Chapter 9

Gauss Map II

9.1 Mean and Gaussian Curvatures of Surfaces in $\mathbb{R}^3$

We'll assume that the curves are in $\mathbb{R}^3$ unless otherwise noted. We start off by quoting the following useful theorem about self adjoint linear maps over $\mathbb{R}^2$:

Theorem 9.1.1 (Do Carmo pp. 216). *: Let V denote a two dimensional vector space over $\mathbb{R}$. Let $A : V \to V$ be a self adjoint linear map. Then there exists an orthonormal basis e_1, e_2 of V such that $A(e_1) = \lambda_1 e_1$, and $A(e_2) = \lambda_2 e_2$ (that is, e_1 and e_2 are eigenvectors, and λ_1 and λ_2 are eigenvalues of A). In the basis e_1, e_2, the matrix of A is clearly diagonal and the elements λ_1, λ_2, $\lambda_1 \geq \lambda_2$, on the diagonal are the maximum and minimum, respectively, of the quadratic form $Q(v) = \langle Av, v\rangle$ on the unit circle of V.*

Proposition 9.1.2. *: The differential $dN_p : T_p(S) \to T_p(S)$ of the Gauss map is a self-adjoint linear map.*

Proof. Since dN_p is linear, it suffices to verify that $\langle dN_p(w_1), w_2\rangle = \langle w_1, dN_p(w_2)\rangle$ for a basis w_1, w_2 of $T_p(S)$. Let $x(u, v)$ be a parametrization of S at P

and x_u, x_v be the associated basis of $T_p(S)$. If $\alpha(t) = x(u(t), v(t))$ is a parametrized curve in S with $\alpha(0) = p$, we have

$$dN_p(\alpha'(0)) = dN_p(x_u u'(0) + x_v v'(0)) \tag{9.1}$$

$$= \frac{d}{dt} N(u(t), v(t))|_{t=0} \tag{9.2}$$

$$= N_u u'(0) + N_v v'(0) \tag{9.3}$$

in particular, $dN_p(x_u) = N_u$ and $dN_p(x_v) = N_v$. Therefore to prove that dN_p is self adjoint, it suffices to show that

$$\langle N_u, x_v \rangle = \langle x_u, N_v \rangle. \tag{9.4}$$

To see this, take the derivatives of $\langle N, x_u \rangle = 0$ and $\langle N, x_v \rangle = 0$, relative to v and u respectively, and obtain

$$\langle N_v, x_u \rangle + \langle N, x_{uv} \rangle = 0, \tag{9.5}$$

$$\langle N_u, x_v \rangle + \langle N, x_{uv} \rangle = 0, \tag{9.6}$$

Thus,

$$\langle N_u, x_v \rangle = -\langle N, x_{uv} \rangle = \langle N_v, x_u \rangle \tag{9.7}$$

□

Now given that dN_p is self-adjoint one can think of the associated quadratic form.

Definition 9.1.3. *The quadratic form II_p defined in $T_p(S)$ by $II_p(v) = -\langle dN_p(v), v \rangle$ is called the second fundamental form of S at p.*

Now that we have two definitions for the second fundamental form we better show that they're equivalent. (Recall from the last lecture that $II_p(\alpha'(0)) = \langle N(0), \alpha''(0) \rangle$ where α is considered as a function of arc length.)

Let $N(s)$ denote the restriction of normal to the curve $\alpha(s)$. We have $\langle N(s), \alpha'(s)\rangle = 0$ Differentiating yields

$$\langle N(s), \alpha''(s)\rangle = -\langle N'(s), \alpha'(s)\rangle. \tag{9.8}$$

Therefore,

$$\begin{aligned} II_p(\alpha'(0)) &= -\langle dN_p(\alpha'(0)), \alpha'(0)\rangle \\ &= -\langle N'(0), \alpha'(0)\rangle \\ &= \langle N(0), \alpha''(0)\rangle \end{aligned} \tag{9.9}$$

which agrees with our previous definition.

Definition 9.1.4. *: The maximum normal curvature k_1 and the minimum normal curvature k_2 are called the* principal curvatures *at p; and the corresponding eigenvectors are called principal directions at p.*

So for instance if we take cylinder $k_1 = 0$ and $k_2 = -1$ for all points p.

Definition 9.1.5. *: If a regular connected curve C on S is such that for all $p \in C$ the tangent line of C is a principal direction at p, then C is said to be a* line of curvature *of S.*

For cylinder a circle perpendicular to axis and the axis itself are lines of curvature of the cylinder.

Proposition 9.1.6. *A necessary and sufficient condition for a connected regular curve X on S to be a line of curvature of S is that*

$$N'(t) = \lambda(t)\alpha'(t)$$

for any parametrization $\alpha(t)$ of C, where $N(t) = N(\alpha(t))$ and λ is a differentiable function of t. In this case, $-\lambda(t)$ is the principal curvature along $\alpha'(t)$

Proof. : Obvious since principal curvature is an eigenvalue of the linear transformation N'. ☐

A nice application of the principal directions is computing the normal curvature along a given direction of $T_p(s)$. If e_1 and e_2 are two orthogonal eigenvectors of unit length then one can represent any unit tangent vector as

$$v = e_1 \cos\theta + e_2 \sin\theta \tag{9.10}$$

The normal curvature along v is given by

$$\begin{aligned} II_p(v) &= -\langle dN_p(v), v\rangle \\ &= k_1 cos^2\theta + k_2 sin^2\theta \end{aligned} \tag{9.11}$$

Definition 9.1.7. *Let $p \in S$ and let $dN_p : T_p(S) \to T_p(S)$ be the differential of the Gauss map. The determinant of dN_p is the* Gaussian curvature *K at p. The negative of half of the trace of dN_p is called the mean curvature H of S at p.*

In terms of principal curvatures we can write

$$K = k_1 k_2, H = \frac{k_1 + k_2}{2} \tag{9.12}$$

Definition 9.1.8. *: A point of a surface S is called*

1. *Elliptic if $K > 0$,*
2. *Hyperbolic if $K < 0$,*
3. *Parabolic if $K = 0$, with $dN_p \neq 0$*
4. *Planar if $dN_p = 0$*

Note that above definitions are independent of the choice of the orientation.

Definition 9.1.9. *Let p be a point in S. An* asymptotic direction *of S at p is a direction of $T_p(S)$ for which the normal curvature is zero. An* asymptotic curve *of S is a regular connected curve $C \subset S$ such that for each $p \in C$ the tangent line of C at p is an asymptotic direction.*

9.2 Gauss Map in Local Coordinates

Let $x(u,v)$ be a parametrization at a point $p \in S$ of a surface S, and let $\alpha(t) = x(u(t), v(t))$ be a parametrized curve on S, with $\alpha(0) = p$ To simplify the notation, we shall make the convention that all functions to appear below denote their values at the point p.

The tangent vector to $\alpha(t)$ at p is $\alpha' = x_u u + x_v v$ and

$$dN(\alpha') = N'(u(t), v(t)) = N_u u' + N_v v' \tag{9.13}$$

Since N_u and N_v belong to $T_p(S)$, we may write

$$\begin{aligned} N_u &= a_{11}x_u + a_{21}x_v \\ N_v &= a_{12}x_u + a_{22}x_v \end{aligned} \tag{9.14}$$

Therefore,

$$dN = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

with respect to basis $\{x_u, x_v\}$.

On the other hand, the expression of the second fundamental form in the basis $\{x_u, x_v\}$ is given by

$$\begin{aligned} II_p(\alpha') &= -\langle dN(\alpha'), \alpha' \rangle \\ &= e(u')^2 + 2fu'v' + g(v')^2 \end{aligned} \tag{9.15}$$

where, since $\langle N, x_u \rangle = \langle N, x_v \rangle = 0$

$$e = -\langle N_u, x_u \rangle = \langle N, x_{uu} \rangle, \tag{9.16}$$

$$f = -\langle N_v, x_u \rangle = \langle N, x_{uv} \rangle = \langle N, x_{vu} \rangle = -\langle N_u, x_v \rangle \tag{9.17}$$

$$g = -\langle N_v, x_v \rangle = \langle N, x_{vv} \rangle \tag{9.18}$$

From eqns. (11), (12) we have

$$-\begin{pmatrix} e & f \\ f & g \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix}$$

From the above equation we immediately obtain

$$K = \det(a_{ij}) = \frac{eg - f^2}{EG - F^2} \tag{9.19}$$

Formula for the mean curvature:

$$H = \frac{1}{2}\frac{sG - 2fF + gE}{EG - F^2} \tag{9.20}$$

Exercise 3. *Compute H and K for sphere and plane.*

Example 6. *Determine the asymptotic curves and the lines of curvature of the helicoid $x = v\cos u, y = v\sin u, z = cu$ and show that its mean curvature is zero.*