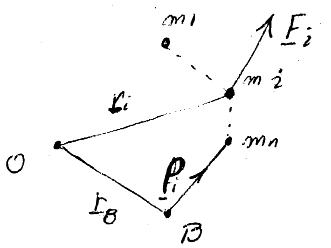
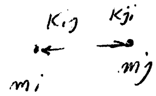
II Dynamics of systems of particles



- A system of n masses
- B: Point in the intertial fram
- F_i: resultant force on m_i

$$\underline{F}_i = \underline{F}_i^{ext} + \underline{F}_i^{int}$$

$$F_i^{int} = \sum_{\substack{j=1 \\ j \neq i}}^{n} \underline{K}_{ij}$$

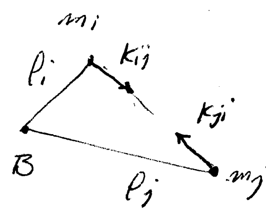


Newton III ⇓

$K_{ij} + K_{ji} = \underline{0}$ $\Rightarrow$ $\sum_{i=1}^{n} F_i^{int} = \underline{0}$

Also. $\sum_{i=1}^{n} \underline{\rho}_i \times \underline{F}_i^{int} = \underline{0}$

Eg. for two masses



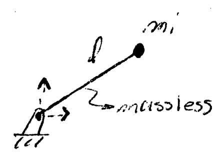
$$\underline{\rho}_i \times \underline{K}_{ij} + \underline{\rho}_j \times K_{ji}$$

$$= (\underline{\rho}_i - \underline{\rho}_j) \times K_{ji} = 0$$

Constraints. geometric limitation on the alsalute or relative motion of particle

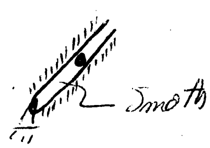
eg. (1)

(3D)



$x_i^2 + y_i^2 + z_i^2 = l^2$

(2)



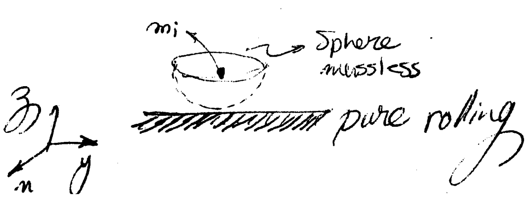
$\begin{Bmatrix} x_2 \\ y_2 \\ z_2 \end{Bmatrix} = \mu(t)\, \rho(t)$ ↳ prescribed

Give Can eliminate $\mu(t)$ and get 2 Constraints

(3)

$z_i = 0 \Rightarrow 1$ Constraint

(4)



z_i = Const

Degrees of freedom

• Degrees of Freedom   Could be time but Not velocity

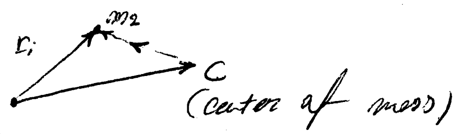
$$\# DOF = \underbrace{3n}_{\substack{\text{DOF for}\\\text{unconstraint}\\\text{motion}}} - (\# \text{ of indep. Scalar restriction on position})$$

or

$\#$ of Constraints

• Total mass $M = \sum_{i=1}^{n} m_i$

• Center of mass: geometric point w.r.t which the total mass moment is zero



$$\sum_{i=1}^{n} m_i (\underline{r}_i - \underline{r}_c) = 0$$

(a) Linear momentum principle   (*) $\dot{\underline{P}}_i = F_i$; $\underline{P}_i = m_i \dot{\underline{r}}_i$

$= F_i^{ext} + F_i^{int}$

Define: $\underline{P} = \sum_{i=1}^{n} \underline{P}_i = M \frac{1}{M} \sum_i m_i \underline{v}_i =$

$$= M \underbrace{\frac{d}{dt} \underbrace{\left(\frac{1}{M} \sum m_i \underline{r}_i\right)}_{r_c}}_{V_c}$$

$= M \underline{V}_c$

Do $\sum$ on (*) $\dot{\underline{P}} = F^{ext}$; $F^{ext} = \sum_{i=1}^{n} \underline{F}_i^{ext}$

linear momentum principle

if $\underline{F}^{ext} = \sum_i F_i^{ext} = 0 \Rightarrow \underline{P} = \text{Const}$ (Conservation of linear momentum)

b) Angular momentum Principle $\dot{\underline{P}}_i = \underline{F}_i^{ext} + \underline{F}_i^{internal}$

$$\Rightarrow \sum_i \underline{\rho}_i \times \dot{\underline{P}}_i = \frac{d}{dt} \underbrace{\sum \underline{\rho}_i \times \underline{P}_i}_{\text{def } \underline{H}_B} - \underbrace{\sum_i \dot{\underline{\rho}}_i \times \underline{P}_i}_{\dot{H}_B'}$$

$= (\dot{\underline{r}}_i - \dot{\underline{r}}_B) \times \underline{P}_i$
$= \cancel{\dot{\underline{r}}_i \times \underline{P}_i} - \dot{\underline{r}}_B \times \underline{P}_i$

ok

$= \dot{H}_B + \underline{V}_B \times \underline{P}$

$$\Rightarrow \dot{\underline{H}}_B + \underline{V}_B \times \underline{P} = \underline{M}_B \qquad \underline{M}_B = \sum_{i=1}^{n} \underline{\rho}_i \times \underline{F}^{ext}$$

If $M_B = 0$
And
- B is fixed
- $\underline{V}_B$ is parallel to $\underline{P}$ $\quad \underline{V}_B \parallel \underline{P}$

or
- B is the Center of mass

$\Rightarrow H_B = Cte$ Conservation of angular momentum

c) Work-Energy Principle

Have seen: $$W_{12}^i = \int_1^2 \underline{F}_i \cdot d\underline{r}_i$$

$$T^i = \frac{1}{2} m |\underline{v}_i|^2$$

Define
$$W_{12} = \sum_i W_{12}^i$$
$$T = \sum_i T^i$$

therefor, by $W_{12}^i = T_2^i - T_1^i$

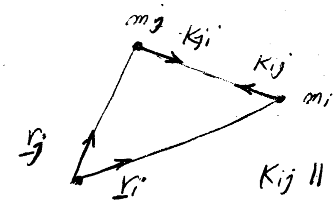
$W_{12} = T_2 - T_1$ Work-Energy principle, includes W_{12}^{int}

$$W_{12} = \sum_{i=1}^n \int_1^2 \underline{F}_i^{ext} \cdot d\underline{r}_i + \sum_{i=1}^n \int \underline{F}_i^{int} \cdot d\underline{r}_i \qquad d\underline{r}_i = \underline{v}_i\, dt$$

$$\sum_{i=1}^n \int_{t_1}^{t_2} \underline{F}_i^{int}\, \underline{v}_i\, dt = \int_{t_1}^{t_2} \sum_{\substack{ij \\ i \neq j}} (\underline{K}_{ij}\, \underline{v}_i + \underline{K}_{ji}\, \underline{v}_j)\, dt$$

$$= \int_{t_1}^{t_2} \sum_{\substack{ij \\ i \neq j}} \underline{K}_{ij} \cdot (\underline{v}_i - \underline{v}_j)$$

Important special case
$$\underline{K}_{ij} \cdot (\underline{v}_i - \underline{v}_j) = 0 \quad \text{for all } ij$$



$K_{ij} \parallel \underline{r}_i - \underline{r}_j$

$$\Rightarrow (\underline{r}_i - \underline{r}_j) \parallel \underbrace{(\underline{v}_i - \underline{v}_j)}_{\frac{d}{dt}(r_i - r_j)}$$

$$\frac{d}{dt}(\underline{r}_i - \underline{r}_j)\,(r_i - r_j) = 0 \Rightarrow \boxed{\frac{d}{dt}(r_i - r_j)^2 = 0} \quad \text{for all } ij$$

Definition: Systems of particles with $|r_i - r_j| = Const$ are called rigid-body

For such systems: $W_{12}^{ext} = T_2 - T_1$

If furthermore all external forces are potential, i.e.,

$$\underline{F}_i^{ext} = -\nabla V_i^i(\underline{r}_j, t)$$

Then

$$T_2 - T_1 = W_{12}^{ext} = \sum_i \int_1^2 \underline{F}_i^{ext} \, d\underline{r}_i = \sum_i V_1^i - V_2^i = V_1 - V_2$$

Where $V_1 = \sum_i V_i^1$

$V_2 = \sum_i V_i^2$

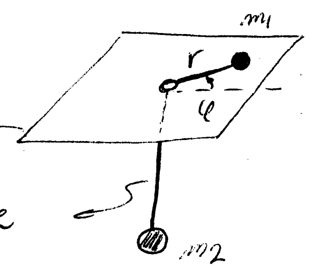
$T + V = \text{Const}$

Conservation of Energy. (rigid body) external forces are potential

Example

Smoth

inextensible



Assume $\dot{r}(0) = 0$

$r(0) = r_0$

$vr \stackrel{?}{=} Cte$

$\frac{v^2}{r} = \frac{(\frac{K}{r})^2}{r} = \frac{K^2}{r^3}$

Question: minimum Value of r

maximum Value String force