The Polynomial Method, Fall 2012, Project List

Very optional questions you could explore. Generally I don't know the answers.

1. Distinct lines. Suppose that we have A distinct lines in $\mathbb{F}_q^n$. Let X be the union of the lines. How small can X be? Using a bush-type argument, prove that $|X| \gtrsim A^{1/2} q$. In some cases this is sharp. For example, if $A = q^2$, we get a lower bound of q^2. A plane has q^2 points in it and contains q^2 distinct lines. But what about $A = q^3$? The bush lower bound is now $|q|^{5/2}$, but there is no such thing as a (5/2)-dimensional plane. We can find q^3 (or even q^4) lines in a 3-dimensional plane, giving examples where $|X| = q^3$. Can you find a better example or prove a better lower bound?

2. Furstenberg-type problem in finite fields. (This is similar to a question that Nate mentioned to me.) Suppose that $X \subset \mathbb{F}_q^n$, and that for every direction, there is a line l in the given direction so that $|X \cap l| \geq A$. In terms of n and A, how small can $|X|$ be? If we take $A = q$, then X is a Kakeya set, but it would be interesting to understand what happens for lower values of A like $q^{1/2}$ or even 10.

3. Can you formulate and investigate a version of the Elekes-Sharir conjectures for lines in $\mathbb{R}^4$?

4. Can you prove a general version of Bezout's theorem with an argument along the lines of the one in lecture 13.

5. Degree reduction in 5 dimensions? In the problem set, you will explore degree reduction for sets in 4 dimensions. There are some new issues that occur in 5 dimensions... Could be good for someone with some background in commutative algebra.