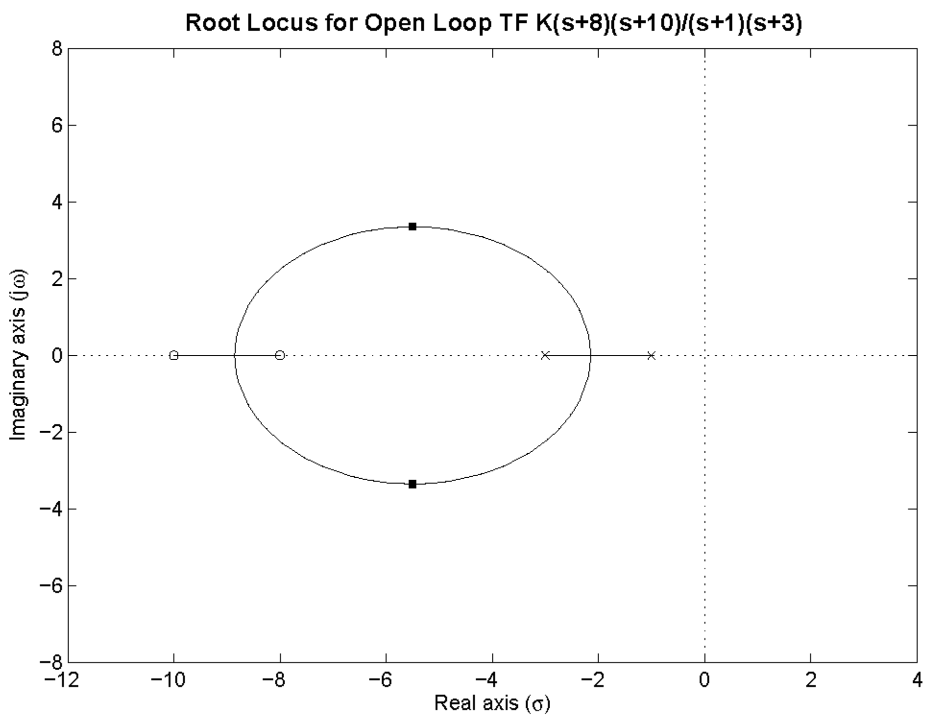
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

2.004 Dynamics and Control II
Fall 2007

Quiz 2 Solution

Posted on Monday, December 3, 2007

1. The root locus of a feedback system with open–loop poles at −1, −3 and open–loop zeros at −8, −10 is shown below. Study the diagram, then answer the following two questions:



1.a) (10%) Can the response of this system be tuned (approximately) to a settling time of 1 sec with an appropriate choice of feedback gain K? Justify your answer graphically. If your answer is "yes," there is no need to compute the value of K that would give this settling time.

Answer: Settling time of 1 sec requires

$$T_s = \frac{4}{\zeta\omega_n} = 1 \Rightarrow \zeta\omega_n = 4$$

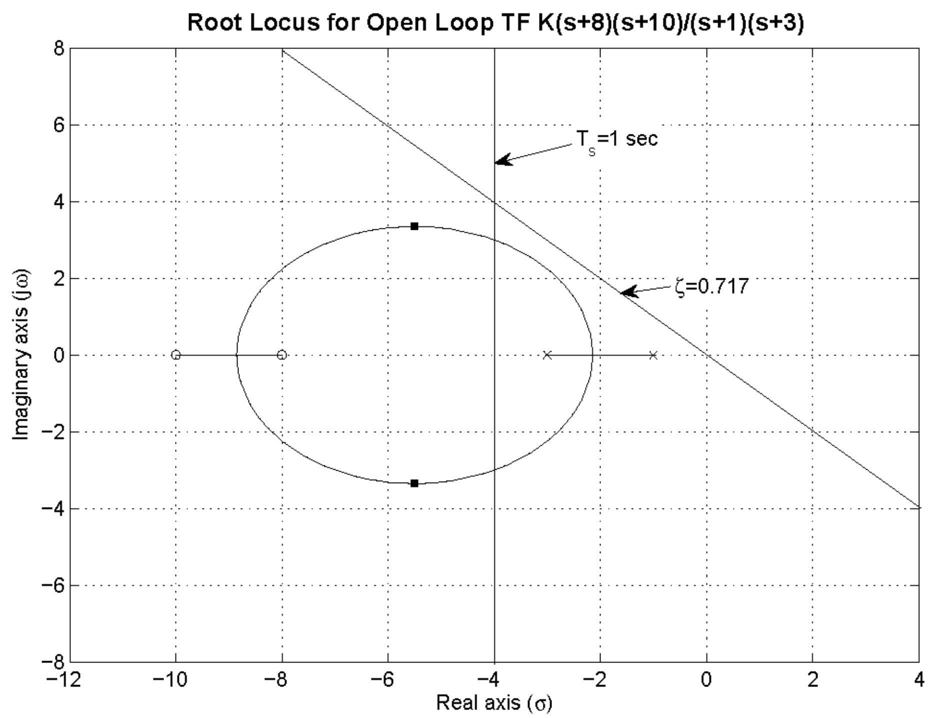
The given root locus does contain closed–loop poles whose real part equals $-\zeta\omega_n = -4$ (see annotated root locus on next page;) therefore, it is possible to achieve settling time of 1 sec by appropriate choice of feedback gain in the given system.

1.b) (10%) Can the response of this system be tuned (approximately) to an overshoot of 4.32% with an appropriate choice of feedback gain K? Justify your answer graphically. If your answer is "yes," there is no need to compute the value of K that would give this amount of overshoot. Hint: A damping ratio of $\zeta = 1/\sqrt{2}$ would approximately yield the desired overshoot value.

Answer: Damping ratio of $\zeta = 1/\sqrt{2}$ requires that the closed–loop poles subtend angle θ to the origin such that

$$\cos\theta = \zeta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45^\circ.$$

The given root locus does not intersect the line $\theta = 45^\circ$ (see annotated root locus below;) therefore, there are no closed–loop poles for any value of feedback gain in the given system that yield the desired overshoot.



2. We are given a feedback system whose open–loop transfer function is

$$\frac{K\,(s+20)}{(s+2)\,(s+4)\,(s+10)},$$

where K is the feedback gain. In this problem, we will evaluate this system's closed–loop behavior using the root locus technique.

2.a) (15%) How many asymptotes are there in this system's root locus? What are the asymptote angles?

Answer: This system has $\#p = 3$ finite poles and $\#z = 1$ finite zero; therefore, it should two open–loop zeros at infinity. The closed–loop poles should approach the open–loop zeros at infinity (as the feedback gain increases to infinity) along $\#p - \#z = 2$ asymptotes at angles

$$\theta_a = \frac{(2m+1)\,\pi}{3-1} = \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}.$$

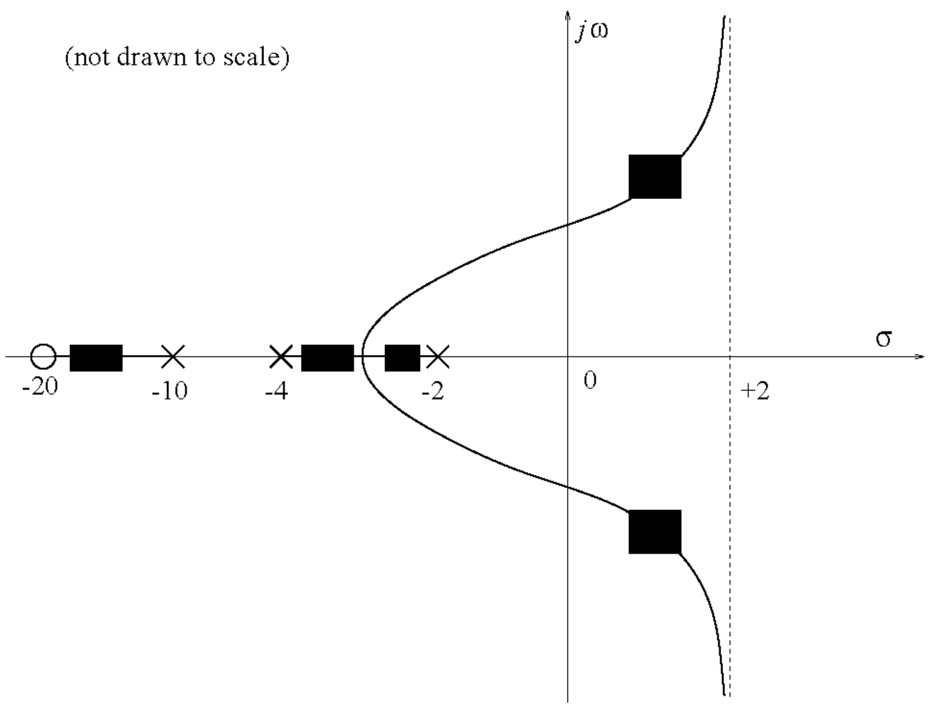
2.b) (15%) Where is the asymptotes' real–axis intercept?

Answer:

$$\sigma_a = \frac{\Sigma p_k - \Sigma z_k}{\#p - \#z} = \frac{-10-4-2-(-20)}{3-1} = +2.$$

2.c) (15%) Sketch the root locus based on the information from the previous questions. There is no need to annotate break–in/away points or imaginary axis intercepts, if any.

Answer:



2.d) (10%) If you had to recommend this system to a customer, what would you advise with respect to increasing the feedback gain K indefinitely?

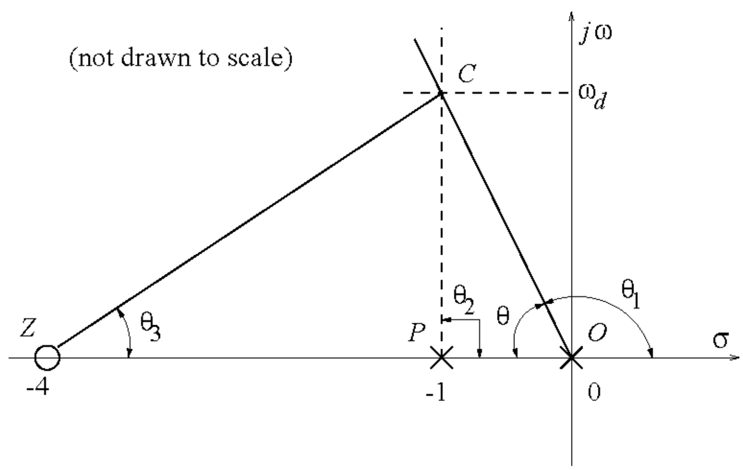
Answer: Since the root locus crosses over to the right–hand half–plane, the system will become unstable for a sufficiently high value of the feedback gain. This is the most important warning one should give to a customer.

In addition, large gain in the stable regime (just before the cross–over to instability) leads to increased overshoot and longer settling time; both of these qualities are generally undesirable.

3. We are given a feedback system with an open–loop pole at the origin (a "free integrator") and another open–loop pole at -1. In this problem, our objective is to evaluate a proposed proportional–derivative (PD) compensator that would approximately exhibit settling time of 4 sec and overshoot of 16.3%. (Hint: This value of overshoot corresponds to a damping ratio $\zeta = 1/2$.) The proposed PD compensator cascades an open–loop zero at -4.

3.a) (15%) Use a graphical construction on the s–plane to verify that the proposed PD compensator indeed meets the design requirements.

Answer:



Since $\cos\theta = \zeta = 1/2 \Rightarrow \theta = 60^\circ$ from the given overshoot requirement. From the settling time requirement,

$$T_s = \frac{4}{\zeta\omega_n} = 4 \Rightarrow \zeta\omega_n = 1;$$

i.e., the real value of the closed–loop pole must be $-\zeta\omega_n = -1$. From the two requirements above, we conclude that the closed–loop pole must be located on the point C, marked on the complex plane in the drawing. (Another closed–loop pole, not shown, will be conjugate to C, *i.e.* with the same real part and opposite–sign imaginary part.) We must now verify that point C belongs to the root locus of the given system.

Generally, a point on the complex plane belongs to the root locus (*i.e.*, it can be a closed–loop pole for an appropriate value of feedback gain) if the angles θ_z, θ_p that this point subtends to the open–loop zeros and open–loop poles, respectively, satisfy the relationship $\Sigma\theta_z - \Sigma\theta_p = (2m+1)\pi$. The given system has two open–loop poles at $s = 0$ and $s = -1$ due to the plant, and one open–loop zero at $s = -4$ due to the PD compensator. Therefore, we must compute the angles subtended by C towards the points Z, P, O on the complex plane. These angles are denoted, respectively, as θ_3, θ_2, θ_1 in the diagram, and they are progressively computed as:

$$\begin{aligned}\theta_1 &= 180^\circ - \theta = 180^\circ - 60^\circ = 120^\circ;\\ \theta_2 &= 90^\circ, \text{ since } C \text{ has the same ordinate as } P;\end{aligned}$$

and, noting that $\omega_d = (PO)\tan\theta = \sqrt{3}$,

$$\tan\theta_3 = \frac{(PC)}{(ZP)} = \frac{\omega_d}{4-1} = \frac{1}{\sqrt{3}} \Rightarrow \theta_3 = 30^\circ.$$

We can now confirm

$$\theta_3 - \theta_2 - \theta_1 = 30^\circ - 90^\circ - 120^\circ = -180^\circ;$$

therefore, C belongs to the root locus.

3.b) (10%) Using the same graphical construction from the previous question, show that the feedback gain required to meet the design requirements in the PD–compensated system is $K = 1$.

Answer: For any complex number s that can be a closed–loop pole (in other words, for any complex number that belongs to the root locus), the following relationship must be satisfied:

$$K\frac{s+4}{s(s+1)} = -1 \Rightarrow K\frac{|s+4|}{|s|\;|s+1|} = 1.$$

For C in particular, the geometric interpretation of the above relationship is

$$\frac{K\,(CZ)}{(OC)\,(CP)} = 1 \Rightarrow K = \frac{(OC)\,(CP)}{(CZ)}$$

Using the drawing,

$$\begin{aligned}(CP) &= \omega_d = \sqrt{3}\\ (OC) &= \sqrt{(OP)^2 + (PC)^2} = \sqrt{1^2 + \left(\sqrt{3}\right)^2} = 2;\\ (CZ) &= \sqrt{(ZP)^2 + (PC)^2} = \sqrt{3^2 + \left(\sqrt{3}\right)^2} = 2\sqrt{3}.\end{aligned}$$

Therefore, we confirm

$$K = \frac{2\times\sqrt{3}}{2\sqrt{3}} = 1.$$
