9.13 Another conjectured instance of tightness

The following problem is posed, by Andrea Montanari, in [Mon14], a description also appears in [Ban15a]. We briefly describe it here as well:

Given a symmetric matrix $W \in \mathbb{R}^{n\times n}$ the positive principal component analysis problem can be written as

$$\begin{array}{ll} \max & x^T W x \\ \text{s. t.} & \|x\| = 1 \\ & x \geq 0 \\ & x \in \mathbb{R}^n. \end{array} \tag{100}$$

In the flavor of the semidefinite relaxations considered in this section, (100) can be rewritten (for $X \in \mathbb{R}^{n\times n}$) as

$$\begin{array}{ll} \max & \mathrm{Tr}(WX) \\ \text{s. t.} & \mathrm{Tr}(X) = 1 \\ & X \geq 0 \\ & X \succeq 0 \\ & \mathrm{rank}(X) = 1, \end{array}$$

and further relaxed to the semidefinite program

$$\begin{array}{ll} \max & \mathrm{Tr}(WX) \\ \text{s. t.} & \mathrm{Tr}(X) = 1 \\ & X \geq 0 \\ & X \succeq 0. \end{array} \tag{101}$$

This relaxation appears to have a remarkable tendency to be tight. In fact, numerical simulations suggest that if W is taken to be a Wigner matrix (symmetric with i.i.d. standard Gaussian entries), then the solution to (101) is rank 1 with high probability, but there is no explanation of this phenomenon. If the Wigner matrix is normalized to have entries $\mathcal{N}(0, 1/n)$, it is known that the typical value of the rank constraint problem is $\sqrt{2}$ (see [MR14]).

This motivates the last open problem of this section.

Open Problem 9.5 *Let W be a gaussian Wigner matrix with entries $\mathcal{N}(0, 1/n)$. Consider the following Semidefinite Program:*

$$\begin{array}{ll} \max & \mathrm{Tr}(WX) \\ \textit{s. t.} & \mathrm{Tr}(X) = 1 \\ & X \geq 0 \\ & X \succeq 0. \end{array} \tag{102}$$

Prove or disprove the following conjectures.

1. *The expected value of this program is $\sqrt{2} + o(1)$.*

2. *With high probability, the solution of this SDP is rank 1.*

References

[Ban15a] A. S. Bandeira. *Convex relaxations for certain inverse problems on graphs*. PhD thesis, Program in Applied and Computational Mathematics, Princeton University, 2015.

[Mon14] A. Montanari. Principal component analysis with nonnegativity constraints. *http://sublinear.info/index.php?title=Open_Problems:62*, 2014.

[MR14] A. Montanari and E. Richard. Non-negative principal component analysis: Message passing algorithms and sharp asymptotics. *Available online at arXiv:1406.4775v1 [cs.IT]*, 2014.