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18.917 Topics in Algebraic Topology: The Sullivan Conjecture  
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# Introduction (Lecture 1)

Let  $X$  be an algebraic variety defined over a field  $k$ . If  $k$  is the field  $\mathbf{C}$  of complex numbers, then  $X$  has an underlying topological space  $X(\mathbf{C})$ . We can therefore study  $X$  using the methods of algebraic topology: for example, for any commutative ring  $R$  we can consider algebraic invariants such as the cohomology ring

$$H^*(X(\mathbf{C}); R).$$

Algebraic geometers have expended a great deal of effort in trying to reconstruct these sorts of invariants in a purely algebraic way. For example, if  $X$  is an elliptic curve over the complex numbers, then the homology group

$$H_1(X(\mathbf{C}), \mathbf{Z}/n\mathbf{Z})$$

is canonically isomorphic to the group of  $n$ -torsion points of  $X$ . This latter definition has the advantage that it makes sense over any field (and behaves well over any algebraically closed field whose characteristic does not divide  $n$ ).

Building on this example, Grothendieck and his school constructed the theory of *étale cohomology*. If  $X$  is an algebraic variety defined over any separably closed field  $k$  and  $R$  is any commutative ring, then one can consider the étale cohomology

$$H_{et}^*(X; R).$$

These groups behave well provided that the ring  $R$  has finite cardinality  $n$ , and the characteristic of  $k$  does not divide  $n$ . Moreover, if  $k$  is the field of complex numbers, then there is a canonical isomorphism

$$H_{et}^*(X; \mathbf{Z}/p\mathbf{Z}) \simeq H^*(X(\mathbf{C}), \mathbf{Z}/p\mathbf{Z})$$

for every integer  $p$ . In other words, étale cohomology provides a purely algebraic recipe for extracting the cohomology groups  $H^*(X(\mathbf{C}); \mathbf{Z}/p\mathbf{Z})$ . This raises the question: to what extent can we recover the topological space  $X(\mathbf{C})$  itself in purely algebraic terms?

Of course, algebro-topological invariants like cohomology do not contain enough information to determine a space like  $X(\mathbf{C})$  up to homeomorphism. The best we might hope for is to recover  $X(\mathbf{C})$  up to homotopy equivalence. However, even this is generally too much to ask for: étale cohomology is generally only well-behaved when the coefficient ring  $R$  is a finite ring, such as  $\mathbf{Z}/p\mathbf{Z}$ . To what extent can a topological space  $M$  be recovered from the cohomology ring  $H^*(M; \mathbf{Z}/p\mathbf{Z})$ ? To answer this question, it is convenient to introduce the notion of  *$p$ -adic completion* for topological spaces, following ideas of Sullivan.

Let  $M$  be a topological space and  $p$  a prime number. For simplicity, we will assume that  $M$  is simply connected, and that each of the homology groups  $H_n(M; \mathbf{Z})$  is finitely generated. In this case, there exists a  *$p$ -adic completion*  $M^\vee$  of  $M$ . It is characterized up to homotopy equivalence by the following properties:

- (1) There is a map  $f : M \rightarrow M^\vee$ .
- (2) The space  $M^\vee$  is simply connected.
- (3) For each  $n > 1$ , the induced map  $H_n(M; \mathbf{Z}) \rightarrow H_n(M^\vee; \mathbf{Z})$  exhibits  $H_n(M^\vee; \mathbf{Z})$  as the  $p$ -adic completion of the (finitely generated) abelian group  $H_n(M; \mathbf{Z})$ .

**Remark 1.** The  $p$ -adic completion of a space  $M$  is, in some sense, determined up to homotopy equivalence by the mod- $p$  cohomology  $H^*(M; \mathbf{Z}/p\mathbf{Z})$ . Of course, for this to be true one must consider  $H^*(M; \mathbf{Z}/p\mathbf{Z})$  as endowed with more structure than just a graded vector space. The precise statement is that  $M^\vee$  can be reconstructed from the cochain complex  $C^*(M; \mathbf{Z}/p\mathbf{Z})$ , viewed as an  $E_\infty$ -algebra over the finite field  $\mathbf{Z}/p\mathbf{Z}$ . This follows from the  $p$ -adic homotopy theory of Mandell and Dwyer-Hopkins, which we will study later in this course.

Building on the work of the Grothendieck school, several mathematicians (Sullivan, Artin-Mazur, Friedlander, and others) have explained how to use the formalism of étale cohomology to extract an  *$p$ -adic étale homotopy type* from an algebraic variety  $X$  defined over an arbitrary separably closed  $k$  of characteristic different from  $p$ . In particular, if  $k$  is the field of complex numbers and the topological space  $X(\mathbf{C})$  is simply connected, the  $p$ -adic étale homotopy type of  $X$  coincides with the  $p$ -adic completion of  $X(\mathbf{C})$  as defined above. This is a strong sense in which one can recover a close approximation to  $X(\mathbf{C})$  by purely algebraic methods.

Suppose that the algebraic variety  $X$  is defined instead over the field  $\mathbb{R}$  of *real* numbers. In this case, one has an underlying topological space of real points  $X(\mathbb{R})$ . To what extent can *this* topological space be reconstructed in purely algebraic terms? We note that there is a canonical inclusion

$$X(\mathbb{R}) \subseteq X(\mathbf{C}).$$

The group  $\mathbf{Z}/2\mathbf{Z}$  acts on  $X(\mathbf{C})$ , via complex conjugation, and we can identify  $X(\mathbb{R})$  with the fixed set  $X(\mathbf{C})^{\mathbf{Z}/2\mathbf{Z}}$  with respect to this action. For every prime number  $p$ , étale homotopy theory allows us to reconstruct the  $p$ -adic completion  $X(\mathbf{C})^\vee$ . Moreover, complex conjugation determines an action of  $\mathbf{Z}/2\mathbf{Z}$  on  $X(\mathbf{C})^\vee$ . It therefore seems natural to consider the fixed point set

$$(X(\mathbf{C})^\vee)^{\mathbf{Z}/2\mathbf{Z}}.$$

However, it is important to keep in mind that the  $p$ -adic completion  $X(\mathbf{C})^\vee$  is defined only up to homotopy equivalence. Consequently, the fixed point set  $(X(\mathbf{C})^\vee)^{\mathbf{Z}/2\mathbf{Z}}$  is not well-defined: given a  $\mathbf{Z}/2\mathbf{Z}$ -equivariant map  $M \rightarrow N$  which is a homotopy equivalence of topological spaces, the induced map

$$M^{\mathbf{Z}/2\mathbf{Z}} \rightarrow N^{\mathbf{Z}/2\mathbf{Z}}$$

need not be a homotopy equivalence. (In fact, we can always arrange that the left side is empty.) To rectify the situation, it is convenient to introduce the notion of *homotopy fixed points*.

Let  $G$  be a group acting on a topological space  $M$ . The fixed point set  $M^G$  can be identified with space of  $G$ -equivariant maps  $\text{Map}_G(*, M)$ ; here  $*$  denotes a point with  $G$  acting trivially. As noted above, the functor  $M \mapsto M^G$  need not preserve homotopy equivalences. One explanation for this is that the  $G$ -space  $*$  is badly behaved. To get a better functor, we need to replace  $*$  by a better  $G$ -space.

**Definition 2.** Let  $G$  be a group. We let  $EG$  denote a contractible space on which  $G$  acts freely, and  $BG$  the quotient space  $EG/G$ .

The topological space  $EG$  always exists, and is unique up to  $G$ -equivariant homotopy equivalence.

**Example 3.** Let  $G$  be the group  $\mathbf{Z}/2\mathbf{Z}$ . Then we can choose  $EG$  to be an infinite dimensional sphere  $S^\infty$ , with  $G$  acting via the antipodal map. In this case, the quotient  $BG = EG/G$  can be identified with the infinite dimensional real projective space  $\mathbf{R}P^\infty$ .

**Definition 4.** Let  $G$  be a group acting on a topological space  $M$ . The *homotopy fixed set*  $M^{hG}$  is the space of  $G$ -equivariant maps  $\text{Map}_G(EG, M)$ .

The homotopy fixed set construction does not suffer the defect of the usual fixed point construction: given a  $G$ -equivariant map  $M \rightarrow N$  which is a homotopy equivalence, the induced map

$$M^{hG} \rightarrow N^{hG}$$

is again a (weak) homotopy equivalence. Moreover, the homotopy fixed point construction is closely related to the usual fixed point construction: every  $G$ -fixed point on a space  $M$  determines a (constant)  $G$ -equivariant map  $EG \rightarrow M$ . This construction yields a natural transformation

$$M^G \rightarrow M^{hG}.$$

Let us now return to our algebraic variety  $X$ , defined over the field of real numbers  $\mathbb{R}$ . We have a sequence of maps

$$X(\mathbb{R}) \simeq X(\mathbf{C})^{\mathbf{Z}/2\mathbf{Z}} \rightarrow X(\mathbf{C})^{h\mathbf{Z}/2\mathbf{Z}} \rightarrow (X(\mathbf{C})^\vee)^{h\mathbf{Z}/2\mathbf{Z}}.$$

Here  $X(\mathbf{C})^\vee$  denotes the  $p$ -adic completion of  $X(\mathbf{C})$ . The left hand side of the above diagram is what we are interested in: the space of  $\mathbb{R}$ -valued points of the algebraic variety  $X$ . The right hand side is what we can understand in purely algebraic terms, using étale homotopy theory. It is natural to ask what happens in between: how good an approximation is  $(X(\mathbf{C})^\vee)^{h\mathbf{Z}/2\mathbf{Z}}$  to  $X(\mathbb{R})$ ? If  $p$  is odd, then  $X(\mathbf{C})$  is generally not a very good approximation at all:

**Example 5.** Let  $X$  be the projective line  $\mathbb{P}^1$ . Then the space  $X(\mathbf{C})$  is isomorphic to the two-sphere  $S^2$ . If  $p$  is an odd prime, then the homotopy groups of the  $p$ -adic completion  $X(\mathbf{C})^\vee$  are all 2-divisible. In this setting, the homotopy fixed point construction behaves like an exact functor: we have canonical isomorphisms

$$\pi_n(X(\mathbf{C})^\vee)^{h\mathbf{Z}/2\mathbf{Z}} \simeq (\pi_n X(\mathbf{C})^\vee)^{\mathbf{Z}/2\mathbf{Z}}.$$

In particular,  $X(\mathbf{C})^\vee)^{h\mathbf{Z}/2\mathbf{Z}}$  is simply connected. However, the actual fixed point set  $X(\mathbf{C})^{\mathbf{Z}/2\mathbf{Z}} \simeq X(\mathbb{R})$  is homeomorphic to a circle, which is definitely not simply connected.

What if  $p = 2$ ? In this case, the algebraic answer  $(X(\mathbf{C})^\vee)^{h\mathbf{Z}/2\mathbf{Z}}$  turns out to be a reasonably close approximation to the topological space  $X(\mathbb{R})$ . This is a consequence of the following conjecture of Sullivan:

**Conjecture 6** (Sullivan). Let  $p$  be a prime number. Let  $M$  be a topological space with an action of a finite  $p$ -group  $G$ . Assume that  $M$  is sufficiently nice (for simplicity, a simply connected finite  $G$ -CW complex). Then the canonical map

$$M^G \rightarrow (M^\vee)^{hG}$$

induces an isomorphism on mod- $p$  cohomology.

This conjecture is interesting (and highly nontrivial) even in the case where the group  $G$  acts trivially on the space  $M$ . In this case, the homotopy fixed set  $(M^\vee)^{hG}$  can be identified with the space of maps  $\text{Map}(BG, M^\vee)$ , and the actual fixed point set can be identified with  $M$  itself. In this case, the conjecture is a consequence of the following theorem of Haynes Miller:

**Theorem 7** (Miller). *Let  $M$  be a finite dimensional CW complex, and let  $G$  be a finite group. Then the space of maps  $\text{Map}(BG, M)$  is homotopy equivalent to  $M$ .*

The general case of Sullivan's conjecture has also been proven (by Carlsson, Lannes, and Miller). The ultimate goal of this course is to give proofs of Conjecture 6 (and Theorem 7). Let us outline our strategy. For simplicity, let us consider the proof of Theorem 7 in the special case where  $G = V \simeq (\mathbf{Z}/p\mathbf{Z})^n$  is an elementary abelian  $p$ -group and  $M$  is simply connected. To prove that the mapping space  $\text{Map}(BG, M)$  is equivalent to  $M$ , it suffices to prove the result after completing  $M$  at each prime number  $q$ . The essential case is that in which  $p = q$ . In this case, the essence of the problem is to compute the cohomology ring

$$H^*(\text{Map}(BG, M^\vee); \mathbf{Z}/p\mathbf{Z}),$$

and to show that it agrees with the cohomology ring

$$H^*(M^\vee; \mathbf{Z}/p\mathbf{Z}) \simeq H^*(M, \mathbf{Z}/p\mathbf{Z}).$$

This raises the question of whether it is possible to compute the cohomology of the mapping space

$$H^*(\text{Map}(BV, Y); \mathbf{Z}/p\mathbf{Z}).$$

Of course, we would expect the answer to be at least as complicated as the cohomology of  $Y$  (since, in the case where  $V$  is the trivial group, we recover precisely the cohomology of  $Y$ ). So the best we might hope for is some recipe which will allow us to recover  $H^*(\text{Map}(BV, Y); \mathbf{Z}/p\mathbf{Z})$  from  $H^*(Y; \mathbf{Z}/p\mathbf{Z})$ .

It turns out that there is such a recipe. However, it requires a very thorough knowledge of the cohomology groups  $H^*(Y; \mathbf{Z}/p\mathbf{Z})$ . To be precise, we need to introduce the notion of a *cohomology operation*.

**Definition 8.** Fix a nonnegative integer  $k$ . A *stable cohomology operation (of degree  $k$ )* is a collection of maps

$$H^m(X, Y; \mathbf{Z}/p\mathbf{Z}) \rightarrow H^{m+k}(X, Y; \mathbf{Z}/p\mathbf{Z}),$$

defined for every pair of spaces  $(X, Y)$  and every integer  $m$ . We require that these operations depend functorially on the pair  $(X, Y)$ , and behave well with respect to the boundary maps in long exact sequences.

The collection of stable cohomology operations forms a graded ring, where multiplication is given by composition. This ring is called the *mod- $p$  Steenrod algebra* and is usually denoted by  $\mathcal{A}$ . If  $Y$  is any topological space, then the cohomology  $H^*(Y; \mathbf{Z}/p\mathbf{Z})$  has the structure of a module over  $\mathcal{A}$ . In fact, a bit more is true:  $H^*(Y; \mathbf{Z}/p\mathbf{Z})$  is an *unstable* module over the Steenrod algebra (we will discuss this condition later). The collection of unstable modules over the Steenrod algebra can be organized into a category  $\mathcal{U}$ .

**Theorem 9.** [Lannes] *Let  $p$  be a prime number, and let  $V$  be an elementary abelian  $p$ -group. There exists a functor  $T_V : \mathcal{U} \rightarrow \mathcal{U}$  with many pleasant properties:*

- (1) *The functor  $T_V$  is exact.*
- (2) *The functor  $T_V$  commutes with tensor products.*
- (3) *The functor  $T_V$  commutes with suspension.*
- (4) *For every topological space  $Y$ , there is a canonical map*

$$T_V H^*(Y; \mathbf{Z}/p\mathbf{Z}) \rightarrow H^*(\text{Map}(BV, Y), \mathbf{Z}/p\mathbf{Z}).$$

*Moreover, this map is an isomorphism if  $Y$  is a sufficiently nice  $p$ -complete space.*

The functor  $T_V$  is called *Lannes'  $T$ -functor*. Because of its many good properties, Lannes'  $T$ -functor is an extremely useful tool. In particular, it can be used to give very elegant proofs of Conjecture 6 and Theorem 7. We now sketch how to use Theorem 9 to prove Theorem 7 in a special case. Assume that  $M$  is the  $p$ -adic completion of a simply connected finite CW-complex. We wish to show that the canonical map

$$M \rightarrow \text{Map}(BV, M)$$

induces an equivalence on mod- $p$  cohomology. In view of Theorem 9, it will suffice to show that the canonical map

$$T_V H^*(M; \mathbf{Z}/p\mathbf{Z}) \rightarrow H^*(M; \mathbf{Z}/p\mathbf{Z})$$

is an isomorphism. This is purely an assertion about the object  $H^*(M; \mathbf{Z}/p\mathbf{Z}) \in \mathcal{U}$ . By assumption,  $H^*(M; \mathbf{Z}/p\mathbf{Z})$  is finite dimensional, and therefore admits a filtration by one-dimensional objects of  $\mathcal{U}$ . Each one dimensional object is a suspension of the trivial module  $H^*(*; \mathbf{Z}/p\mathbf{Z}) \in \mathcal{U}$ . Since  $T_V$  is exact and commutes with suspensions, we can reduce to the case where  $M$  is a single point, where the result is obvious.

Let us now conclude by giving a rough outline of what we will cover in this course (a more detailed outline, which may turn out to be grossly inaccurate, is given in the syllabus). We will begin by giving a construction of the Steenrod algebra  $\mathcal{A}$ , and establishing its basic properties. Though our ultimate goal is to

prove Sullivan’s conjecture, we will take the scenic route: for example, we will begin not with the classical Steenrod algebra but with the “generalized Steenrod algebra” of May, which acts on the homotopy groups of any  $E_\infty$ -algebra over the field  $\mathbf{Z}/p\mathbf{Z}$ . Once we understand the Steenrod algebra well enough, we will proceed to study its category  $\mathcal{U}$  of unstable modules, and introduce the functor  $T_V$ . To establish the basic properties of  $T_V$  (such as exactness), we will need to have a good understanding of the structure of the category  $\mathcal{U}$ ; in particular, we will make a thorough study of injective objects of  $\mathcal{U}$ . Once we are done, we will turn to the proof of Theorem 9. It turns out that the hypothesis that  $X$  be “sufficiently nice” can be dropped if we work in the setting of *p-profinite homotopy theory*. After explaining these ideas, we apply them to give proofs of Conjecture 6 (in the  $p$ -profinite setting) and Theorem 7 (in the setting of classical homotopy theory). If time permits, we will then go on to study the algebraic structure of the category  $\mathcal{U}$  in more detail.

**Warning 10.** The mod- $p$  Steenrod algebra  $\mathcal{A}$  has a somewhat different structure at the prime 2 than at odd primes. It is often difficult to give uniform arguments which apply at all primes simultaneously. To simplify the exposition, *we will assume that the prime  $p$  is equal to 2 whenever it is convenient to do so*. Generally speaking, we will do all calculations at the prime 2, though more conceptual arguments will usually work for any prime  $p$ .