PROFESSOR: We've got three lectures left-- today, Thursday, and next Tuesday. So today we're going to talk a little bit more about modal analysis, and we did initial conditions last time. This time we'll do harmonic excitation mostly with a little review there. Thursday-- let me say this slightly differently.

Modal analysis looks at the vibration of a system with many degrees of freedom and looks at one mode at a time, but you can also just solve the whole thing at once. You don't have to break it down into the individual modes, so you can come up with a-- if you have a harmonic excitation on a system with many degrees of freedom, if you put in a harmonic force, the whole thing is going to shake, and you can solve it in one go, and that's essentially using a transfer function approach. So this is breaking it down one mode at a time. Here we're going to do this concept of transfer functions for Hij, so response at location i due to an excitation at j. In a multiple degree of freedom system, you get many different combinations. So this is steady state response, but done all at once.

And Tuesday-- and also this has an application to something I called dynamic absorbers. I'll just be able to scratch the surface of that, but the Hancock Building across the way-- I told you a couple lectures back about it. In the 1970s, it was brand new, and the windows were falling out. And there was lots of windows falling out at the bottom and none at the top, and the thing was bending back and forth like just a cantilever, a reed in the wind, and one of the fixes for it is they put a thing called the dynamic absorber on the 58th floor. There are two 300 ton blocks of lead on a pressurized oil film so they can slide back and forth on the 58th floor of that building. They're called dynamic absorbers.

And at the expense of letting these big blocks of lead slide back and forth, it keeps the building from vibrating. It keeps the windows from falling out. So I hope to get to talking a little bit about dynamic absorbers-- one other way of stopping problem vibration. And on Tuesday, we'll talk about strings and beams. Just a brief introduction to continuous systems. And that'll be our last lecture, and probably give you a very quick review then of what's going to be covered on the final, but mostly the final will be covering this last third of the course on vibration.

So let's turn to modal analysis, and posted on Stellar is a little too page handout that gives you a just step by step cookbook approach to conducting a modal analysis. And we're going to hit the highlights of that this morning as a way of reviewing what we learned last time and moving on to calculating the response to harmonic forces. So we begin with some n degree of freedom system linearized. We're only dealing with linear equations of motion.

And our first step is we need equations of motion. So you write them m-- in general, you'd write out your equations of motion for the system, and these x's here are just my generalized coordinates. They can be rotations, deflections, whatever makes sense in the problem. That's the first step. You need your equations of motion.

Second step, find undamped natural frequency-- is the omega i's-- I'll just call it omega i's-- and mode shapes. And the mode shapes we put into a matrix that we call a mode shape matrix. I write it as u. So that's the next step in the problem.

The third is basically to apply or invoke the modal expansion theorem. This is the key to the whole thing, and that is to say that you can write the generalized motions of the system-- the responses of the system-- as a weighted sum of the individual modes of the system. So the q's are the modal amplitudes. The u's are the mode shape-- each mode, like the first mode, has some amplitude and time dependence, and its motion is distributed to the whole system according to its mode shape, and that's what that statement is. And another way of writing it which is more intuitive is this is a summation then of i equals 1 over the degrees of freedom of the system of the mode shapes times qi of t, the modal amplitudes of the response of that single degree of freedom system that describes each of the modes.

Now, I should-- the qi's are the solution to n equations of the form mi qi double dot plus ci qi dot plus ki qi equals the excitation for that single degree of freedom system. So the modal expansion theorem says we're going to take each mode of the system, treat it like a single degree of freedom problem. And so for the first mode, I would be 1. You would have a modal mass, damping, stiffness, and a force for that mode. And you know how to solve for the response of a single degree of freedom system to initial conditions, or the steady state response to some harmonic input, and you would do that for each of the modes. And to get back to the final answer, you sum them back up again this way.

So I've put up here on the board where that-- the description of that demo, and we went through all this information last time so you have it in your notes for last time. If you missed it last time, I put it up again. So here's the demo on the table. This is my diagram of it. Mass 1, mass 2, spring, lowercase k1, lowercase k2-- so the lower cases I'm going to use to describe system parameters. Uppercase K's and that sort of thing will be my modal parameters.

So this has masses, stiffnesses, and damping, and this damper is connected to the non-moving frame, and so is this one to model the damping sliding up and down on that shaft. So the equations of motion of the system are here in your generalized coordinates. And I've included the possibility that I could have a force in the first mass, F1, a force in the second mass. So this is a completely general set up for this problem.

Then you need to find the natural frequencies and mode shapes, so you assume a solution of the form e to i omega t. Plug in, and you get this algebraic equation. Minus omega squared m plus k times some vector, which will turn out to be the mode shapes. And either this is 0, which is trivial, or the determinant of this is 0, and this gives you the roots of this determinant, give you the natural frequencies of the system. If it's a four degree of freedom problem, you get four roots.

So we found for this particular problem that the natural frequencies are 5.65 and 17.69, and the mass matrix, if you weren't here last time, in kilograms, 3.193, 0.63.

And the stiffness matrix in newtons per meter for that system is this. Notice, stiffness matrices are always symmetric, and this one minus 36, minus 36.

And many of you have been taking 2001. Did anybody invoke something called Maxwell's reciprocal theorem? Have you run into that? Well, this comes from structural analysis, this notion of the symmetry of the K matrix. The mass matrix is also going to be symmetric. So we need to find-- to do modal analysis, you make this calculation. You transpose mu. This is the transpose of the mode shape matrix mu, and if you do that calculation with these numbers and those mode shapes, you get a new mass-- you get the modal mass matrix, which is guaranteed to be diagonal, even if it wasn't to begin with. This is the numbers you get.

The stiffness matrix certainly wasn't diagonal to begin with. It looked like that. And we multiply u transpose times k times u, and we get a diagonal matrix. This element here we'd call capital K1. It's the modal stiffness for mode 1. This is the modal mass for mode 1, and they represent a-- they're the numbers we need to write a single degree of freedom equation of motion of the kind at the bottom of that board up there.

So we want to write M1q1 double dot plus some c1q1 dot plus k1q1 equals, in general, a-- this is a 1-- some excitation, but it's a single degree of freedom system. If it's a single degree of freedom system, what's its natural frequency using the parameters in this equation?

AUDIENCE: k1/m1.

PROFESSOR: Right, so omega 1 had better be the square root of k1/m1, and that's the square root of-- k1 is 91-- nope. That's the real [INAUDIBLE]. I need my u transpose k here. 113.71. 113.71. And the modal mass for system one is 3.556. So that's 3 or so and 100 and something. That's neighborhood of 35, 36. Square root of 36 is about 6. So it needs to give you back exactly the correct natural frequency, which you found in the beginning when you solved for the natural frequencies and mode shapes of the system. So I repeated myself here. And if you write the same thing-- omega 2-- it better be equal to k2/m2. So these are just ways of verifying that you've done your arithmetic correctly.

And I haven't done the damping matrix yet, because the damping matrices have to be treated rather carefully. They don't automatically uncouple, and so we'll address it in just a second. The damping matrix, however, would be written utcu would give me what I hope is a diagonalized modal damping matrix. I'm trying to make these look like caps or something to make them look a little different from the c's of the dashpots themselves. We hope to find a diagonalized damping matrix, but we have to make some effort to make that happen.

Finally, one thing we haven't dealt with before. In order to-- when we derive the single degree of freedom modal systems, we had to multiply through by this u transpose, the original equations of motion, and so what we haven't worked with yet is how do you get the modal forces in the system? And they come from the calculation u transpose times the generalized forces in the system. So that's a vector times a matrix gives you a new vector, which are the generalized forces. And we're going to do that example in more detail in a few minutes.

So last time we did the initial conditions problem, and we need the results of that to do-- we need the damping that we learned from that to do the force vibration problem. So let's go back and review a little bit about what we did. So let's let, for a moment, the generalized forces be 0. So no excitation. So we're only doing an initial conditions problem, and let's assume that we have some set of initial conditions on the generalized displacements at time 0. So those initial conditions on displacements-- they'll look something like x10 down to xn0, and we just have a two degree of freedom system in our example. And also you could have an x dot at t equals 0, and that would be some vector of v10's down to vn0's. And for today, we're going to let those be 0s. So we're just going to have an initial deflection in the system and see what happens.

And we learned last time that, since the modal expansion theorem is what makes this whole thing work, the idea that x can be written as u times q-- then we ought to be able to say the same thing for-- if these are initial conditions, then we should be able to get the initial conditions in modal coordinates. So x at 0 here is also u at the q's at 0, but we normally would specify these. We need to know those. Well, to get these I just multiply through by u inverse so that the q, the vector of the initial displacements in modal coordinates, is just equal to u inverse times the initial conditions-- the initial deflections in the generalized coordinates. And q dot at 0, if you had non-zero initial velocities, would be u inverse times x dot at time 0.

So let me tell you where we're going. I'm not going to go back all the way through the initial conditions problem. That was last lecture. I'm going to review the results of it, because what I want to get to is, how do you compute the response by modal analysis-- the response to a harmonic force? For example, if you get close to resonance for a single degree of freedom system, what controls the height of that peak? If you drive a single degree of freedom system at resonance, what's the most important parameter?

AUDIENCE: Damping [INAUDIBLE].

PROFESSOR: Damping. So we really need to know damping. And the way we get damping is-- one of the ways is measure it. So I want to measure the right damping so that I can do-- what I'm trying to get to is to do the force vibration problem, but I need to get some estimates of damping to do it.

Now, I said damping can sometimes be a problem, so I'm going to show you-- I alluded to this damp last time, but I wasn't able to kind of really clearly go through it. So I'm going to make a damping model, my damping matrix for my system such that it is proportional to the original mass and stiffness matrix of the system. So alpha is just a parameter that I get to choose. Beta is another one that I get to choose, and this is m and k, the original mass and stiffness matrices of this problem or any n degree of freedom problem. We're doing these two degree of freedom examples because they're tractable on the board.

So if I make my damping matrix look like that, it is guaranteed to work when I do u transpose times that times u, because we know the mass and stiffness matrices give me diagonals, so I'm just essentially doing that again. So this is guaranteed to give me a diagonal matrix. It's what I want. Yes?

AUDIENCE: So are those the original mass?

PROFESSOR: Yes, these are the original ones. Usually, if I'm really trying to write the modal ones, I'll draw them with a diagonal through them or something just to-- so these are m and k, just like up here. Right there. They're right from the original system. They're the ones on the top of this board. There's the mass matrix, damping matrix, stiffness matrix for the original equations of motion.

So I am saying that my unspecified damping matrix, which is written as c1, c2 there-- I'm going to represent it this way, so that u transpose cu, which is the calculation I need to be able to do-- what that gives me is an alpha, and here's my modal mass, diagonalized modal mass matrix that results, plus a beta times my diagonalized stiffness matrices. So Christina, these guys are the modal ones after doing u transpose mu, u transpose ku.

So we have a two degree of freedom system, and when I do this calculation, I'll get modal mass m1, m2, k1, k2. So the final diagonalized stiffness matrix will end up looking like some capital C10, capital C2 here. That's my diagonalized damping matrix. It'll come from these, and I'll just write them out. So the C1 will be an alpha m1 plus beta k1, and C2 will be alpha m2 plus beta k2.

So I have two free parameters with which I can fit-- I can fit those parameters to give me the damping that I want, and I'm going to measure the damping in the system. And then I'm going to find the two parameters that make that work. That's why I'm going through this. All right, running out of room.

So zeta 1, the damping ratio for mode 1-- if it's a single degree of freedom system, you say, oh, well that's the damping constant over 2 omega 1 m1. That's how we define damping ratio for a single degree of freedom system, but we know what these quantities are. This then is an alpha m1 over 2 omega 1 m1 plus a beta k1 over 2 omega 1 m1.

k1/m1 is omega 1 squared. So put omega 1 squared up here. Cancel with that.

These two gives you-- this gives you alpha over 2 omega 1 plus beta omega 1 over 2, and I can do the same thing for zeta 2. Be my c2 over 2 omega 2 m2, and that'll give me an alpha over 2 omega 2 plus beta omega 2 over 2. So if I can measure a value for this damping and a value for that damping, I have. These are known then. I have two equations and two unknowns, alpha and beta-- just algebraic equations that I can solve.

So now I need to conduct my experiment. I have this system. Without external excitation, it's just a free vibration system-- typical equations of motion. The first mode would look like m1q1 double dot plus c1q1 dot plus k1q1 equals 0. I'm looking for solutions to that, and there's a similar one for the second mode.

And what I want to do is excite only one of these modes at a time. Now, you've seen this demo done before, but at a certain combination of deflections, it'll respond only in mode 1, and a different one will respond only in mode 2. And it's guaranteed that, if you deflect a system in the shape of one of its modes and let it go, it will only vibrate in that mode, but let's prove to ourselves that that actually is going to work.

So I'm going to let, for example, initial conditions-- x10, x20-- be in the shape of mode 1. Well mode 1 is 1, 2.2667-- is the mode shape of mode 1. I'm just going to let that be. I'm going to deflect it in the shape of mode one, and I'm going to let x10 dot and x20 dot-- those are 0. And I need to know, if I do that, what are the resulting initial conditions in modal coordinates? Well, we know q10 and q20-- I can obtain them by doing u inverse times this. So this will be u inverse-- last time-- I'm going to write this. Last time I made a mistake. So u inverse-- I left out a 0. So 0.0898, 0.9102, 0.4016, minus 0.4016. So that's u inverse, and I'm going to multiply it by one of the mode shapes. 1 and 2.2667.

So I'm saying that my initial conditions are going to look exactly like one mode shape. To compute the equivalent modal initial conditions, I multiply the generalized the coordinate initial conditions by u inverse. Here's u inverse times that, and if I do that calculation, I get exactly 1, 0. And if instead of the first mode shape I put in the second mode shape, the 1 minus 0.2236-- if I did that, I would get exactly 0, 1.

So I just wanted to go through that just so you'd see it-- that the math bears it out. If you put in a deflection that's exactly the shape of a mode, then you will get back an equivalent initial condition in the modal coordinates for only that mode and everything else will be 0. So that's q. This is q10-- is that guy. And this q20 in this case is 0. OK? So now we're ready to do the experiment.

So if I deflected in the shape of mode 1-- come back here. And now would be a good time to lower the lights a little bit. And unfortunately we have that white chalk in the background to distract us, but it's now deflected only in the shape of mode 1, and now it's going to behave like a single degree of freedom system, right? So how would you estimate-- do a quick estimate of the damping of mode 1. I gave you a little quick, easy rule you could use a few times back. What was it?

AUDIENCE: Do the 50% thing?

PROFESSOR: Do the 50% thing she suggests. Now, can you be a little more specific. Pardon?

AUDIENCE: The half life.

PROFESSOR: Yeah, how many-- what am I looking for? How many--

AUDIENCE: Cycles.

PROFESSOR: Cycles it takes for the thing to decay 50%. So the reference line is here. I've deflected it that far, and it'll start vibrating, and when the top of this on a vibration gets to the halfway point between here and the reference-- reference is where it starts-- that'll be my 50%. So one, two, three, four. Got about four cycles when it only went down halfway.

Now, when I did it in my office the other day, I only got 2 and 1/2 cycles. So we're going to use 2 and 1/2 cycles because I ran the numbers for that. This thing is very sensitive if it's inclined a little bit, because that changes the friction on the shaft. So in my office the other day, there was a lot more damping. So anyway, pretend it's 2 and 1/2.

Now, while we're at it, let's do the other case. So the other case we want to deflect it in the shape of mode 2. So I go down, say, a unit amount. It's now deflected downward some amount and upwards such that the ratio is in the mode shape of mode 2. So every unit I went down, I go up minus 22% of that. So that's what's been done here. So when I release this one, now the reference line for the second one is up here down to here. So when this upper one decays to halfway-- about here-- that'll be the number of cycles. 1, 2, 3, 4, 5, 6, 7, 8, 9, 10. Anyway, in my office I got about 10. We're going to use my numbers, but that one is clearly a lot less damped. That thing just went on, an on, and on, and on, and on.

So in our experiment then-- so we conducted our experiment. Zeta 1 is approximately equal to point 1.1 over the number of cycles to decay 50%, and when I did the experiment in my office, that was 0.11 over 2.5. So it was really decaying fast, and so when you only have two or three cycles, it helps to use even fractional cycles. And it's OK to use fractional cycles. This is just an estimate. That's going to give us a number.

Zeta 2 is approximately 0.11 over the number of cycles to decay 50%, and in this case, that's going to be 10. And so that gives me 0.011, or what's known as 1.1% damping. And the other one, 0.11/2.5, gives me 0.044, or 4.4% damping. Percent of what? Anybody remember what happens when you're at 1? Have a damping ratio of 1.0?

AUDIENCE: Critical damping.

PROFESSOR: That's critical damping. That's that crossover point. If you have 1 or greater, you get no oscillation. It just goes and stops. Less than 1, it will actually across 0 and oscillate a little bit. Well, now I have values for 0.044 0.011 that I can plug in to these two equations, and I now have two equations and two unknowns, alpha and beta, because I know omega 1 and omega 2. So solve for alpha, and for alpha I'm not going to do that on the board. It's kind of a waste of good lecture time. 10.71. And beta, minus 0.033.

So they can be positive or negative to make these work out, but those are the two values you need, and that says that we're going to model the damping matrix of the system as 10.71m plus minus 0.033k. And we'll compute the diagonalized damping matrix, which comes from utcu, and did I write that down? I didn't, but I don't need it, because I know the damping that I'm after 0.044 and 0.011, because in order to complete this problem, if I wanted to go to completion and have the transient decay of the system, I would have it in for each of the two modal systems. q of t for the first mode would be some e to the minus zeta 1 omega 1t cosine omega 1-- I left out my q10. q10 cosine omega 1d times t-- that's the most of the initial displacement part-- plus q10 dot plus zeta 1 omega 1 q10. All of that over omega 1d. All of this times sine omega 1d times time.

So this is just the response to initial conditions for a single degree of freedom system, and you'd have a similar one for q2 of t, except now all the ones would be replaced by 2's. So you have an e to the minus zeta 2 omega 2t times-- and those are your two-- that would be in general the response to initial conditions in modal terms. And if you want to get back to the final response in your generalized coordinates, then it's just summation i equals 1 to 2, in this case, of u mode shape, or mode 1, q1 plus mode shape for mode 2 q2 of t. So these two added together would give you total response.

So that's the response. Kind of a quick review of what we learned before. Response to initial conditions, but now along the way we've learned how to experimentally fix the damping matrix, so it'll work, and actually get it to give us exact accurate results if we want to do a response to any initial conditions problem. But now we also have damping ratios for this system, and we can now go on to do the force vibration problem. And the force vibration problem will be pretty easy at this point.

So that was a quick review of response to initial conditions by modal analysis-- how to find a damping matrix that works. We've also proven that, if you deflect a system in the shape of a mode, it responds only in that mode. That's all pretty use-- and that's generally true of vibration systems.

So we know-- I banged on this before. We know these things like to shake a little.

They're flexible, and this one has a mode shape. It's kind of a cantilever. It's first mode. It bends like that a little bit, and this thing has to bend with it. But if I were able to just give this thing an initial deflection just in the shape of that mode and let it go, it will vibrate in just that mode. And if I do it in some other contorted way that its initial shape isn't just one mode and let it go, it'll vibrate in a couple different modes.

AUDIENCE: Is there a way to visualize the modal coordinates? Would they be the center of a mass of certain things, or would they would be the center masses of the springs or something?

PROFESSOR: Yeah, I taught a vibration course for many, many years, and I scratched my head long and hard to try to come up with an example where you could do exactly what you asked. I know of one example that works. In general, the modal coordinates-- it's a coordinate transformation into some system that it's very hard to place yourself physically so you can see what's going on, but to answer your question I'll show you one and you still have enough time to get through the other part.

Imagine this eraser. It's a car, a car automobile suspension system. So here's the car. Sits on springs, the tires and its suspension system. Got a center of mass that's here somewhere. And I'm only going to consider vertical motion of this thing. Now, you can imagine that it could go up and down, but you can imagine it can also pitch back and forth.

So I'm going to have two generalized coordinates. One is the vertical deflection of its center of mass, and the other is the angular rotation of the center of mass with respect to my initial horizontal. Now, you get equations of motion of this. One, you'll get an mx double dot kind of equation by summing the forces on this and the forces from the springs. And of course you'll have dampers and those kinds of things. So you'll have one force equation by summing the forces on it. You'll have another equation that's sum of torques, and it's an i theta double dot kind of equation.

So it'll be a two degree of freedom system. You'll get two natural frequencies, and two mode shapes. And the two natural frequencies look approximate-- the two mode shapes look-- here is the original undeflected system. One mode shape the system moves up and rotates up. It has both positive rotation and some positive deflection. And I need to draw this over here a little further. So here's the original system. It moves up in its mode shape like that, and when it goes down to its max negative, it's like that. So it goes through a motion.

And if you think about it, if you extend these lines out here, you come to a point that this intersects. And so in fact, in the first mode of vibration of this thing, if you went and sat right here, you would just see this thing go through an angle. And that, in fact, is the modal coordinate for mode 1.

And the second mode I've kind of forgotten. It goes has a sign change, and so when you're going-- deflecting upwards, it goes-- positive flexion upwards-- it tips downwards. And when it goes the other way, it goes like that. So this one, it goes up and rotates up. And the second mode, when it goes up, it rotates down. And out here is the point at which you could go sit there, and you would just see this thing rotate up, rotate down if you went and sat there.

So you move yourself to a place where you can see with a single coordinate-- with only the angle measured by this point, you can completely describe that modal motion. And with only the angle measured around this point, you can describe the modal motion. But in general, you can't do it. So if I'm out here-- if my eye is at this point, I'll see this thing go like that. That help?

AUDIENCE: Yeah, a lot.

PROFESSOR: So one of the hardest things for me as a lecturer is, since I really like vibration and I've taught it for 35 years or so, to try to cram everything I know about vibration into the one third of this course. So I obviously don't do that. So I'm trying to give you-- my goal then becomes give you some basic insight about vibration so that, when you do get out there in the real world and you need to know something, need to solve a vibration problem, you'll know the fundamentals, and you'll know where to go look it up. And if you want to take another course or something in vibration, you can do that.

All right, now we've got to do, in relatively short order, response of our initial system over here to some harmonic excitation. And the one I've chosen to do-- let's imagine that I just-- I'm just going to put a force, a harmonic force, on this mass, and I want the steady state response. And you can imagine, if I do it close to the natural frequency of one of the modes, you're going to get a lot of that, but if I do it close to the natural frequency of the other mode, I can do it too.

So let's do the steady state harmonic excitation problem. So we're going to do this problem. F2 of t-- so my generalized forces are 0 some magnitude Ft e to the i omega t. And the modal forces are u transpose F, and that would be the calculation one and 2.2667, and 1, and minus 0.2236-- my two mode shapes transposed multiplied by 0 and F2. So that says that q1 is F2.2667 F2e to the i omega t. And q2, the modal force for mode 2, is this times that-- is minus 0.2236 F2 e to the i omega t.

So I've put on just a force on one mass, but it gets distributed in a way that it'll excite both modes, and it'll excite mode one in an amount 2.2 times F2. And it'll excite mode 2 in an amount minus 0.22. So mode 1 is going to get more excitation in this particular case because of the shape of the modes. The bigger the modal deflection is at the point of application of the force, the more that mode is going to get.

Well, now these gave you two equations of motion. m1q1 double dot plus c1q1 dot k1q1 equals, in the case of mode 1, 2.2667 F2 e to the i omega t. And that's a single degree of freedom system excited by a harmonic force. We worked that problem. We know what the answer looks like.

So for example, the magnitude of the response q1 is given by the magnitude of the force q2 times the magnitude of a transfer function, which is the response q1 per unit input force q1 evaluated at whatever frequency I evaluate it at. So the magnitude of the response magnitude of the modal force times the transfer function, and this transfer function is this exactly the same form as when we did just the single degree of freedom system as x/F. That's what I called the response x over input force F. Same thing, where just now the response is q, and the input force is capital Q.

Therefore, that looks like magnitude of Q-- and I made a mistake here. Q1. The magnitude of force Q1, which is this, times 1/k1-- this should look familiar-- 1 minus omega squared over omega 1 squared squared plus 2 zeta 1 omega over omega 1 quantity squared square root. So there's that transfer function expression. At resonance, for example, all this goes to 0. Omega over omega 1 goes to 1. This whole denominator turns into 1/2 times the damping ratio, for example. And Q1 over k1 is the static deflection of the system under the load-- under a static load of that large.

So I'll do the problem. If the excitation frequency happens to be right on the natural frequency for mode one, then I can evaluate this at 1 omega over omega 1 is 1, and I've worked that out. That says that magnitude of q1 here-- 2.2667F2/k1 times 1 over 2 zeta one. That's what all this condenses to. We know what k1 is. We know what zeta 1 is. F2 is the thing we're specifying. And so zeta 1 for example is what-- it was 0.044. So you plug in your 0.044 here, and this whole thing works out to be-- actually I'll write down the numbers here-- 2.2667F2/k1 is 10984, and not 109. That's k2. 113.71 times 11.36, which is 1/2 times zeta. 1/0.088 is 11.36. The final analysis-- q1 is 0.227F2. So it's just a single degree of freedom system excited by harmonic force, and you can figure out how big its response is. I could do the same thing for mode 2. Is mode 2 resonant? No.

So this is instructive. The magnitude of the responsive q2 is the magnitude of the force times the magnitude of the Hq2 per unit input 2q. It looks like that, but this time this is minus 0.2236F2/109.84, which is k2. And then all of the denominator involving one minus-- and this will now be omega-- over omega 2 squared squared plus 2 zeta 2 omega over omega 2 squared square root. This number here is k2.

The reason I'm going to this is this one is not resonant, and in fact, what's omega? Well, in fact, what is the excite-- I started this problem. I said let the excitation frequency be what? Natural frequency of mode 1. So omega in this case equals omega 1. Therefore, for this problem, omega over omega 2 is omega 1 over omega

2, and that's 5.65 something over 17 point something. It's about 1/3.

So we are exciting the second mode at about 1/3 its natural frequency. And so this is 1 minus 1/3 quantity squared and so forth. And if I run the numbers, I just want you to see what happens with the numbers. This turns out to be 0.806, and this turns out to be-- remember this is 0.01 something here. This number turns out to be 5 times 10 to the minus fifth. That's a really small number. And this whole square root of this stuff turns out to be 0.898.

And in the end, q2 works out- the magnitude of q2 works out to be 0.0023F2. I did this on purpose. If, when you're at resonance, meaning the excitation frequency is close to the natural frequency for any linear vibration system, then this term is important, and it'll be the most important term in the denominator, because this term goes to 0. It's the only term in the denominator, and it's likely be quite small. That's why it gives you one over that big response, but when you're not-- when this denominator-- if you're not at resonance, this term almost always is negligible, and it's this term that governs it.

So here you are at 0.8 versus 10 to the minus fifth. So away from resonance, this term is important, and on resonance, that one is important. And now, how would you get back to-- we've now got the two responses. How do we compute the total system response? How would you do it? The modal expansion theorem.

AUDIENCE: Put it back into x [INAUDIBLE].

PROFESSOR: Yeah, so x equals u cubed. That's where we started. And q1, since this is a harmonic excitation problem, if the input had been e to the i omega t, then the output is some e to the i omega t minus a phase angle. So this is going to look like some q1 amplitude. Say, cosine omega 1t minus phi 1 but at resonance, we know the phase angle is pi over 2. And q2 of t is going to be the amplitude q2 times some cosine omega 1t-- because that's the excitation frequency-- minus phi 2.

And remember, each of these systems-- these transfer functions, Hq/Q for whichever one it happens to be looks like this. Different amounts of damping give you different heights of the peak at resonance. And this is omega over omega i. So when you're at resonance, you're at 1.

So what we've done is we have a two degree of freedom system. We're exciting it at the natural frequency of mode 1. So it means for mode 1, we're right here. Let's say that's where our damping is, so that's going to be our transfer function for that mode, but drawn on the same figure, where are we for mode 2? We're at omega 1 over omega 2, which is somewhere in the neighborhood of 0.3. So we are in here. So this is omega over omega 2. This is omega over omega 1. So down here, we're at about 0.3.

Here we're at 1.0, so we're at resonance for one of the modes, and we're here in what's called the stiffness controlled region for the other mode. This mode-- mode 2 basically acts like a spring. The dynamic amplification is about 1. It just gives you the static response for mode 2 and the resonant response for mode 1.

I know what I was going to draw to remind you. This figure has a phase diagram that goes with it, and for lightly damped systems, it goes from 0 to pi. And at resonance, all of them cross pi over 2. When it's a response to a force, a simple force, at resonance, the phase angle is pi over 2, so the response lags the input by 90 degrees.

When you're down here in this region, the response moves with the input. The phase angle is basically 0, and up in here the phase angle is 180 degrees. It acts like a driving mass. So our two responses look like that, and to get back into modal coordinates, x1, x2 is going to look like u1 q1 of t plus u2 q2 of t. It's harmonic. Steady state response. You know the amplitudes, q1 and q2. We figured them out. One of them is 2-- q1 turned out to be 0.227F2, and q2 is 0.0023F2. And this one, point cosine cosine.

So if we do that, if you put your excitation only on here, it's trying to tell you that mostly you'll get mode 1, and not much mode 2, but that's primarily caused because you chose to put the excitation frequency at where? The natural frequency of mode one. So if I had made my excitation here the same place, but made the frequency close to the natural frequency of mode 2, which of the two modes would have dominated? Mode 2.

So I've got another little demo. We're doing well. Out of necessity, to do it on the blackboard and in relatively short time, we've only talked about a two degree of freedom two rigid body system. This is a continuous system. It's a taut string. It's your violin string, and I've put some white tags on it so that you could see it against black backgrounds.

Everything that we've learned about the behavior of this two degree of freedom system will apply to a three, or four, or five degree of freedom system, but actually the basic lessons apply to continuous systems, too. So the lesson we just learned is, if you excite a system, this has many natural frequencies. And in fact, if the first one is at 1 Hertz, which this just about is-- maybe a little more than that. First mode is like that. Maybe 2 Hertz. The second mode is twice that. Third mode is three times that. It happens to be really simple.

So if I put a force-- the only force in this problem-- I'm going to do the analogous problem here. Harmonic excitation-- I'm going to do it in one little place right here, and if I drive this system at the natural frequency of mode 1, what do you see? What mode is dominating response? Mode 1. Very, very small responses of other modes. So now if I can get the system to stop shaking, if I drive it at exactly the same place at a different natural frequency, I see 3/2 sine waves there. So I was driving it at the natural frequency of?

AUDIENCE: Mode 3.

PROFESSOR: Mode 3. And there's no mode 1. No mode 2. Now I need a helper. Can somebody come hang onto this for me? You've got to keep the tension on it.

So now I'm going to teach you a really important lesson about systems exciting systems. The second mode-- I might have a hard time driving it here. Let me see if I can get the second mode going. There's second mode. I has a node right here. There's a point right here with no motion. If I sit here, and I'm going to let the system stop-- if I drive this system at the second mode natural frequency right here, what will happen? Pardon?

AUDIENCE: It would be mode 1.

PROFESSOR: Maybe [INAUDIBLE] 1. But how much mode 2 will I get? So another lesson here is that continuous systems have nodes, points of no motion, and the second mode happens to have a node right here. This point doesn't move one when it's vibrating in the second mode. And the modal force looks something like the generalized force times a mode shape, u transpose F. That's how we got the-- you have to multiply the generalized external force times the mode shape to get the modal force.

Well, here's my generalized force. What's the amplitude of the mode shape here? 0. F times 0 is 0. There is just no way I can get this thing to vibrate in the second mode by driving it at a node, and that's just generally true. Take it to the bank and remember that. Thank you. All right.

Something else I wanted to do. I need you back. So when you give a system an initial deflection, because when we want to get initial conditions, we said that the equivalent modal initial conditions were u inverse times the u inverse times to the initial conditions in generalized coordinates. So for a continuous system, this thing has mode shapes that look like sine and pi x over l. Those are the mode shapes. And if I take this thing and grab it about in the middle, wherever that node was, and give it an initial deflection, the shape that initial deflection is kind of a triangle.

You've all had Fourier series. This a triangular shape. Could you express this shape as a Fourier sine series? Sure. And it just happens that sine waves are the mode shapes. So you would be coming up-- the Fourier coefficients are the modal amplitudes of n initial conditions. And so whatever the Fourier coefficients for this are are the modal amplitudes for each of the modes. So which mode do you think is going to-- which Fourier sign component is going to be largest in this one?

AUDIENCE: Mode 1?

PROFESSOR: Mode 1. By the way, all even numbered modes would be 0 because they're asymmetric. This is a symmetrically shaped pulse. The modal initial conditions for 2, 4, 6, 8, 10 are 0. You will get non-zero Fourier coefficients for 1, 3, 5, 7. The biggest one is mode 1. So what do you expect to see if I let go of this? Vibration primarily at?

AUDIENCE: [INAUDIBLE].

PROFESSOR: And a little bit of some others. Now, let's say they all had about the same damping. Let's say they all had 10% damping. No, excuse me. All had 1% damping. That means they'll go through about 10 cycles to decay to halfway. So if they all have the same damping, even if there are several other modes present, which ones are going to last longer in time? Mode 1, because it takes just longer in time to get to 10 cycles. The other modes get there quicker.

If you take a guitar string and plunk it, or a violin, or a piano, you'll hear the basic tone. And what makes it sound nice, you have those nice overtones, but if you listen carefully, the overtones die out usually, and you're left with the fundamental at the end. If you smack a piano key hard, you'll get an interesting sound at the beginning, and then it'll mellow out. And you'll hear just a pure tone at the end, and that's because the higher frequencies damp out quicker because they get in more cycles per unit of time. That's the other quick lesson.

I have one thing I want to explain to you which will help maybe a little conceptual understanding about-- this came up in a homework discussion-- and that is just a note about stiffness matrices. There's a really fast easy way to assemble stiffness matrices. So here's a three mass system and a spring, spring, spring. And I'm going to put a spring here and one here. k1, k2, k3, k4, k5, and k6-- and I want to get my stiffness matrix.

The stiffness matrix-- this is a three degree of freedom system. It'll be a 3 by 3, and it'll have elements up here, which I'll call k11, k12, k13, k21, k23, k31, k22, and so forth. So what's the new meaning of kij? So ki11-- i as 1. J is 1. If you can understand the interpretation of what a stiffness matrix is, it'll help you make it much easier for you to find them. So kij is the force required at i due to a unit deflection at j due to sounds kind of like a causal thing. I don't quite mean that, but so let's think about this.

What's k11? So k11 is the force required at 1 per unit deflection at 1. So if I take this system and I make it move over one unit, and the other-- this is now x1 here, by the way. x1, x2, x300 so x2 and x3 are 0. I do this one a time. I move this over one unit. How much force does it take to make that happen? A real system. You're grabbing it. You're pulling it over one unit. You're making some springs move. How much force does it take to move that one one unit holding these still?

Well, you're going to-- one, force on a spring is kx. If x is one, the force for trying to stretch this spring is k1 times 1. The force required to push on that spring as k5 times 5-- k5 times 1. k2 times one. The k11, this first element, is the sum of all of the springs connected to it. k1 plus k2 plus k5. So how about k12? k12 is the force required at 1-- actually, let me do k21. Make a little more sense-- is the force required at 2, because I've moved the system one unit at 1. That's what the problem-- that's this thing I've done. I've moved this one unit. In order to keep this one from moving, do I have to apply force to it? How big?

AUDIENCE: k2.

PROFESSOR: k2 or minus k2? This thing is pushing, going over one unit. It compresses that spring. It's pushing on this thing. I say it cannot move. What do I have to do? Push back minus k2, and you can go through-- and then how about number 3? What's the force required at 3 because I've moved the one at 1 by 1 unit? Move this over. Are there any springs connected to mass 3 that are affected by that motion?

AUDIENCE: k5.

PROFESSOR: k5, and it pushes on it through that spring, so I have to resist by-- so k31 equals minus k5. So now then you go on. If you want to get the next ones, OK, you go to the next system. This is now can't move. This can't move. We're going to let this be unit deflection. So unit deflection at two, add up the springs. k2 plus k3 plus k6-- and that's all there is. So k22, k2 plus k3 plus k6, and then you go through all the ones that it affects, and you'll get minus this and that.

So that's the meaning. Each of the elements of that stiffness matrix have that meaning to it. And to give this closure, we have equation of motion, but the stiffness matrices also applies to the statics problem. So you have mx double dot plus cx dot plus kx equals F. What if I only want to do a statics problem? I'm going to put a static force on here, and I want to know the deflections, or I'm going to cause deflections, and I want to know what force it takes to do it.

Well, the static problems-- let this be 0, this be 0. And it says kx equals F. So in this three degree of freedom system, what forces are required to cause the deflection 0, 0, 1. I want to deflect it one unit on the third mass only. What forces do I apply it to the system to make it happen?

Well, you just multiply it out. This is F1, F2, F3. So it'll only be this one. This times this, this, and this. The only ones that matter are these three here, and this will end up being k31 times 1. This will be k-- I guess I've got to round-- 13. And we get 21, 22, 23, k33. Those are the three forces required. You put on those forces. Force is equal to these amounts. You will get that deflection.

So that's just a little help. That gives you a little insight as to what stiffness matrices mean. So you can do them by inspection. Once you understand that, you can actually just fill them in by inspection, just by doing unit displacements at each place and adding up the forces. See you on Thursday.