18.S096 Problem Set 7 Fall 2013
Factor Models
Due Date: 11/14/2013

1. Consider a bivariate random variable:

$$\boldsymbol{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

with mean and covariance:

$$E[X] = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}, \text{ and } Cov[\boldsymbol{X}] = \Sigma = \begin{bmatrix} \Sigma_{1,1} & \Sigma_{1,2} \\ \Sigma_{1,2} & \Sigma_{2,2} \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

where $\sigma_1 = \sqrt{\Sigma_{1,1}}$, $\sigma_2 = \sqrt{\Sigma_{2,2}}$ and ρ is the correlation between X_1 and X_2.

Conduct the Principal Components Analysis (PCA) of X:

1(a) Compute the **eigenvalues** $\boldsymbol{\Sigma}$: $\lambda_1 \geq \lambda_2 \geq 0$.

1(b) Compute the **eigenvectors** $\boldsymbol{\gamma}_1, \boldsymbol{\gamma}_2$:

$$\begin{aligned} \boldsymbol{\Sigma}\boldsymbol{\gamma}_i &= \lambda_i \boldsymbol{\gamma}_i, \quad i = 1, 2 \\ \boldsymbol{\gamma}_i'\boldsymbol{\gamma}_i &= 1, \quad i = 1, 2 \\ \boldsymbol{\gamma}_1'\boldsymbol{\gamma}_2 &= 0, \end{aligned}$$

1(c) Demonstrate that:

$$\boldsymbol{\Sigma} = \sum_{i=1}^{m} \lambda_i \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i'$$

1(d) Define the **Principal Component Variables:**

$$p_i = \boldsymbol{\gamma}_i'(\boldsymbol{x} - \boldsymbol{\alpha}), \quad i = 1, 2.$$

Prove that

- $E[p_i] = 0$, $i = 1, 2$.
- $Var(p_i) = \lambda_i$, $i = 1, 2$.
- $Cov(p_1, p_2) = 0$.

1(e) Are the eigenvectors in 1(b) unique? If so, explain why; if not, explain the relationship between different solutions.

2. Let $T : \Re^2 \to \Re^2$ be an affine transformations of $\boldsymbol{X}$ in 2-dimensional space that preserves distances between points. Then, for some orthogonal (2×2) matrix Φ with columns ϕ_1, ϕ_2 ($\Phi = [\phi_1 : \phi_2]$, $\Phi^{-1} = \Phi^T$) and some 2-vector $\mu = (\mu_1, \mu_2)^T$,

$$T(X) = \Phi^T(X - \mu).$$

The transformation T translates the origin to μ and rotates the coordinate axes by an angle specified by Φ (whose elements are cosines/sines of the new coordinate axes relative to the original ones).

2(a) Let ϕ_1 be the orthonormal two-vector which maximizes $Var(\phi_1^T X)$

- Solve for ϕ_1.
- Show that $Var(\phi_1^T X) = \lambda_1$ in problem 1(a).

2(b) Let ϕ_2 be the orthonormal two-vector which minimizes $Var(\phi_2^T X)$

- Solve for ϕ_2
- Show that $Var(\phi_2^T X) = \lambda_2$ in problem 1(a).

2(c) Provide an equivalent definition/specification of principal components analysis in terms of affine transformations of variables and their variances/expectations.

3. Consider the daily yield rate data for constant maturity US Treasury securities, taken from the Federal Reserve Economic Database (FRED); see case study for the lecture on factor modeling. To analyze changes in yields, the daily changes in yield, in basis point units; 1 basis point (BP) $= 0.01 \times 1\%$ were computed for the 9 securities.

   Two principal components analyses are conducted on the data for 2001-2005, the first using the sample covariance matrix and the second using the sample correlation matrix. The results are:

US Tresaury Yield Data: 2001-2005

Principal Components Analysis of Yield Changes Covariance Matrix

Summary:

Importance of components:

	Comp.1	Comp.2	Comp.3	Comp.4	Comp.5
Standard deviation	0.1618033	0.05323436	0.02988340	0.01918045	0.013461432
Proportion of Variance	0.8494434	0.09194833	0.02897475	0.01193650	0.005879525
Cumulative Proportion	0.8494434	0.94139169	0.97036644	0.98230294	0.988182464

	Comp.6	Comp.7	Comp.8	Comp.9
Standard deviation	0.011196400	0.009568498	0.009009979	0.008131889
Proportion of Variance	0.004067397	0.002970621	0.002633948	0.002145570
Cumulative Proportion	0.992249861	0.995220482	0.997854430	1.000000000

Loadings:

Loadings:

	Comp.1	Comp.2	Comp.3	Comp.4	Comp.5	Comp.6	Comp.7	Comp.8	Comp.9
DGS3MO	0.109	-0.523	0.543	-0.541	0.211	0.287			
DGS6MO	0.158	-0.482	0.251	0.256	-0.258	-0.734			
DGS1	0.253	-0.411		0.669		0.524	-0.167		
DGS2	0.390	-0.207	-0.475		0.453		0.503	0.285	0.181
DGS3	0.420		-0.402	-0.296		-0.205	-0.555	-0.420	-0.190
DGS5	0.421	0.108		-0.210	-0.593	0.184	-0.182	0.478	0.345
DGS7	0.401	0.229	0.148		-0.217		0.387		-0.753
DGS10	0.370	0.284	0.271				0.290	-0.619	0.486
DGS20	0.310	0.362	0.394	0.232	0.529	-0.152	-0.375	0.347	

Principal Components Analysis of Yield Changes Correlation Matrix

Summary:

Importance of components:

	Comp.1	Comp.2	Comp.3	Comp.4	Comp.5
Standard deviation	2.6327753	1.1887500	0.56497709	0.39691846	0.26927753
Proportion of Variance	0.7707849	0.1571400	0.03549501	0.01751896	0.00806317
Cumulative Proportion	0.7707849	0.9279249	0.96341989	0.98093885	0.98900202

	Comp.6	Comp.7	Comp.8	Comp.9
Standard deviation	0.207897889	0.148434327	0.13658957	0.122439822
Proportion of Variance	0.004806244	0.002450046	0.00207463	0.001667059
Cumulative Proportion	0.993808264	0.996258311	0.99833294	1.000000000

Loadings:

Loadings:

	Comp.1	Comp.2	Comp.3	Comp.4	Comp.5	Comp.6	Comp.7	Comp.8	Comp.9
DGS3MO	-0.208	0.629	0.583	-0.410	0.228				
DGS6MO	-0.283	0.515		0.490	-0.629	0.118			
DGS1	-0.337	0.270	-0.407	0.378	0.664	-0.233			
DGS2	-0.363		-0.401	-0.360		0.471	-0.377	0.408	-0.216
DGS3	-0.366		-0.272	-0.400	-0.161	0.135	0.343	-0.621	0.284
DGS5	-0.367	-0.162		-0.172	-0.192	-0.528	0.430	0.301	-0.468
DGS7	-0.361	-0.230	0.144			-0.278	-0.221	0.347	0.735
DGS10	-0.352	-0.266	0.255	0.126		-0.175	-0.599	-0.475	-0.331
DGS20	-0.329	-0.334	0.413	0.338	0.195	0.552	0.378		

3(a) Compare the loadings (eigen-vectors) of the first principal component variable for the two cases. The loadings are all positive for the covariance case and all negative for the correlation case. Is the difference in sign meaningful? (Hint: consider the eigen-vector/value decomposition of the matrices; does the decomposition change if any eigen-vector is multiplied by -1?)

3(b) The magnitudes of the loadings for the first principal component has a larger range from smallest to largest for the covariance matrix case compared to the correlation matrix case. The loading on the least variable yield change (DGS3MO) is higher in magnitude for the correlation matrix case. Also, the loadings on the highly variable yield changes are lower in magnitude for correlation matrix case.

Provide a logical explanation for why the range of magnitudes is larger for the covariance case. (Recall that the first principal component variable is the normalized weighted average of the yield-change variables which has the highest variance (the normalized weights have sum-of-squares equal to 1). For the covariance matrix case, the yield-change variables are the original variables while for the correlation matrix case, these yield-change variables have been scaled to have mean 0 and variance 1.)

3(c) Provide an interpretion of the first three principal component variables for the correlation matrix case. Compare these to an interpretation of those for the covariance matrix case.

3(d) If the analysis objective is to model dynamics of the term-structure of interest rates across all tenors, argue why the principal components analysis of the correlation matrix might be preferred to that of the covariance matrix.

4. For the correlation-matrix case of the principal components analysis in problem 3, an order-3 vector autoregression was fit to the first 3 principal component variables ("scores"). The results are as follows:

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VAR Estimation Results:
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Endogenous variables: Comp.1, Comp.2, Comp.3
Deterministic variables: const
Sample size: 1245
Log Likelihood: -5903.314
Roots of the characteristic polynomial:
0.4383 0.4383 0.3602 0.3529 0.3529 0.3364 0.3298 0.3298 0.1959
Call:
VAR(y = obj.princomp0.cor$scores[, 1:3], p = 3)


Estimation results for equation Comp.1:
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========================================
Comp.1 = Comp.1.l1 + Comp.2.l1 + Comp.3.l1 + Comp.1.l2 + Comp.2.l2 + Comp.3.l2 +

          Estimate Std. Error t value Pr(>|t|)
Comp.1.l1  0.05268    0.02835   1.858   0.0634
Comp.2.l1 -0.13399    0.06451  -2.077   0.0380
Comp.3.l1 -0.13215    0.13173  -1.003   0.3160
Comp.1.l2 -0.05296    0.02833  -1.869   0.0618
Comp.2.l2 -0.05416    0.06440  -0.841   0.4005
Comp.3.l2  0.18253    0.13138   1.389   0.1650
Comp.1.l3 -0.01690    0.02820  -0.599   0.5490
Comp.2.l3  0.01300    0.06335   0.205   0.8375
Comp.3.l3  0.08300    0.13131   0.632   0.5274
const     -0.01140    0.07379  -0.154   0.8772


Residual standard error: 2.603 on 1235 degrees of freedom
Multiple R-Squared: 0.01337,    Adjusted R-squared: 0.006181
F-statistic:  1.86 on 9 and 1235 DF,  p-value: 0.05411


Estimation results for equation Comp.2:
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Comp.2 = Comp.1.l1 + Comp.2.l1 + Comp.3.l1 + Comp.1.l2 + Comp.2.l2 + Comp.3.l2 +

           Estimate Std. Error t value Pr(>|t|)
Comp.1.l1 -0.020291   0.012327  -1.646    0.100
Comp.2.l1  0.138152   0.028048   4.926 9.55e-07
Comp.3.l1 -0.083300   0.057279  -1.454    0.146
Comp.1.l2 -0.007222   0.012320  -0.586    0.558
Comp.2.l2  0.037021   0.028001   1.322    0.186
Comp.3.l2 -0.019483   0.057123  -0.341    0.733
Comp.1.l3 -0.008461   0.012260  -0.690    0.490
Comp.2.l3 -0.040270   0.027546  -1.462    0.144
Comp.3.l3  0.014819   0.057093   0.260    0.795
const      0.012147   0.032083   0.379    0.705


Residual standard error: 1.132 on 1235 degrees of freedom
Multiple R-Squared: 0.0288,     Adjusted R-squared: 0.02173
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F-statistic:  4.07 on 9 and 1235 DF,  p-value: 3.656e-05


Estimation results for equation Comp.3:
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Comp.3 = Comp.1.l1 + Comp.2.l1 + Comp.3.l1 + Comp.1.l2 + Comp.2.l2 + Comp.3.l2 +

            Estimate Std. Error t value Pr(>|t|)
Comp.1.l1 -0.0029211  0.0060770  -0.481   0.6308
Comp.2.l1 -0.0142645  0.0138269  -1.032   0.3024
Comp.3.l1  0.0162730  0.0282366   0.576   0.5645
Comp.1.l2 -0.0135254  0.0060731  -2.227   0.0261
Comp.2.l2 -0.0037062  0.0138035  -0.268   0.7884
Comp.3.l2 -0.0522528  0.0281599  -1.856   0.0638
Comp.1.l3 -0.0019415  0.0060438  -0.321   0.7481
Comp.2.l3 -0.0030298  0.0135795  -0.223   0.8235
Comp.3.l3 -0.0711803  0.0281452  -2.529   0.0116
const      0.0001768  0.0158157   0.011   0.9911


Residual standard error: 0.558 on 1235 degrees of freedom
Multiple R-Squared: 0.01317,    Adjusted R-squared: 0.005976
F-statistic: 1.831 on 9 and 1235 DF,  p-value: 0.05868



Covariance matrix of residuals:
         Comp.1   Comp.2    Comp.3
Comp.1 6.777587  0.07254  0.001976
Comp.2 0.072540  1.28138 -0.013479
Comp.3 0.001976 -0.01348  0.311395

Correlation matrix of residuals:
        Comp.1   Comp.2   Comp.3
Comp.1 1.00000  0.02462  0.00136
Comp.2 0.02462  1.00000 -0.02134
Comp.3 0.00136 -0.02134  1.00000
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4(a) Interpret the estimation results for equation Comp.1

4(b) Interpret the estimation results for equation Comp.2

4(c) Interpret the estimation results for equation Comp.3

4(d) For each equation, comment on whether there is evidence of mean-reversion or momentum in the model for the principal component variable. Explain your reasoning.