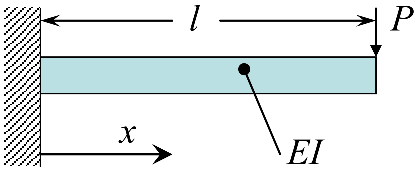
Lecture 8

Energy Methods in Elasticity

Problem 8-1: Consider an elastic cantilever beam loaded at its tip.



a) Specify the boundary conditions.

b) Derive the load-tip displacement relation using four methods presented in class.

Method I Solving uncoupled problems

Method II Solving coupled problem (direct integration)

Method III Castigliano Theorem

Method IV Ritz Method

Problem 8-1 Solution:

(a) Boundary conditions

$$w(0) = 0$$
$$w'(0) = 0$$

(b) Derive the load-tip displacement relation using four methods

1. Solving second order uncoupled problems

We will use

moment-curvature relationship $\quad M = EI\kappa$

curvature-displacement relationship $\quad \kappa = -w''$

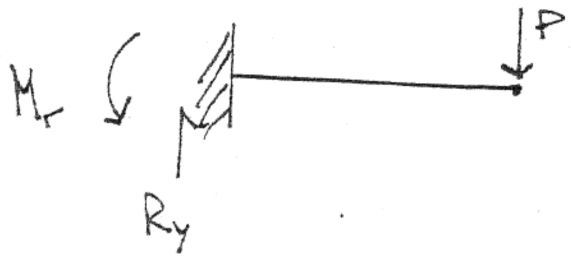
to solve the problem.

These relationships combine to give:

$$-EIw'' = M$$

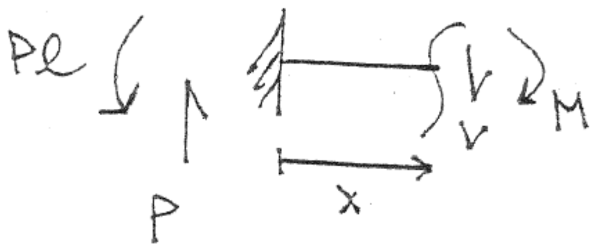
We need the moment.

Find reactions



$$\sum F_y = R_y - P = 0$$
$$R_y = P$$
$$\sum M_r = -(M_r - Pl) = 0$$
$$M_r = Pl$$

Find moment in beam base on force balance and moment balance



$$\sum F_y = P - V = 0$$
$$V = P$$
$$\sum M_0 = -(Pl - Vx - M) = 0$$
$$M = -Pl + Vx$$

Combing the above results from force balance and moment balance

$$M = -Pl + Px$$

Integrate moment once

$$-EIw'' = M$$

$$\int w' dx = -\int -\frac{1}{EI}\left(Pl - Px\right)$$

$$w' = \frac{1}{EI}\left(Plx - \frac{Px^2}{2}\right) + C_1$$

Use B.C.

$$w'(0) = 0$$
$$\Rightarrow C_1 = 0$$

Integrate once

$$w = \int \frac{1}{EI}\left(Plx - \frac{Px^2}{2}\right)$$

$$w = \frac{1}{EI}\left(\frac{Plx^2}{2} - \frac{Px^3}{6}\right) + C_2$$

Use B.C.

$$w(0) = 0$$
$$\Rightarrow C_2 = 0$$

$$\boxed{\begin{aligned} w(x) &= \frac{1}{EI}\left(\frac{Plx^2}{2} - \frac{Px^3}{6}\right) \\ w(l) &= \frac{Pl^3}{3EI} \end{aligned}}$$

2. Solve coupled problems using 4th order direct integration

We use the 4th order differentiated equation:

$$EIw^{\mathrm{IV}} = q = 0$$

And we need four boundary conditions:

$$w(0) = 0$$
$$w'(0) = 0$$
$$M(0) = -Pl = -EIw''$$
$$M(l) = 0 = -EIw''$$

Integrate and use boundary conditions to determine the constants

$$\int EIw^{\mathrm{IV}}dx = 0$$

$$EIw''' = C_1$$

$$EIw'' = C_1 x + C_2$$

Use B.C.

$$M(0) = -Pl$$

$$M(l) = 0$$

We get

$$C_1 = -P$$

$$C_2 = Pl$$

Then

$$EIw'' = -Px + Pl$$

$$EIw' = -\frac{Px^2}{2} + Plx + C_3$$

Use B.C.

$$w'(0) = 0$$

We get

$$C_3 = 0$$

Then

$$EIw = -\frac{Px^3}{6} + \frac{Plx^2}{2} + C_4$$

Use B.C.

$$w(0) = 0$$

We get

$$C_4 = 0$$

Finally

$$\boxed{\begin{gathered} w(x) = \frac{1}{EI}\left(\frac{Plx^2}{2} - \frac{Px^3}{6}\right) \\ w(l) = \frac{Pl^3}{3EI} \end{gathered}}$$

3. Castigliano Theorem

Find moment in beam

$$\boxed{M = -Pl + Px}$$

Calculate strain energy

$$\begin{aligned} U &= \frac{1}{2}\int \frac{M^2}{EI}dx \\ &= \frac{1}{2}\int \frac{\left(Px - Pl\right)^2}{EI}dx \\ &= \frac{P^2 l^3}{6EI} \end{aligned}$$

Use Castigliano Theorem

$$w(l) = \frac{\partial U}{\partial P}$$

$$\boxed{w(l) = \frac{Pl^3}{3EI}}$$

4. Ritz method

Assume shape function

$$w(x) = C_1 x^3 + C_2 x^2 + C_3 x + C_4$$

Use boundary conditions

$$w(0) = 0$$

$$w'(0) = 0$$

We get

$$C_3 = C_4 = 0$$

$$w(x) = C_1 x^3 + C_2 x^2$$

Also

$$w(l) = C_1 l^3 + C_2 l^2 = w_o$$

Find total potential energy

$$\Pi = \frac{1}{2}\int_0^l EI(w'')^2 dx - Pw_o$$
$$= \frac{1}{2}\int_0^l EI(6C_1 x + 2C_2)^2 dx - Pw_o$$
$$= \frac{EI}{2}\left(12C_1^{\,2} l^3 + 12C_1 C_2 l^2 + 4C_2^{\,2} l\right) - Pw_o$$

Minimize Π

$$\frac{\partial \Pi}{\partial C_1} = 0 \text{ and } \frac{\partial \Pi}{\partial C_2} = 0$$

$$\frac{\partial \Pi}{\partial C_1} = \frac{EI}{2}\left(24C_1 l^3 + 12C_2 l^2\right) - Pl^3 = 0$$
$$\frac{\partial \Pi}{\partial C_2} = \frac{EI}{2}\left(12C_1 l^2 + 8C_2 l\right) - Pl^2 = 0$$

Solve for C_1 and C_2 using the above two equations

$$C_1 = -\frac{P}{6EI}$$
$$C_2 = \frac{Pl}{2EI}$$

Finally

$$\boxed{\begin{array}{c} w(x) = \dfrac{1}{EI}\left(\dfrac{Plx^2}{2} - \dfrac{Px^3}{6}\right) \\ w(l) = \dfrac{Pl^3}{3EI} \end{array}}$$

Note: we obtain the exact solution with

$$w(x) \sim O(3)$$

If we were to assume shape function of lower order, we can only obtain an approximate solution.

Problem 8-2: Consider a beam of length L and bending rigidity EI which is fully clamped on both ends shown in Figure 1. The beam is subjected to a point force P at the midspan. Solve the problem (find the expression between the load and the deflection under the load) using:

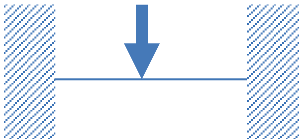


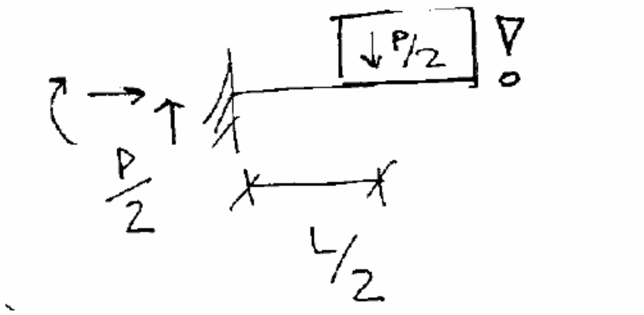
Figure 1

a) The direct integration of beam equation with the suitable boundary conditions (exact solution).

b) The Ritz Method (approximate solution).

c) Compare the results and calculate the relative error of the Ritz Method.

Problem 8-2 Solution:



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Solve for half the beam. Boundary conditions are:

$$w(0) = 0 \tag{1}$$

$$w'(0) = 0 \tag{2}$$

$$w'(\frac{l}{2}) = 0 \tag{3}$$

$$V = EIw'''(0) = -\frac{P}{2} \tag{4}$$

a) Direct integration

We use the 4th order differentiated equation:

$$EIw^{IV} = q$$

where $q = 0$

Integrate once

$$w''' = C_1$$

Use B.C. (4), we have

$$w'''(0) = -\frac{P}{2EI} = C_1$$
$$\Rightarrow w''' = -\frac{P}{2EI}$$

By integration

$$w'' = -\frac{P}{2EI}x + C_2$$
$$w' = -\frac{P}{4EI}x^2 + C_2 x + C_3$$

Use B.C. (2), we have

$$w'(0) = C_3 = 0$$
$$\Rightarrow w' = -\frac{P}{4EI}x^2 + C_2 x$$

Use B.C. (3),

$$w'\left(\frac{l}{2}\right) = 0$$
$$\Rightarrow C_2 = \frac{Pl}{8EI}$$
$$\Rightarrow w' = -\frac{P}{4EI}x^2 + \frac{Pl}{8EI}x$$

Integrate once

$$w = -\frac{P}{12EI}x^3 + \frac{Pl}{16EI}x^2 + C_4$$

Use B.C. (1),

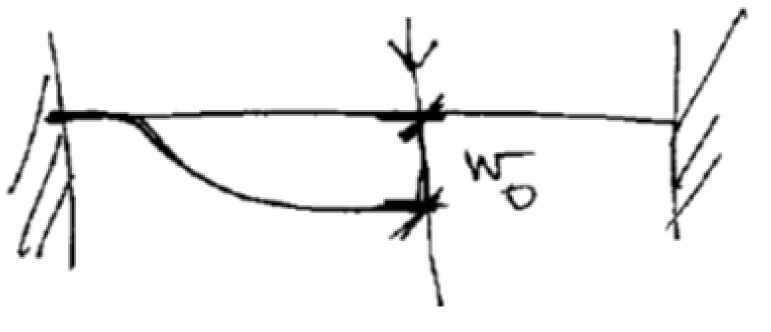
$$w(0) = 0$$
$$\Rightarrow C_4 = 0$$

$$\Rightarrow \boxed{w = -\frac{P}{12EI}x^3 + \frac{Pl}{16EI}x^2, 0 < x < \frac{l}{2}}$$

Mid-span deflection

$$\boxed{w\left(\frac{l}{2}\right) = \frac{Pl^3}{192EI}}$$

b) Ritz method



Assume shape function

$$y(x) = C_1x^3 + C_2x^2 + C_3x + C_4$$

Boundary conditions are:

$$y(0) = 0 \tag{5}$$

$$y'(0) = 0 \tag{6}$$

$$y'\left(\frac{l}{2}\right) = 0 \tag{7}$$

$$y\left(\frac{l}{2}\right) = w_0 \tag{8}$$

Use B.C. (5) and (6),

$$C_3 = C_4 = 0$$
$$\Rightarrow y(x) = C_1x^3 + C_2x^2$$

Apply B.C. (7) and (8),

$$y'(\frac{l}{2}) = 0$$

$$\Rightarrow 3C_1 \frac{l^2}{4} + 2C_2 \frac{l}{2} = 0$$

$$y(\frac{l}{2}) = w_0$$

$$\Rightarrow C_1 \frac{l^3}{8} + C_2 \frac{l^2}{4} = w_0$$

Combine the above two equations, we can solve for C_1 and C_2

$$C_1 = -\frac{16}{l^3} w_0$$

$$C_2 = \frac{12}{l^2} w_0$$

Calculate $\prod = U - W$

$$y''(x) = \frac{24 w_0}{l^2}\left(-\frac{4x}{l} + 1\right)$$

$$U = \frac{1}{2} EI \int_0^l \left(\frac{24 w_0}{l^2}\left(-\frac{4x}{l} + 1\right)\right)^2 dx = 96 EI \frac{{w_0}^2}{l^3}$$

Minimize $\prod$

$$\frac{\partial \prod}{\partial w_0} = 96 EI \frac{2 w_0}{l^3} - P = 0$$

$$\boxed{w_0 = \frac{Pl^3}{192 EI}}$$

Finally

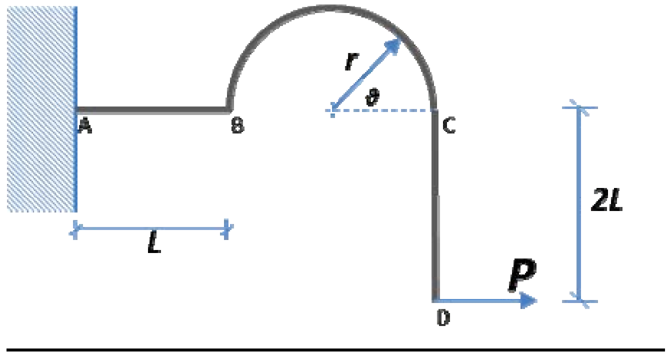
$$\boxed{w = -\frac{P}{12EI} x^3 + \frac{Pl}{16EI} x^2, 0 < x < \frac{l}{2}}$$

c) Compare Ritz method and exact solution

$$w_0\big|_{\text{exact}} = \frac{Pl^3}{192EI}$$
$$w_0\big|_{\text{Ritz}} = \frac{Pl^3}{192EI}$$

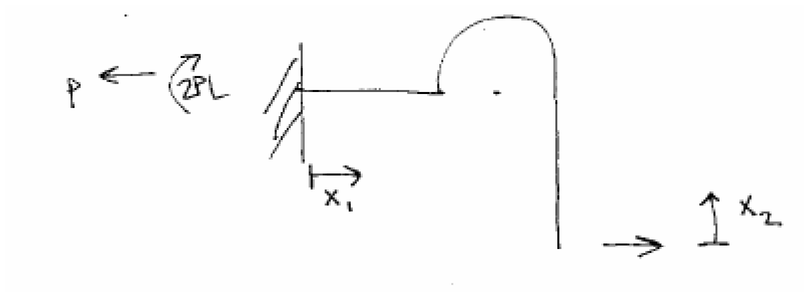
Because we guess the correct order of the shape function, we got the exact solution. %error=0

Problem 8-3: Use Castigliano's Theorem to calculate the horizontal deflection at point D in Figure



Problem 8-3 Solution:

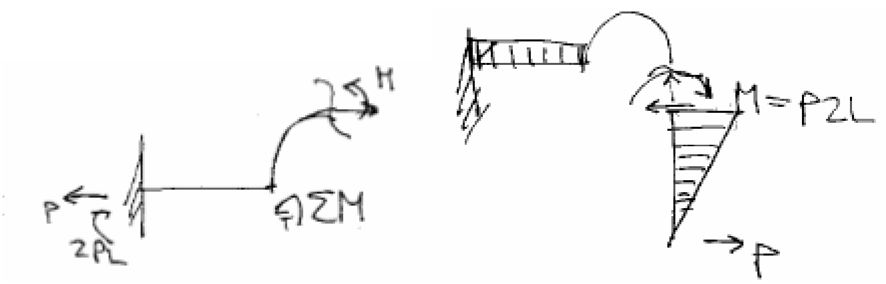
Find reaction



$$R_x = -P$$
$$M_R = 2PL$$

where R_x and M_R are reaction force and reaction moment

Find internal moment distribution at each section



Section AB

$$M_1 = 2PL$$

Section BC

$$M_2 = P\left(2L + r\sin\theta\right)$$

Section CD

$$M_3 = Px_2$$

Calculate strain energy

$$\begin{aligned}
U &= U_1 + U_2 + U_3 \\
&= \frac{1}{2}\int_0^L \frac{(2PL)^2}{EI}dx_1 + \frac{1}{2}\int_0^\pi \frac{\left(P(2L + r\sin\theta)\right)^2}{EI} r d\theta + \frac{1}{2}\int_0^{2L} \frac{(Px_2)^2}{EI}dx_2 \\
&= \frac{1}{2EI}\left\{(2PL)^2 L + rP^2\int_0^\pi \left[4L^2 + 2(2Lr\sin\theta) + r^2\sin^2\theta\right]d\theta + \frac{8P^2L^3}{3}\right\} \\
&= \frac{P^2}{2EI}\left(\frac{20L^3}{3} + 4\pi L^2 r + 8Lr^2 + \frac{\pi r^3}{2}\right)
\end{aligned}$$

Use Castigliano

$$\boxed{w_0 = \frac{\partial U}{\partial P} = \frac{2P}{2EI}\left(\frac{20L^3}{3} + 4\pi L^2 r + 8Lr^2 + \frac{\pi r^3}{2}\right)}$$

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