Lecture 2: The Concept of Strain

Strain is a fundamental concept in continuum and structural mechanics. Displacement fields and strains can be directly measured using gauge clips or the Digital Image Correlation (DIC) method. Deformation patterns for solids and deflection shapes of structures can be easily visualized and are also predictable with some experience. By contrast, the stresses can only be determined indirectly from the measured forces or by the inverse engineering method through a detailed numerical simulation. Furthermore, a precise determination of strain serves to define a corresponding stress through the work conjugacy principle. Finally the equilibrium equation can be derived by considering compatible fields of strain and displacement increments, as explained in Lecture 3. The present author sees the engineering world through the magnitude and shape of the deforming bodies. This point of view will dominate the formulation and derivation throughout the present lecture note. Lecture 2 starts with the definition of one dimensional strain. Then the concept of the three-dimensional (3-D) strain tensor is introduced and several limiting cases are discussed. This is followed by the analysis of strains-displacement relations in beams (1-D) and plates (2-D). The case of the so-called moderately large deflection calls for considering the geometric non-linearities arising from rotation of structural elements. Finally, the components of the strain tensor will be re-defined in the polar and cylindrical coordinate system.

2.1 One-dimensional Strain

Consider a prismatic, uniform thickness rod or beam of the initial length l_o. The rod is fixed at one end and subjected a tensile force (Fig. (2.1)) at the other end. The current, deformed length is denoted by l. The question is whether the resulting strain field is homogeneous or not. The concept of *homogeneity* in mechanics means independence of the solution on the spatial coordinates system, the rod axis in the present case. It can be shown that if the stress-strain curve of the material is convex or linear, the rod deforms uniformly and a homogeneous state of strains and stresses are developed inside the rod. This means that local and average strains are the same and the strain can be defined by considering the total lengths. The displacement at the fixed end $x = 0$ of the rod is zero, $u(x = 0)$ and the end displacement is

$$u(x = l) = l - l_o \tag{2.1}$$

The strain is *defined as a relative displacement.* Relative to what? Initial, current length or something else? The definition of strain is simple but at the same time is non-unique.

$$\epsilon \overset{\text{def}}{=} \frac{l - l_o}{l_o} \quad \text{Engineering Strain} \tag{2.2a}$$

$$\epsilon \overset{\text{def}}{=} \frac{1}{2}\frac{l^2 - l_o^2}{l^2} \quad \text{Cauchy Strain} \tag{2.2b}$$

$$\epsilon \overset{\text{def}}{=} \ln\frac{l}{l_o} \quad \text{Logarithmic Strain} \tag{2.2c}$$

Each of the above three definitions satisfy the basic requirement that strain vanishes when $l = l_o$ or $u = 0$ and that strain in an increasing function of the displacement u.

Consider a limiting case of Eq.(2.1) for small displacements $\frac{u}{l_o} \ll 1$, for which $l_o + l \approx 2l_o$ in Eq.(2.2b). Then, the Cauchy strain becomes

$$\epsilon = \frac{l - l_o}{l_o}\frac{l + l_o}{2l_o} \cong \frac{l - l_o}{l_o}\frac{2l}{2l_o} \cong \frac{l - l_o}{l_o} \tag{2.3}$$

Thus, for small strain, the Cauchy strain reduces to the engineering strain. Likewise, expanding the expression for the logarithmic strain, Eq.(2.2c) in Taylor series around $l - l_o \cong 0$,

$$\ln \frac{l}{l_o}\bigg|_{l/l_o=1} \cong \frac{l - l_o}{l_o} - \frac{1}{2}\left(\frac{l - l_o}{l_o}\right)^2 + \cdots \approx \frac{l - l_o}{l_o} \tag{2.4}$$

one can see that the logarithmic strain reduces to the engineering strain.

The plots of ϵ versus $\frac{l}{l_o}$ according to Eqs.(2.2a)-(2.2c) are shown in Fig.(2.1).

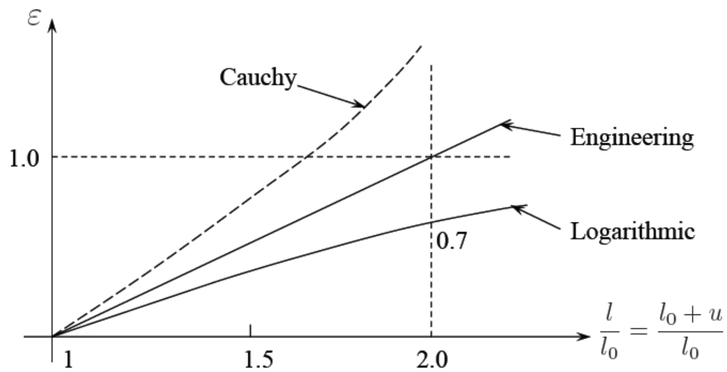

Figure 2.1: Comparison of three definitions of the uniaxial strain.

Inhomogeneous Strain Field

The strain must be defined locally and not for the entire structure. Consider an infinitesimal element dx in the undeformed configuration, Fig.(2.2).

After deformation the length of the original material element becomes $\mathrm{d}x + \mathrm{d}u$. The engineering strain is then

$$\epsilon_{\text{eng}} = \frac{(\mathrm{d}x + \mathrm{d}u) - \mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \tag{2.5}$$

The spatial derivative of the displacement field is called the displacement gradient $\boldsymbol{F} = \frac{\mathrm{d}u}{\mathrm{d}x}$. For uniaxial state the strain is simply the displacement gradient. This is not true for general 3-D case.

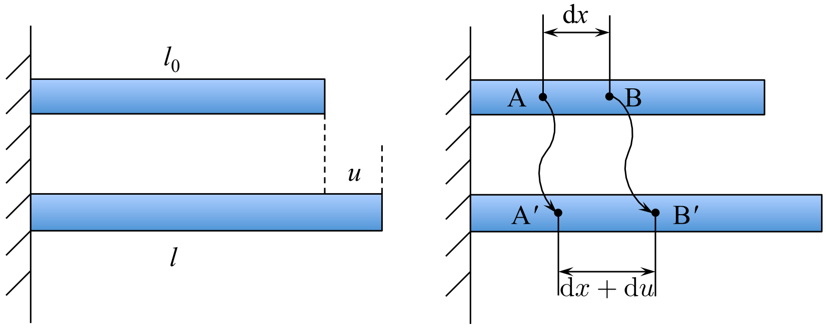

Figure 2.2: Undeformed and deformed element in the homogenous and inhomogeneous strain field in the bar.

The local Cauchy strain is obtained by taking relative values of the difference of the square of the lengths. As shown in Eq. (2.3), in order for the Cauchy strain to reduce to the engineering strain, the factor 2 must be introduced in the definition. Thus

$$\epsilon_{\mathrm{c}} = \frac{1}{2}\frac{(\mathrm{d}x + \mathrm{d}u)^2 - \mathrm{d}x^2}{\mathrm{d}x^2} = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 \tag{2.6}$$

or $\epsilon_{\mathrm{c}} = \boldsymbol{F} + \frac{1}{2}\boldsymbol{F}^2$. For small displacement gradients,

$$\epsilon_{\mathrm{c}} = \epsilon_{\mathrm{eng}} \tag{2.7}$$

2.2 Extension to the 3-D case

Equation (2.5) can be re-written in an alternative form

$$\mathrm{d}u = \epsilon\, \mathrm{d}x \tag{2.8}$$

Consider an Euclidian space and denote by $\boldsymbol{x} = \{x_1, x_2, x_3\}$ or x_i the vector representing a position of a generic point of a body. In the general three-dimensional case, the displacement of the material point is also a vector with components $\boldsymbol{u} = \{u_1, u_2, u_3\}$ or u_i where $i = 1, 2, 3$.

Recall that the increment of a function of three variables is a sum of three components

$$\mathrm{d}u_1(x_1, x_2, x_3) = \frac{\partial u_1}{\partial x_1}\mathrm{d}x_1 + \frac{\partial u_1}{\partial x_2}\mathrm{d}x_2 + \frac{\partial u_1}{\partial x_3}\mathrm{d}x_3 \tag{2.9}$$

In general, components of the displacement increment vector are

$$\mathrm{d}u_i(x_i) = \frac{\partial u_i}{\partial x_1}\mathrm{d}x_1 + \frac{\partial u_i}{\partial x_2}\mathrm{d}x_2 + \frac{\partial u_i}{\partial x_3}\mathrm{d}x_3 = \sum_{j=1}^{3}\frac{\partial u_i}{\partial x_j}\mathrm{d}x_j \tag{2.10}$$

where the repeated j is the so called "dummy" index. The displacement gradient

$$\boldsymbol{F} = \frac{\partial u_i}{\partial x_j} \tag{2.11}$$

is not a symmetric tensor. It also contains terms of rigid body rotation. This can be shown by re-writing the expression for $\boldsymbol{F}$ in an equivalent form

$$\frac{\partial u_i}{\partial x_j} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.12}$$

Strain tensor ϵ_{ij} is defined as a "symmetric" part of the displacement gradient, which is the first term in Eq. (2.12).

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{2.13}$$

Now, interchange (transpose) the indices i and j in Eq.(2.13):

$$\epsilon_{ji} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \tag{2.14}$$

The first term in Eq.(2.14) is the same as the second term in Eq.(2.13). And the second term in Eq.(2.14) is identical to the first term in Eq.(2.13). Therefore the strain tensor is symmetric

$$\epsilon_{ij} = \epsilon_{ji} \tag{2.15}$$

The reason for introducing the symmetry properties of the strain tensor will be explained later in this section. The second terms in Eq.(2.12) is called the *spin tensor* ω_{ij}

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \tag{2.16}$$

Using similar arguments as before it is easy to see that the spin tensor is antisymmetric

$$w_{ij} = -w_{ji} \tag{2.17}$$

From the definition it follows that the diagonal terms of the spin tensor are zero, for example $w_{11} = -w_{11} = 0$. The components, of the strain tensor are:

$$i = 1, j = 1 \quad \epsilon_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1} \tag{2.18a}$$

$$i = 2, j = 2 \quad \epsilon_{22} = \frac{\partial u_2}{\partial x_2} \tag{2.18b}$$

$$i = 3, j = 3 \quad \epsilon_{33} = \frac{\partial u_3}{\partial x_3} \tag{2.18c}$$

$$i = 1, j = 2 \quad \epsilon_{12} = \epsilon_{21} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \tag{2.18d}$$

$$i = 2, j = 3 \quad \epsilon_{23} = \epsilon_{32} = -\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \tag{2.18e}$$

$$i = 3, j = 1 \quad \epsilon_{31} = \epsilon_{13} = -\frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) \tag{2.18f}$$

For the geometrical interpretation of the strain and spin tensor consider an infinitesimal square element $(\mathrm{d}x_1, \mathrm{d}x_2)$ subjected to several simple cases of deformation. The partial derivatives are replaced by finite differences, for example

$$\frac{\partial u_1}{\partial x_1} = \frac{\Delta u_1}{\Delta x_1} = \frac{u_1(x_1) - u_1(x_1 + h)}{h} \tag{2.19}$$

Rigid body translation

Along x_1 axis:

$$u_1(x_1) = u_1(x_1 + h) \tag{2.20a}$$

$$u_2 = u_3 = 0 \tag{2.20b}$$

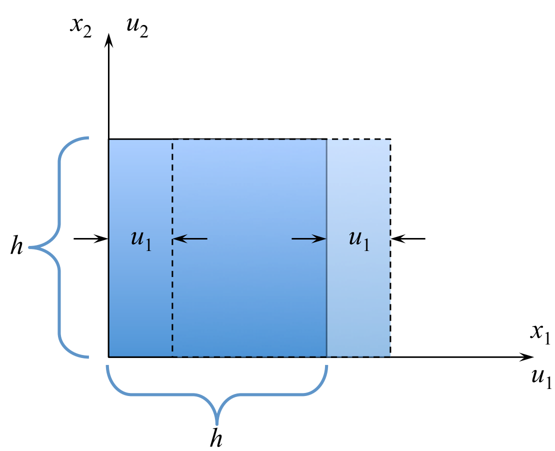

Figure 2.3: Rigid body translation of the infinitesimal square element.

It follows from 2.18a that the corresponding strain component vanishes, $\epsilon_{11} = 0$. The first component of the spin tensor is zero from the definition, $\omega_{11} = 0$.

Extension along x_1 axis

At $x_1 : u_1 = 0$.
At $x_1 + h : u_1 = u_o$.

The corresponding strain is $\epsilon_{11} = \dfrac{u_o}{h}$.

Pure shear on the $x_1 x_2$ plane

At $x_1 = 0$ and $x_2 = 0$: $u_1 = u_2 = 0$
At $x_1 = h$ and $x_2 = 0$: $u_1 = 0$ and $u_2 = u_o$
At $x_1 = 0$ and $x_2 = h$: $u_1 = u_o$ and $u_2 = 0$

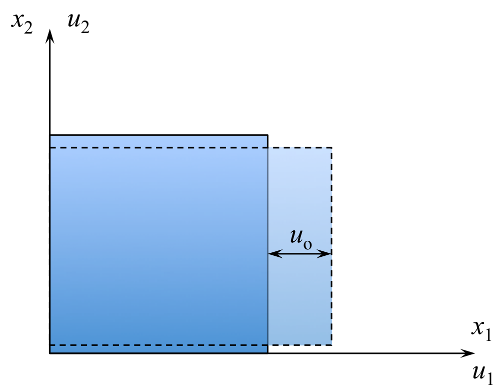

Figure 2.4: The square element is stretched in one direction.

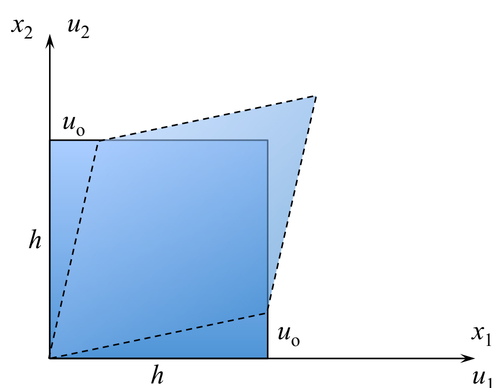

Figure 2.5: Imposing constant deformative gradients.

It follows from Eq. (2.17) and Eq. (2.13) that:

$$\epsilon_{12} = \frac{1}{2}\left(\frac{u_o}{h} + \frac{u_o}{h}\right) = \frac{u_o}{h} \tag{2.21}$$

$$\omega_{12} = \frac{1}{2}\left(\frac{u_o}{h} - \frac{u_o}{h}\right) = 0 \tag{2.22}$$

The resulting strain is representing change of angles of the initial rectilinear element.

Rigid body rotation

At $x_1 = 0$ and $x_2 = 0$: $u_1 = u_2 = 0$
At $x_1 = h$ and $x_2 = 0$: $u_1 = 0$ and $u_2 = u_o$
At $x_1 = 0$ and $x_2 = h$: $u_1 = -u_o$ and $u_2 = 0$

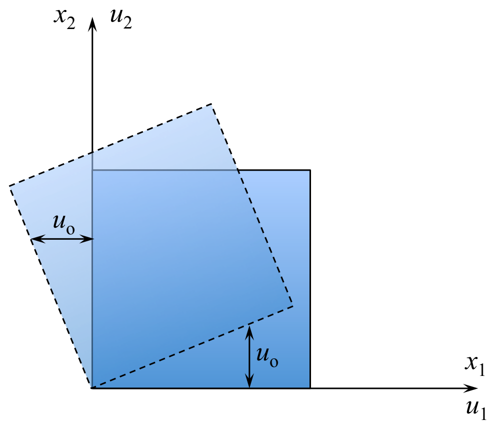

Figure 2.6: An infinitesimal square element subjected to rigid body rotation.

Changing the sign of u_1 at $x_1 = 0$ and $x_2 = h$ from u_o to $-u_o$ results in non-zero spin but zero strain

$$\epsilon_{12} = \frac{1}{2}\left(\frac{u_o}{h} + \left(-\frac{u_o}{h}\right)\right) = 0 \tag{2.23}$$

$$\omega_{12} = \frac{1}{2}\left(\frac{u_o}{h} + \left(-\frac{u_o}{h}\right)\right) = \frac{u_o}{h} \tag{2.24}$$

The last example provides an explanation why the strain tensor was defined as a symmetric part of the displacement gradient. The physics dictates that rigid body translation and rotation should not induce any strains into the material element. In rigid body rotation displacement gradients are not zero. The strain tensor, defined as a symmetric part of the displacement gradient removes the effect of rotation in the state of strain in a body. In other words, strain described the change of length and angles while the spin, element rotation.

2.3 Description of Strain in the Cylindrical Coordinate System

In this section the strain-displacement relations will be derived in the cylindrical coordinate system (r, θ, z). The polar coordinate system is a special case with $z = 0$.

The components of the displacement vector are $\{u_r, u_\theta, u_z\}$. There are two ways of deriving the kinematic equations. Since strain is a tensor, one can apply the transformation rule from one coordinate to the other. This approach is followed for example on pages 125-128 of the book on “A First Course in Continuum Mechanics” by Y.C. Fung. Or, the expression for each component of the strain tensor can be derived from the geometry. The latter approach is adopted here. The diagonal (normal) components ϵ_{rr}, $\epsilon_{\theta\theta}$, and ϵ_{zz} represent the change of length of an infinitesimal element. The non-diagonal (shear) components describe the change of angles.

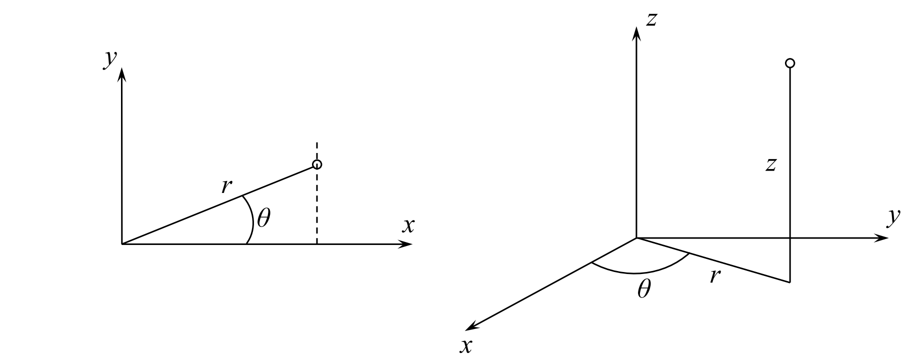

Figure 2.7: Rectangular and cylindrical coordinate system.

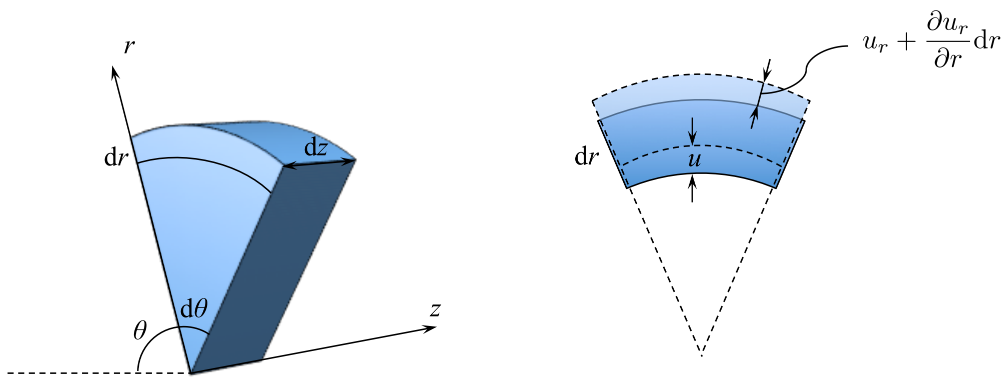

Figure 2.8: Change of length in the radial direction.

The radial strain is solely due to the presence of the displacement gradient in the r-direction

$$\epsilon_{rr} = \frac{\left\{u_r + \dfrac{\partial u_r}{\partial r}\mathrm{d}r - u_r\right\}}{\mathrm{d}r} = \frac{\partial u_r}{\partial r} \tag{2.25}$$

The circumferential strain has two components

$$\epsilon_{\theta\theta} = \epsilon_{\theta\theta}^{(1)} + \epsilon_{\theta\theta}^{(2)} \tag{2.26}$$

The first component is the change of length due to radial displacement, and the second component is the change of length due to circumferential displacement.

From Fig.(2.9) the components $\epsilon_{\theta\theta}^{(1)}$ and $\epsilon_{\theta\theta}^{(2)}$ are calculated as

$$\epsilon_{\theta\theta}^{(1)} = \frac{(r + u_r)\mathrm{d}\theta - r\mathrm{d}\theta}{r\mathrm{d}\theta} = \frac{u_r}{r} \tag{2.27a}$$

$$\epsilon_{\theta\theta}^{(2)} = \frac{u_\theta + \dfrac{\partial u_\theta}{\partial \theta}\mathrm{d}\theta - u_\theta}{r\mathrm{d}\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \tag{2.27b}$$

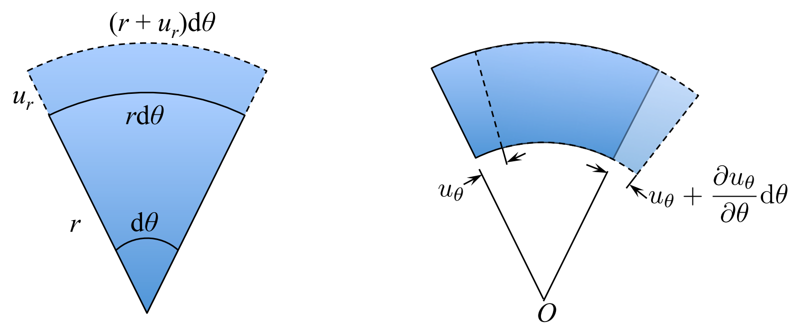

Figure 2.9: Two deformation modes responsible for the circumferential (hoop) strain.

The total circumferential (hoop) component of the strain tensor is

$$\epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial d\theta} \tag{2.28}$$

The strain components in the z-direction is the same as in the rectangular coordinate system

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{2.29}$$

The shear strain $\epsilon_{r\theta}$ describes a change in the right angle.

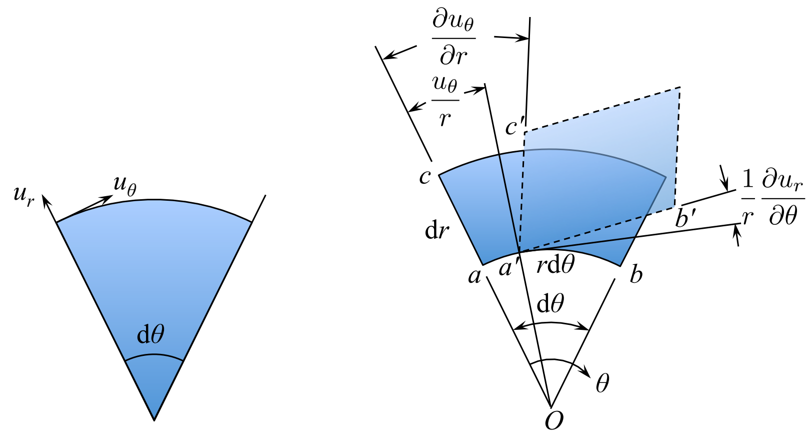

Figure 2.10: Construction that explains change of angles due to radial and circumferential displacement.

From Fig.(2.10) the shear strain over the $\{r,\theta\}$ plane is

$$\epsilon_{r\theta} = \frac{1}{2}\left[\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r} + \frac{1}{r}\frac{\partial u_r}{\partial \theta}\right] \tag{2.30}$$

On the $\{r,z\}$ plane, the ϵ_{rz} shear develops from the respective gradients, see Fig.(2.11).

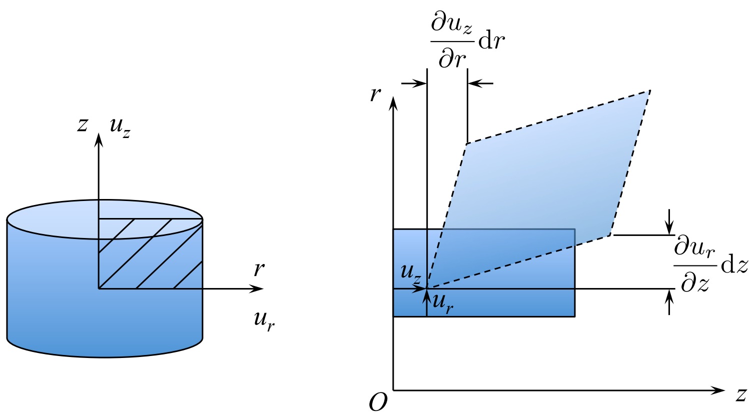

Figure 2.11: Change of angles are $\{r, z\}$ plane.

From the construction in Fig.(2.10), the component ϵ_{rz} is

$$\epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{2.31}$$

Finally, a similar picture is valid on the (tangent) $\{z, \theta\}$ plane

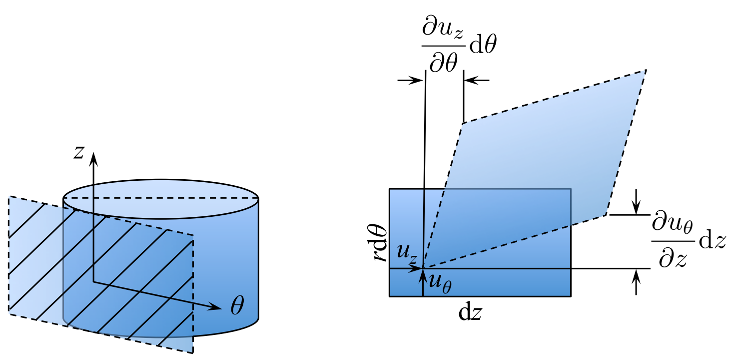

Figure 2.12: Visualization of the strain component $\epsilon_{\theta z}$.

The component $\epsilon_{\theta z}$ of the strain tensor is one half of the change of angles, i.e.

$$\epsilon_{\theta z} = \frac{1}{2}\left(\frac{\partial u_z}{r\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) \tag{2.32}$$

To sum up the derivation, the six components of the infinitesimal strain tensor in the cylindrical coordinate system are

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} \tag{2.33a}$$

$$\epsilon_{\theta\theta} = \frac{u_r}{r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} \tag{2.33b}$$

$$\epsilon_{zz} = \frac{\partial u_x}{\partial z} \tag{2.33c}$$

$$\epsilon_{r\theta} = \epsilon_{\theta r} = \frac{1}{2}\left(\frac{1}{r}\frac{\partial u_r}{\partial \theta} + \frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \tag{2.33d}$$

$$\epsilon_{\theta z} = \epsilon_{z\theta} = \frac{1}{2}\left(\frac{\partial u_z}{r\partial \theta} + \frac{\partial u_\theta}{\partial z}\right) \tag{2.33e}$$

$$\epsilon_{zr} = \epsilon_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{2.33f}$$

Considerable simplifications are obtained in the case of axial (rotational symmetry for which $u_\theta = 0$ and $\frac{\partial}{\partial \theta}[\ ] = 0$

$$\epsilon_{rr} = \frac{\partial u_r}{\partial r} \qquad \epsilon_{r\theta} = 0 \tag{2.34a}$$

$$\epsilon_{\theta\theta} = \frac{u_r}{r} \qquad \epsilon_{\theta z} = 0 \tag{2.34b}$$

$$\epsilon_{zz} = \frac{\partial u_z}{\partial z} \qquad \epsilon_{zr} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{2.34c}$$

The application of the above geometrical relations for axi-symmetric loading of circular plates and cylindrical shells will be given in subsequent chapters.

2.4 Kinematics of the Elementary Beam Theory

The word “kinematics” is derived from the Greek word “kinema”, which means movements, motion. Any motion of a body involves displacements u_i, their increments du_i and velocities $\dot{u}_i$. If the rigid body translations and rotations are excluded, strains develop. We often say “Kinematic assumption” or “Kinematic boundary conditions” or “Kinematic quantities” etc. All it means that statements are made about the displacements and strains and/or their rates. By contrast, the word “static” is reserved for describing stresses and/or forces, even though a body could move. The point is that *for statically determined* structures, one could determine stresses and forces without invoking motion. Such expressions as “static formulation”, “static boundary conditions”, “static quantities” always refer to stresses and forces.

Elementary is another word in the title of this section that requires explanation. A beam is a *slender structure* that can be compressed, extended or bent. The beam must be subjected to a *transverse load* (perpendicular to its axis). Otherwise it becomes something else, as explained in 2.13.

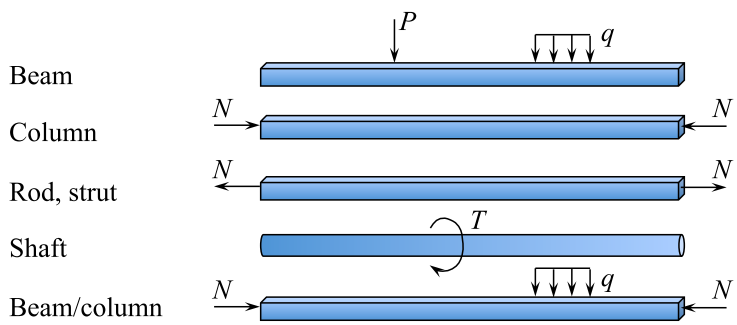

Figure 2.13: The type of loading distinguishes between *five* different types of structures.

All the above structures may have a similar slenderness. How slender the structure must be to become a beam. The slenderness is defined as a length to thickness ratio $\frac{l}{h}$. If $\frac{l}{h} > 20$, the beam obeys the simplified kinematic assumptions and it is called an *"Euler beam"*. Much shorter beams with $\frac{l}{h} < 10$ develop considerable shear stresses in addition to bending stresses and must be treated by a different set of assumptions. Such beams are referred to as *Timoshenko beams*. The intermediate range $10 < \frac{l}{h} < 20$ is a grey area where the simplifying assumptions of the elementary beam theory gradually lose validity.

This section deals with a solid section beams, as opposed to thin-walled sections. In the present lecture notes, the rectangular right handed coordinate system (x, y, z) is consistently used. The x-axis is directed along the length of the beam with an origin at a convenient location, usually the end of the center of the beam. The y-axis is in the width direction with its origin on the symmetry plane of the cross-section, 2.14. Finally, the z-axis is pointing out down and it is measured from the *centroidal axis* of the cross-section (see Recitation 2 for the definition of a centroidal axis).

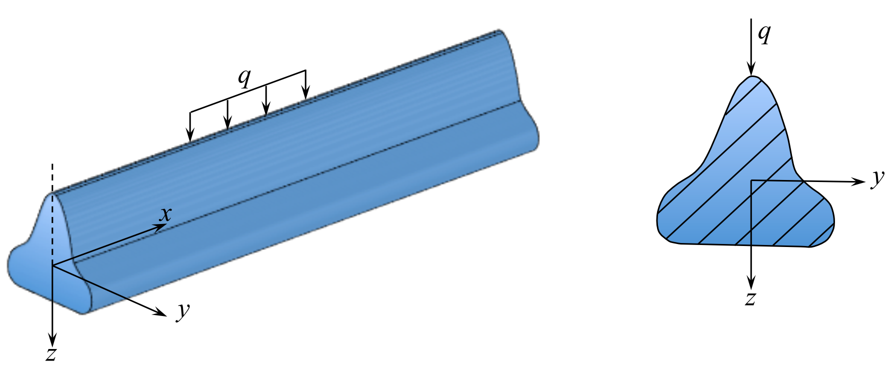

Figure 2.14: A prismatic slender beam with a symmetric cross-section.

In structural mechanics the components of the displacement vector in x, y, and z directions are denoted respectively by (u, v, w). The development of elementary beam theory is based on three kinematic assumptions. Additional assumptions on the stress state will be introduced later.

2.5 Euler-Bernoulli Hypothesis

In this section reference is often made to the *beam axis*. The meaning of the beam axis is intuitive for a prismatic beam with a rectangular cross-section. It is the *middle axis*. Other terms, such as: *neutral axis, bending axis and centroidal axis* are also frequently used. They all express the same property that no axial stresses σ_{xx} should develop on the axis under pure bending.

Hypothesis 1: Plan Remains Plane

This is illustrated in 2.15 showing an arbitrary cross-section of the beam before and after deformation.

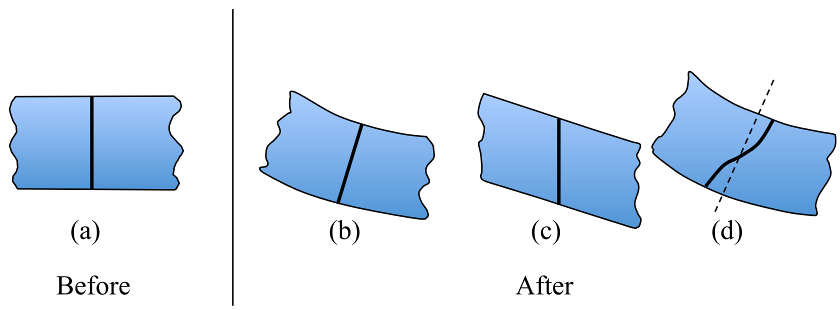

Figure 2.15: Flat (b) and (c) and warped (d) cross-sections after deformations.

Imagine a straight cut made through the undeformed beam. The plane-remains-plane hypothesis means that all material points on the original cut align also on a plane in the deformed beam. The cases (b) and (c) obey the hypothesis but the warped section (d) violates it.

Hypothesis 2: Normal Remains Normal

If the initial cut were made at right angle of the undeformed beam axis as in Fig.(2.16(a)), it should remain normal to the deformed axis, see Fig.(2.16(b)).

In the sketch on Fig. (2.16(c)) the hypothesis is violated when the angle $\alpha \neq 90°$.

The Euler-Bernoulli hypothesis gives rise to an elegant theory of infinitesimal strains in beams with arbitrary cross-sections and loading in two out-of-plane directions. The interested reader is referred to several monographs with a detailed treatment of the subject, of bi-axial loading of beams. The present set of notes on beams is developed under the

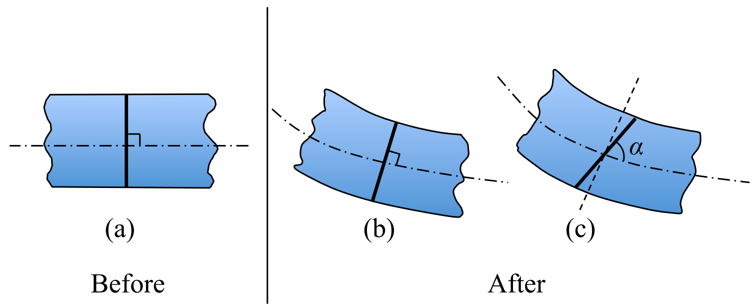

Figure 2.16: Testing the normal-remains-normal hypothesis.

assumption of *planar deformation*. This means that the beam axis motion is restricted only to one plane.

Mathematically, the Hypothesis 1 is satisfied when the u-component of the displacement vector is a linear function of z.

$$u(z) = u^{\circ} - \theta z \ \text{ at any } x \tag{2.35}$$

The constant first term, u° is the displacement of the beam axis (due to axial force). The second term is due to bending alone, Fig. (2.17).

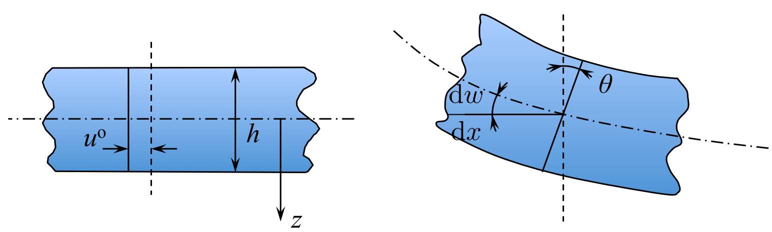

Figure 2.17: Linear displacement field through the thickness of the beams.

The second Euler-Bernoulli hypothesis is satisfied if the rotation of the deformed cross-section θ is equal to the local slope of the bent middle axis $\frac{\mathrm{d}w}{\mathrm{d}x}$

$$\theta = \frac{\mathrm{d}w}{\mathrm{d}x} \tag{2.36}$$

Eliminating the rotation angle θ between equations 2.35 and 2.36 yields

$$u(x, z) = u^{\circ} - \frac{\mathrm{d}w}{\mathrm{d}x} z \tag{2.37}$$

It can be seen from Fig.(2.17) that the displacement at the bottom (tensile) side of the beam is negative, which explains the minus sign in the second term of Eqs. (2.36) and (2.37).

Hypothesis 3

The cross-sectional shape and size of the beam remain unchanged. This means that the vertical component of the displacement vector does not depend on the z-coordinate. All points of the cross-section move by the same amount.

$$w = w(x) \tag{2.38}$$

In the case of planar deformation, which covers most of the practical cases of the beam response, the y-component of the displacement vector vanishes

$$v \equiv 0 \tag{2.39}$$

We are now in the position to calculate all components of the strain tensor from Eq. (2.17)

$$\epsilon_{xx} = \frac{\mathrm{d}u_x}{\mathrm{d}x} = \frac{\mathrm{d}u}{\mathrm{d}x} \tag{2.40a}$$

$$\epsilon_{yy} = \frac{\mathrm{d}u_y}{\mathrm{d}y} = \frac{\mathrm{d}v}{\mathrm{d}y} = 0 \text{ on account of } 2.42 \tag{2.40b}$$

$$\epsilon_{zz} = \frac{\mathrm{d}u_{zz}}{\mathrm{d}z} = \frac{\mathrm{d}w(x)}{\mathrm{d}z} = 0 \text{ from } 2.38 \tag{2.40c}$$

$$\epsilon_{xy} = \frac{1}{2}\left(\frac{\mathrm{d}u_x}{\mathrm{d}y} + \frac{\mathrm{d}u_y}{\mathrm{d}x}\right) = 0 \text{ from } 2.37 \text{ and } 2.42 \tag{2.40d}$$

$$\epsilon_{yz} = \frac{1}{2}\left(\frac{\mathrm{d}u_y}{\mathrm{d}z} + \frac{\mathrm{d}u_z}{\mathrm{d}y}\right) = \frac{1}{2}\left(\frac{\mathrm{d}v}{\mathrm{d}z} + \frac{\mathrm{d}w}{\mathrm{d}y}\right) = 0 \tag{2.40e}$$

$$\begin{aligned}\epsilon_{zx} &= \frac{1}{2}\left(\frac{\mathrm{d}u_z}{\mathrm{d}x} + \frac{\mathrm{d}u_x}{\mathrm{d}z}\right) = \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x} + \frac{\mathrm{d}u}{\mathrm{d}z}\right) \\ &= \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x} - \frac{\mathrm{d}w}{\mathrm{d}x}\right) = 0\end{aligned} \tag{2.40f}$$

It is seen that all components of the strain tensor vanish except the one in the direction of beam axis.

Note that ϵ_{xx} is the only component of the strain tensor in the elementary beam theory. Therefore the subscript "xx" can be dropped and, unless specified otherwise $\epsilon_{xx} = \epsilon$. Introducing Eq.(2.37) into Eq.(2.38) one gets

$$\epsilon(x, z) = \frac{\mathrm{d}u^\circ(x)}{\mathrm{d}x} - \frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2} z \tag{2.41}$$

The first term represents the strain arising from a uniform extension of the entire cross-section

$$\epsilon^\circ(x) = \frac{\mathrm{d}u^\circ(x)}{\mathrm{d}x} \tag{2.42}$$

The second term adds a contribution of bending. Introducing the definition of the curvature of the beam axis

$$\kappa \stackrel{\text{def}}{=} -\frac{\mathrm{d}^2 w(x)}{\mathrm{d}x^2}, \tag{2.43}$$

the expression for strain can be put in the final form:

$$\epsilon(x,z) = \epsilon^{\circ}(x) + z\kappa \tag{2.44}$$

Mathematically, the curvature is defined as a gradient of the slope of a curve. The minus sign in Eq.(2.27b) follows from the rigorous description of the curvature of a line in the assumed coordinate system. Physically, it assumes that strains on the tensile side of the beam are positive.

A quite different interpretation of the Euler-Bernoulli hypothesis is offered by considering a two-term expansion of the exact strain profile in the Taylor series around the point $z = 0$

$$\epsilon(x,z) = \epsilon(x,z)\mid_{z=0} + \left.\frac{\mathrm{d}\epsilon}{\mathrm{d}z}\right|_{z=0} z + \frac{1}{2}\left.\frac{\mathrm{d}^2\epsilon}{\mathrm{d}z^2}\right|_{z=0} z^2 + \cdots \tag{2.45}$$

Taking only the first two terms is a good engineering approximation but leads to some internal inconsistencies of the elementary beam theory. These inconsistencies will be explained in the two subsequent lectures.

2.6 Strain-Displacement Relation of Thin Plates

The present course 2.080 is a prerequisite for a more advanced course 2.081 on Plates and Shells. A complete set of lecture notes for 2.081 is available on OpenCourseWare. The interested reader will find there a complete presentation of the theory of moderately large deflection of plates, derived from first principles. Here only a short summary is given.

Notation

In the lectures on plates and shells two notations will be used. The formulation and some of the derivation will be easier (and more elegant) by invoking the tensorial notation. Here students should flip briefly to Recitation 1 where the above mathematical manipulations are explained. For the purpose of the solving plate problems, the expanded notation will be used.

Points on the middle surface of the plate are described by the vector $\{x_1, x_2\}$ or x_α, $\alpha = 1,2$ in tensor notation or $\{x, y\}$ in expanded notation.

Likewise, the in-plane components of the displacement vector are denoted by $\{u, v\}$.

The vertical component of the displacement vector in the z-direction is denoted by w.

Plate versus Beam Theory

The plate theory requires fewer assumptions and is more self-consistent than the beam theory. For one, there are no complications arising from the concept of the centroidal axis for arbitrarily shaped prismatic beams. The z-coordinate is measured from the *middle plane* which is self explanatory. Finally, the flexural/torsional response of non-symmetric and/or thin-walled cross-section beams is not present in plates. The complexity of the plate formulation comes from the two-dimensionality of the problem. The ordinary differential equations in beams are now becoming partial differential equations.

ADVANCED TOPIC

2.7 Derivation of the Strain-Displacement Relation for Thin Plates

The *Love-Kirchoff* hypothesis extends the one-dimensional Euler-Bernoulli assumptions into plates. A plate can be bent in two directions, forming a double curvature surface. Therefore the plane-remains-plane and normal-remains-normal properties are now required in both directions. Thus, Eq. (2.35) and Eq. (2.36) take the form

$$u_\alpha = u^\circ_\alpha - \theta_\alpha z \tag{2.46a}$$

$$\theta_\alpha = \frac{\partial w}{\partial x_\alpha} \stackrel{\text{def}}{=} w_{,\alpha} \tag{2.46b}$$

where θ_α is the slope (rotation) in x_α-direction. Upon elimination of θ_α between the above equation, one gets the familiar linear dependence of the in-plane components of the displacement vector on the z-coordinate

$$u_\alpha(x_\alpha, z) = u^\circ_\alpha(x_\alpha) - z w_{,\alpha} \tag{2.47}$$

The constant thickness ($w = \dot{w}(x_\alpha)$) is the third kinematic assumption of the plate theory.

Now, watch carefully how the strain components in the plate are calculated. Considering all components of the strain tensor, one can distinguish three in-plane strain components $\epsilon_{\alpha\beta}$ (framed area on the matrix below) and three out-of-plane components.

$$\begin{pmatrix} \boxed{\begin{matrix} \quad & \quad \\ \quad & \epsilon_{\alpha\beta} \end{matrix}} & \begin{matrix} \epsilon_{13} \\ \epsilon_{23} \end{matrix} \\ \square \quad \square & \epsilon_{33} \end{pmatrix}$$

The through thickness strain component vanishes on the assumption of independence of the vertical displacement on the coordinate z

$$\epsilon_{33} = \epsilon_{zz} = \frac{\delta w}{\delta z} = 0 \tag{2.48}$$

The two out-of-plane shear components of the strain tensor $\epsilon_{\alpha 3}$ vanish due to the Love-Kirchoff hypothesis, Eq.(2.47),

$$\begin{aligned} \epsilon_{\alpha 3} &= \frac{1}{2}\left(\frac{\delta u_\alpha}{\delta z} + \frac{\delta w}{\delta x_\alpha}\right) = \frac{1}{2}(u_{\alpha,z} + w_{,\alpha}) \\ &= \frac{1}{2}\left[\frac{\mathrm{d}}{\mathrm{d}z}(u^\circ_\alpha(x_\alpha) - z w_{,\alpha}) + w_{,\alpha}\right] = 0 \end{aligned} \tag{2.49}$$

The non-vanishing components of the strain tensor are the in-plane strain components

$$\epsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) \quad \alpha, \beta = 1, 2 \tag{2.50}$$

where u_α is defined by 2.32. Performing the differentiation one gets

$$\begin{aligned}\epsilon_{\alpha\beta} &= \frac{1}{2}[u^\circ_\alpha - zw_{,\alpha}]_{,\beta} + \frac{1}{2}[u^\circ_\beta - zw_{,\beta}]_{,\alpha} \\ &= \frac{1}{2}(u^\circ_{\alpha,\beta} + u^\circ_{\beta,\alpha}) - \frac{1}{2}z[w_{,\alpha\beta} + w_{,\beta\alpha}]\end{aligned} \tag{2.51}$$

The first term in Eq. (2.51) is the strain $\epsilon^\circ_{\alpha\beta}$ arising from the membrane action in the plate. It is a symmetric gradient of the middle plane displacement u°_α. Since the order of partial differentiation is not important, Eq. (2.51) simplifies to

$$\epsilon_{\alpha\beta}(x_\alpha, z) = \epsilon^\circ_{\alpha\beta}(x_\alpha) - zw_{,\alpha\beta} \tag{2.52}$$

Defining the curvature tensor $\kappa_{\alpha\beta}$ by

$$\kappa_{\alpha\beta} = -w_{,\alpha\beta} = -\frac{\partial^2 w}{\partial x_\alpha \partial x_\beta} \tag{2.53}$$

The strain-displacement relation for thin plates takes the final form

$$\epsilon_{\alpha\beta} = \epsilon^\circ_{\alpha\beta} + z\kappa_{\alpha\beta}, \tag{2.54}$$

where

$$\epsilon^\circ_{\alpha\beta} = \frac{1}{2}(u^\circ_{\alpha,\beta} + u^\circ_{\beta,\alpha}) \tag{2.55}$$

END OF ADVANCED TOPIC

2.8 Expanded Form of Strain-Displacement Relation

Having derived the geometric relations in the tensorial notations, equations (2.54) and (2.55) will be re-written in the coordinate system (x, y) and physical interpretation will be given to each term. Consider first (2.55)

$$\alpha = 1, \beta = 1 \;\; x_1 = x, \;\; \epsilon^\circ_{xx} = \frac{1}{2}\left(\frac{\partial u^\circ_x}{\partial x} + \frac{\partial u^\circ_x}{\partial x}\right) = \frac{\partial u^\circ_x}{\partial x} \tag{2.56a}$$

$$\alpha = 2, \beta = 2 \;\; x_2 = y, \;\; \epsilon^\circ_{xx} = \frac{1}{2}\left(\frac{\partial u^\circ_y}{\partial y} + \frac{\partial u^\circ_y}{\partial y}\right) = \frac{\partial u^\circ_y}{\partial y} \tag{2.56b}$$

$$\alpha = 1, \beta = 2 \;\; x_1 = x, x_2 = y, \;\; \epsilon^\circ_{xy} = \frac{1}{2}\left(\frac{\partial u^\circ_x}{\partial y} + \frac{\partial u^\circ_y}{\partial x}\right) \tag{2.56c}$$

The ϵ°_{xx} and ϵ°_{yy} components denote strains of the middle surface of the plate in the x and y directions, respectfully. The membrane strains are due to the imposed displacements or membrane forces applied to the edges. In the theory of small deflection of plates, lateral pressure loading will not produce membrane strains. By contrast, membrane strains do develop in the theory of moderately large deflection of plates due to transverse loading. This topic will be covered later in Lecture 6.

The third component of the strain tensor is the in-plane shear strain ϵ°_{xy}. It represents the change of angles in the plane of the plate due to the shear loading at the edges. The geometrical interpretation of the membrane strain tensor is similar to that given for the general strain tensor in Figures (2.4) and (2.5).

The curvature tensor $\kappa_{\alpha\beta}$ requires a careful explanation. Consider an infinitesimal segment $\mathrm{d}s$ of a curve and fit into it a circle of an instantaneous radius ρ, Fig. (2.18). Then

$$\mathrm{d}s = \rho \mathrm{d}\theta \tag{2.57}$$

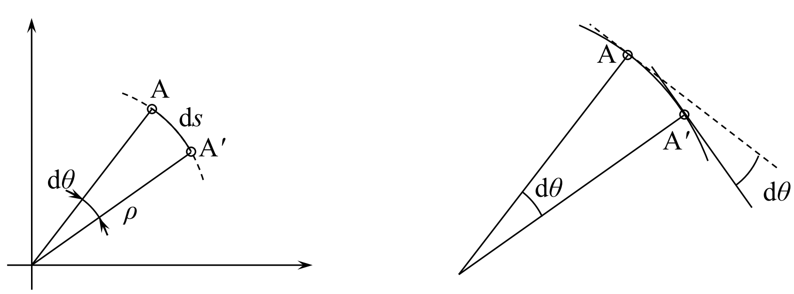

Figure 2.18: Change of slope of a line between two points

Mathematically, the curvature of any line κ is the change of the slope as one moves along the curve

$$\kappa \overset{\text{def}}{=} \frac{\mathrm{d}\theta}{\mathrm{d}s} \tag{2.58}$$

By comparing Eq. (2.58) with Eq. (2.46b), the curvature in $[\frac{1}{\mathrm{m}}]$ is the reciprocity of the radius of curvature $\kappa = \frac{1}{\rho}$. The first component of the curvature tensor, defined by Eq. (2.54) is

$$\alpha = 1, \beta = 1 \;\; x_1 = x \quad \kappa_{xx} = -\frac{\partial^2 w}{\partial x^2} = -\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}\right) = \frac{\partial}{\partial x}(-\theta_x) \tag{2.59}$$

This will be the only component of the curvature tensor if the plate is subject to the so-called *cylindrical bending*.

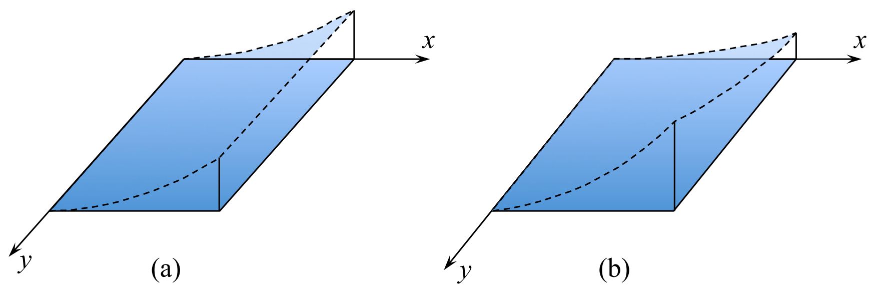

Figure 2.19: (a) cylindrical bending of a plate, and (b) bending with a twist.

The interpretation of the κ_{yy} components of the curvature tensor

$$\alpha = 2, \beta = 2 \;\; x_2 = y \quad \kappa_{yy} = -\frac{\partial^2 w}{\partial y^2} = -\frac{\partial}{\partial y}(-\theta_y) \tag{2.60}$$

is similar as before. More interesting is the mixed component of the curvature tensor κ_{xy}

$$\alpha = 1, \beta = 2 \;\; x_1 = x, x_2 = y \quad \kappa_{xy} = -\frac{\partial^2 w}{\partial x \partial y} = -\frac{\partial}{\partial y}(-\theta_x) \tag{2.61}$$

To detect κ_{xy} one has to check if the slope in one direction, say θ_x changes along the second y-direction. It does not for a cylindrical bending, 2.14(a). But if it does, the plate is twisted, as shown in 2.14(b). Therefore, the component κ_{xy} is called a twist.

An important parameter that distinguishes between these classes of the deformed shape of a plate is the Gaussian curvature, κ_G. The Gaussian curvature is defined as a product of two principal curvatures

$$\kappa_G = \kappa_I \kappa_{II} \tag{2.62}$$

The curvature is a tensor, so its components change by rotating the coordinate system by an angle ψ to a new direction (x', y'). There is one such an angle ψ_{p} for which the twisting components vanish. The remaining diagonal components are called *principal curvature*. The full coverage of the transformation formulae for vectors and tensors are presented in Recitation 2. Using these results, the Gaussian curvature can be expressed in terms of the components of the curvature tensor

$$\kappa_G = \kappa_{xx}\kappa_{yy} - \kappa_{xy}^2 \tag{2.63}$$

For cylindrical bending the twist κ_{xy} as well as one of the principal curvatures vanishes so that the Gaussian curvature is zero. The sign of the Gaussian curvature distinguishes between three types of the deformed plate, the bowl, the cylinder and the saddle, Fig.(2.20).

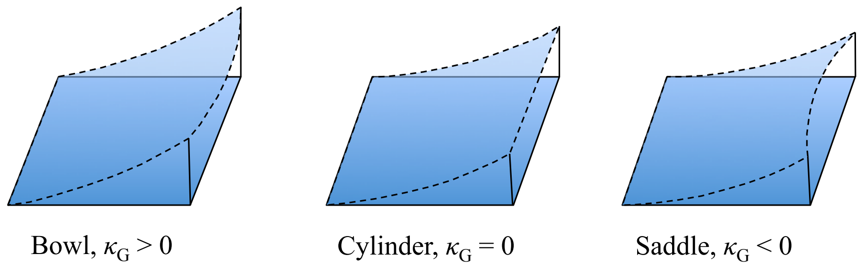

Figure 2.20: Deformed plate with three different classes of shapes.

The consideration of Gaussian curvature introduces important simplifications in formulation and applications of the energy method in structural mechanics. A separate lecture will be devoted to this topic.

2.9 Moderately Large Deflections of Beams and Plates

A complete presentation of the theory of moderately large deflections of plates, derived from first principles is presented in the course 2.081 Plates and Shells. The lecture notes for this course are available on OpenCourseWare. There the strain-displacement relation for the theory of moderately large deflection of beams are derived. Here the corresponding equations for plates are only stated with a physical interpretation. An interested reader is referred to the Plates and Shells notes for more details.

Defining moderately large deflections of beams

What are the "*moderately large deflections*" and how do they differ from the "small deflection". To see the difference, it is necessary to consider the initial and deformed configuration of the beam axis. The initial and current length element in the undeformed and deformed configuration respectively is denoted dx and ds, as in Fig. (2.21)

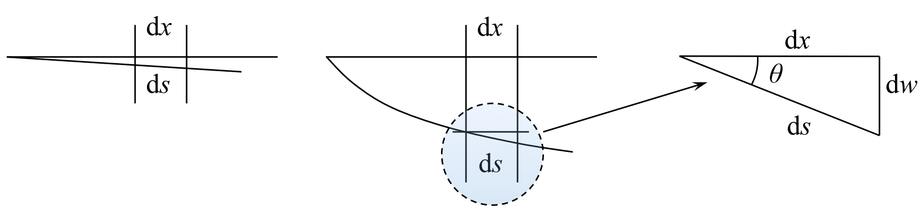

Figure 2.21: Change of length of the beam axis produced by rotation.

From the geometry of the problem

$$\mathrm{d}x = \mathrm{d}s\cos\theta \approx \mathrm{d}s\left[1 - \frac{\theta^2}{2}\right] \tag{2.64}$$

One can distinguish between three theories:

(i) Small deflections, linear geometry $\theta^2 \ll 1$, $\mathrm{d}x \approx \mathrm{d}s$, Fig. (2.21(a)).

(ii) Moderately large deflections. The two-term expansion of the cosine function gives a good approximation for $0 < \theta < 10°$. Relation between $\mathrm{d}x$ and $\mathrm{d}s$ is given by Eq. (2.64), Fig. (2.21(b)).

(iii) For larger rotation, a full nonlinearity of the problem must be considered.

The present derivation refers to case (ii) above. The Cauchy strain measure, defined in Eq. (2.2b) is adopted:

$$\epsilon = \frac{\mathrm{d}s^2 - \mathrm{d}x^2}{2\mathrm{d}x^2} \tag{2.65}$$

The current length $\mathrm{d}s$ can be expressed in terms of $\mathrm{d}x$ and $\mathrm{d}w$, see Fig.(2.21)

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}w^2 \tag{2.66}$$

From the above two equations, the strain of the beam axis due to element rotation, ϵ_{rot} is

$$\epsilon = \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 = \frac{1}{2}\theta^2 \tag{2.67}$$

The beam axis also extends due to the gradient of the axial component of the displacement vector, defined by Eq.(2.42). Therefore the total strain of the beam axis due to the combined extension and rotation is

$$\epsilon^\circ = \frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 \tag{2.68}$$

It can be noticed that the second term in the above equation is always positive while the first term can be either positive or negative. In a special case the two terms can cancel one another even though a beam undergoes large deformation.

The question often asked by students is if the expression for the curvature, given by Eq. (2.43) should also be modified due to larger rotation. From the mathematical point of view the answer is YES. But engineers have a way to get around it.

In the rectangular coordinate system the exact definition of the curvature of the line is:

$$\kappa = \frac{-\dfrac{\mathrm{d}^2w}{\mathrm{d}x^2}}{\left(1 + \left(\dfrac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right)^{3/2}} \tag{2.69}$$

In the limit $\dfrac{\mathrm{d}w}{\mathrm{d}x} \to 0$ the linear definition is recovered from the nonlinear equation Eq. (2.69). The difference between Eq.(2.43) and Eq.(2.69) is small in the case of moderately large deflection.

The total strain at an arbitrary point of a beam undergoing moderately large deflection is

$$\epsilon = \underbrace{\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2}_{\text{membrane strain } \epsilon^\circ} + \underbrace{z\kappa}_{\text{bending strain } z\kappa} \tag{2.70}$$

Extension to Moderately Large Deflection of Plates

In the compact tensorial notation, the nonlinear strain-displacement relation takes the form

$$\epsilon_{\alpha\beta} = \frac{1}{2}(u_{\alpha,\beta} + u_{\beta,\alpha}) + \frac{1}{2}w_{,\alpha}w_{,\beta} + z\kappa_{\alpha\beta} \tag{2.71}$$

By comparing with a similar expression for the small deflection theory, Eq.(2.54) and Eq.(2.55), the new nonlinear term is

$$\frac{1}{2}w_{,\alpha}w_{,\beta} = \frac{1}{2}\frac{\partial w}{\partial x_\alpha}\frac{\partial w}{\partial x_\beta} \tag{2.72}$$

This term forms a 2×2 matrix:

$$\begin{vmatrix} \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 & \frac{1}{2}\frac{\partial w}{\partial w}\frac{\partial x}{\partial y} \\ \frac{1}{2}\frac{\partial w}{\partial y}\frac{\partial w}{\partial x} & \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \end{vmatrix} = \begin{vmatrix} \frac{\theta_x^2}{2}, \frac{1}{2}\theta_x\theta_y \\ \frac{1}{2}\theta_x\theta_y, \ \frac{\theta_y^2}{2} \end{vmatrix} \tag{2.73}$$

The diagonal terms represent square of the slope of the deflection shape in x and y directions. The non-diagonal terms are symmetric and are a product of slopes in the two directions. This term vanishes for cylindrical bending.

2.10 Strain-Displacement Relations for Circulate Plates

The theory of circular plates is formulated in the cylindrical coordinate system (r, θ, z). The corresponding components of the displacement vector are (u, v, w). In the remainder of the notes, the axi-symmetric deformation is assumed, which would require the loading to be axi-symmetric as well. This assumption brings four important implications

(i) The circumferential component of the displacement is zero, $v \equiv 0$

(ii) There are no in-plane shear strains, $\epsilon_{r\theta} = 0$

(iii) The radial and circumferential strains are principal strains

(iv) The partial differential equations for plates reduces to the ordinary differential equation where the radius is the only space variable.

Many simple closed-form solutions can be obtained for circular and annular plates under different boundary and loading conditions. Therefore such plates are often treated as prototype structures on which certain physical principles could be easily explained.

The membrane strains on the middle surface are stated without derivation

$$\epsilon_{rr}^{\circ} = \frac{\mathrm{d}u}{\mathrm{d}r} + \frac{1}{2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^2 \tag{2.74a}$$

$$\epsilon_{\theta\theta}^{\circ} = \frac{u}{r} \tag{2.74b}$$

The two principal curvatures are

$$\kappa_{rr} = -\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} \tag{2.75a}$$

$$\kappa_{\theta\theta} = -\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r} \tag{2.75b}$$

The sum of the bending and membrane strains is thus given by

$$\epsilon_{rr}(r,z) = \epsilon_{rr}^{\circ}(r) + z\kappa_{rr} \tag{2.76a}$$

$$\epsilon_{\theta\theta}(r,z) = \epsilon_{\theta\theta}^{\circ}(r) + z\kappa_{\theta\theta} \tag{2.76b}$$

It can be noticed that the expression for the radial strains and curvature are identical to those of the beam when r is replaced by x. The expressions in the circumferential direction are quite different.

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