 5×5 "grid" in $(F_{\text{normal, applied}}, A_{\text{surface}})$ space. The data is provided to you in the `.mat` file `friction_data_PSet3` as 50×1 arrays `F_fstaticmaxmeas`, `F_normalload`, and `A_surface`. Entry i of `F_fstaticmaxmeas`, `F_normalload`, and `A_surface` provides respectively the measured friction force $F^{\text{max, meas}}_{\text{f, static}}$ (in Newtons), the prescribed normal load $F_{\text{normal, applied}}$ (in Newtons), and the prescribed surface area A_{surface} (in cm^2), for the i^{th} measurement. For example, in the first measurement, $i = 1$, the measured friction force is 0.1080 Newtons, the imposed normal load is 0.9810 Newtons, and the nominal surface area is 1.2903 cm^2.

We next postulate a dependence (or "model")

$$F^{\text{max}}_{\text{f, static}}(F_{\text{normal, applied}}, A_{\text{surface}}; \beta) = \beta_0 + \beta_1\, F_{\text{normal, applied}} + \beta_2\, A_{\text{surface}} \ , \tag{2}$$

where $\beta = (\beta_0, \beta_1, \beta_2)^{\text{T}}$. We expect — but do not *a priori* assume — from Messieurs Amontons and Coulomb that $\beta_0 = 0$ and $\beta_2 = 0$. We shall refer to (2) as the "full model." We assume that the experimental measurements follow the response model

$$F_{\text{f, static}}^{\text{max, meas}} = F_{\text{f, static}}^{\text{max}}(F_{\text{normal, applied}}, A_{\text{surface}}; \beta^{\text{true}}) + \epsilon \tag{3}$$

where β^{true} is the true value of β in the absence of noise, and ϵ is the noise. We implicitly assume by the existence of β^{true} that our model is bias-free. We further assume that the noise is normal, homoscedastic, and uncorrelated (at different $F_{\text{normal, applied}}$ and A_{surface}) per our assumptions (N1), (N2), and (N3) of the text. We would like you to perform a regression analysis to determine estimates and confidence intervals for the coefficients β_0^{true}, β_1^{true}, and β_2^{true}.

Note for each question we identify each deliverable as a subquestion which you should clearly delineate ((a), (b), ...) in your report. All quantities not explicitly defined in the problem statement are taken directly from the text.

Questions

1. (15 pts) Indicate the correspondence between (mapping from) the general formulation and variables of Section 17.2.1 of the text to our particular case here described by equations (2) and (3). We provide the first correspondence: $F_{\text{f, static}}^{\text{max, meas}}$ corresponds to Y. Please fill in the rest:

   (a) $p = ?$;

   (b) x (a p-vector) corresponds to ?;

   (c) $Y_{\text{model}}(x;\beta)$ corresponds to ? ;

   (d) $n = ?$;

   (e) $h_j(x)$, $1 \le j \le n-1$ corresponds to ? ;

   (f) $m = ?$.

   Note in each case the ? should be replaced by variables or numbers provided in this Problem Set 3 statement (*not* the generic variables of Section 17.2.1).[1]

2. (10 pts)

   (a) Complete the (single-line) MATLAB assignment statement `X = ?` which forms the X matrix for your regression analysis.

   Note your statement should be a syntactically correct line of code cut from MATLAB and pasted to your problem set document. The statement should use only standard MATLAB built-in's (if you like) and the arrays `F_fstaticmaxmeas`, `F_normalload`, and `A_surface`.

   (b) What is the `size` of `X`?

[1] *Hint*: Recall that n is the number of regression coefficients which is also equal to the number of basis functions (one basis function per regression coefficient) if we recall that the basis function $h_0(x) = 1$ (multiplied by β_0) is included in this count.

3. (5 pts)

   (a) Complete the (single-line) MATLAB assignment statement `Y = ?` which forms the Y vector for your regression analysis. (Same rules apply as for Question 2. above.)

   (b) What is the `size` of `Y`?

4. (10 pts)

   (a) Provide mathematical expressions (no MATLAB syntax required) for 95% (confidence level) joint confidence intervals I_0^{joint}, I_1^{joint}, and I_2^{joint} for β_0^{true}, β_1^{true}, and β_2^{true}, respectively.

   Your expressions should contain only the variables $\hat{\beta}_0$, $\hat{\beta}_1$, and $\hat{\beta}_2$, the $\widehat{\Sigma}$ matrix, and $s_{\gamma=0.95,k,q}$. Please provide values for k and q, but you need not yet evaluate $s_{\gamma=0.95,k,q}$. Your confidence intervals should be of the form [lower limit, upper limit].

   (b) Provide a numerical value for "$s_{\gamma=0.95,k,q}$" for your chosen k, q. Indicate, briefly, how you obtained this result. Note you may use interpolation in Table 17.1(b) of the text to obtain an appropriate (approximate) "$s_{\gamma=0.95,k,q}$" value.

5. (15 pts) Perform your regression analysis and provide numerical values for

   (a) $\hat{\beta}_0$,

   (b) $\hat{\beta}_1$,

   (c) $\hat{\beta}_2$,

   (d) $\hat{\sigma}$, and

   (e) $\widehat{\Sigma}$ (all entries of this matrix). (Note to find $\widehat{\Sigma}$ for this small problem you may just use the MATLAB matrix inverse routine `inv`.)

6. (12 pts)

   (a) Provide numerical values for the confidence intervals I_0^{joint}, I_1^{joint}, and I_2^{joint}. Your confidence intervals should be of the form [lower limit, upper limit], where now "lower limit" and "upper limit" are numbers evaluated from your formulas of Question 4(a).

   (b) Is the value 0 included in I_0^{joint}?

   (c) Is the value 0 included in I_2^{joint}?

   (d) What do you conclude about the correctness of equation (1) from your answers to 6.(b) and 6.(c)?

7. (4 pts)

   (a) What is the average of the measured friction force data (over the 50 measurements)?

   (b) What is the average of the *predicted friction force* (average of $(X\hat{\beta})_i \equiv \widehat{Y}_i \equiv Y_{\text{model}}(x_i;\hat{\beta})$, $1 \le i \le m$)?

(Your two means (measured, predicted) should be the same. If not, you have a bug.)

8. (9 pts)

   (a) Provide a histogram of $Y - Y$. You may use the MATLAB built-in function `hist` with the default number of bins.

   (b) Create and provide several — at least three — histograms for samples of size 50 from the normal density $\mathcal{N}(0, \hat{\sigma}^2)$. (You should use the `randn` built-in and then the appropriate scaling, as described in the Appendix to Unit II on Estimation: the Normal Density.)

   (c) Ask a friend to pick the "odd character" from the histograms of (a) and (b). Is the histogram of the experimental data chosen?

9. (14 pts) Repeat Questions 1–7 (now 2 pts each) but now rather than the *full* model given by equation (2) consider an *abridged* model corresponding to equation (2) *without* the $\beta_2\, A_{\text{surface}}$ term. Note please mark your answers Q9/1(a), Q9/1(b), ..., Q9/7(a), Q9/7(b) so that we can clearly identify each part.

10. (6 pts)

    (a) Which model, full or abridged, realizes a lower value of $\|Y - Y\|^2$ — and why must this be the case?

    (b) Which model, full or abridged, do you think is better — gives a better prediction for β_1^{true} — and why? (Tricky question, with no right or wrong answer. But your reasoning should be sound in any event.)