LECTURE 31: COMPLETION OF A DEFERRED PROOF, WHITNEY SUM, AND CHERN CLASSES

1. A deferred proof

From the last lecture, I constructed for a sub-Lie group H of a Lie group G a map

$$\phi: EG/H \to BH,$$

and owed you a proof of

Proposition 1.1. The map ϕ is an equivalence, and the map induced by the inclusion

$$i: H \hookrightarrow G$$

is the quotient map

$$i_*: BH \simeq EG/H \rightarrow EG/G = BG.$$

We first state some easy lemmas.

Lemma 1.2. The induction functor

$$\operatorname{Ind}_{H}^{G} = G \times_{H} (-) : H$$
-spaces $\rightarrow G$ -spaces

is left adjoint to the restriction functor

$$\operatorname{Res}_{H}^{G}: G\text{-spaces} \to H\text{-spaces}$$

which regards a $G\operatorname{-space} X$ as an $H\operatorname{-space}.$ In particular, there is a natural isomorphism

$$\operatorname{Map}_G(G \times_H X, Y) \cong \operatorname{Map}_H(X, Y)$$

for an H-space X and a G-space Y.

Given G-bundles $E \to B$ and $E' \to B'$, a G-equivariant map

$$f: E \to E'$$

gives rise to a map of bundles

$$E \xrightarrow{f} E'$$

$$\downarrow \qquad \qquad \downarrow$$

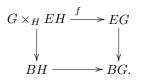
$$B \xrightarrow{f/G} B'$$

Lemma 1.3. There is an equivalence of bundles

$$E \cong (f/G)^* E'.$$

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Sketch proof of Proposition 1.1. We need to construct a map in the opposite direction. The G-bundle $G \times_H EH \to BH$ is classified by a map



Let

$$f: EH \to EG$$

be the *H*-equivariant map adjoint to the map f. Let ψ be the induced map of *H*-orbits:

$$\psi = f/H : BH = EH/H \to EG/H.$$

The composite $\phi \circ \psi$ is seen to be an equivalence because it is covered by the H-equivariant composite

$$EH \xrightarrow{\overline{f}} EG \to EH$$

and thus classifies the universal bundle over BH.

The composite $\psi \circ \phi$ is covered by the *H*-equivariant composite

$$\widetilde{h}: EG \to EH \xrightarrow{f} EG$$

whose adjoint h gives a map of G-bundles

$$\begin{array}{ccc} G \times_H EG \xrightarrow{h} EG \\ & & \downarrow \\ & & \downarrow \\ EG/H \xrightarrow{} BG \end{array}$$

The bundle $G \times_H EG \to EG/H$ is easily seen to be classified by the quotient map $EG/H \to EG/G = BG$. Thus we can conclude that h is G-equivariantly homotopic to the map

$$G \times_H EG \to EG$$

which sends [g, e] to ge. Thus the adjoint \tilde{h} is *H*-equivariantly homotopic to the identity map $EG \to EG$. Taking *H*-orbits, we see that $\psi \circ \phi$ is homotopic to the identity.

2. Whitney sum

Let V be an n-dimensional complex vector bundle over X and W be an m-dimensional complex vector bundle over Y.

Definition 2.1. The *external direct sum* $V \boxplus W$ is the product bundle

$$V \boxplus W = V \times W \to X \times Y$$

where the vector space structure on the fibers is given by the direct sum.

Now assume X = Y.

Definition 2.2. The Whitney sum $V \oplus W$ is the n+m-dimensional complex vector bundle given by the pullback $\Delta^* V \boxplus W$, where

$$\Delta: X \to X \times X$$

is the diagonal.

Let V_{univ}^n be the universal *n*-dimensional complex vector bundle over BU(n). Let

$$f_{n,m}: BU(n) \times BU(m) \to BU(n+m)$$

be the classifying map of $V_{univ}^n \boxtimes V_{univ}^m$. Then if

$$f_V: X \to BU(n)$$
$$f_W: X \to BU(m)$$

classify V and W, respectively, the composite

$$X \xrightarrow{f_V \times f_W} BU(n) \times BU(m) \xrightarrow{f_{n,m}} BU(n+m)$$

classifies $V \oplus W$.

3. Chern classes

Our computation of $H^*(BU(n))$ allows for the definition of characteristic classes for complex vector bundles.

Definition 3.1. Let $V \to X$ be a complex *n*-dimensional vector bundle, with classifying map

$$f_V: X \to BU(n).$$

We define the *i*th Chern class $c_i(V) \in H^{2i}(X;\mathbb{Z})$ to be the induced class $f_V^*(c_i)$ for $1 \leq i \leq n$. We use the following conventions:

$$c_0(V) := 1$$

 $c_i(V) := 0$ for $i > n$.

These classes are *natural*: for a map $f: Y \to X$ we have

$$c_i(f^*V) = f^*c_i(V).$$