LECTURE 30: INDUCED MAPS BETWEEN CLASSIFYING SPACES, $H^*(BU(n))$

1. Induced maps between classifying spaces

Let G be a topological group.

Definition 1.1. Let X be a right G-space and Y be a left G-space. The *Borel construction* is the quotient

$$X \times_G Y = X \times Y/(xg, y) \sim (x, gy).$$

Observe:

$$\begin{aligned} G \times_G Y &= Y \\ * \times_G Y &= Y/G. \end{aligned}$$

Suppose that $\alpha : G \to H$ is a homomorphism of topological groups. We can get an induced map

$$\alpha_* : BG \to BH$$

by providing a natural transformation between the functors that these spaces represent:

$$\alpha_* : \{G\text{-bundles over } X\} \to \{H\text{-bundles over } X\}.$$

Namely, send a G-bundle $E \to X$ to the H-bundle

$$H \times_G E \to X$$

where G acts on H through the homomorphism α.

Thus B may be viewed as a functor

$$\text{Topological groups} \to \text{homotopy category of CW-complexes}.$$

There is one particular case where the induced map admits a more explicit description. Suppose that H is a sub-Lie group of a Lie group G. On a homework problem you showed that $G \to G/H$ was an H-bundle. It follows that the quotient

$$EG \to EG/H$$

is an H-bundle, and is therefore classified by a map ϕ:

$$\begin{array}{ccc} EG & \longrightarrow & EH \\ \downarrow & & \downarrow \\ EG/H & \xrightarrow[\phi]{} & BH \end{array}$$

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Proposition 1.2. The map ϕ is an equivalence, and the map induced by the inclusion

$$i : H \hookrightarrow G$$

is the quotient map

$$i_* : BH \simeq EG/H \to EG/G = BG.$$

A proof of this proposition is deferred to the next lecture.

Corollary 1.3. There is a fiber sequence

$$G/H \to BH \xrightarrow{i_*} BG.$$

2. Classifying vector bundles

Given a paracompact space X, there are $1-1$ correspondences

$$\{U(n)\text{-bundles over } X\}$$
$$\updownarrow$$
$$\{\text{hermitian } n\text{-dimensional complex vector bundles over } X\}$$
$$\updownarrow$$
$$\{n\text{-dimensional complex vector bundles over } X\}.$$

The first correspondence associates to a principle $U(n)$-bundle P over X the vector bundle

$$P \times_{U(n)} \mathbb{C}^n \to X$$

with a fixed hermitian structure on $\mathbb{C}^n$.

Using partitions of unity, every vector bundle admits a hermitian structure, unique up to isomorphism.

3. Calculation of $H^*(BU(n))$

Theorem 3.1. There is an isomorphism

$$H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, \dots, c_n].$$

The generator c_i lies in degree $2i$, and is called the *ith* Chern class. The induced map

$$H^*(BU(n)) \to H^*(BU(n-1))$$

is the quotient by the ideal (c_n).

Proof. We prove this theorem by induction on n, using the Serre spectral sequence of the fiber sequence

$$S^{2n-1} \to BU(n-1) \to BU(n).$$

The E_2-term takes the form

$$E_2^{*,*} = H^*(BU(n)) \otimes \Lambda[e_{2n-1}]$$

where $H^s(BU(n))$ lies in $E_2^{s,0}$ and $|e_{2n-1}| = (0, 2n-1)$. Because this is a 2-line spectral sequence, the only possible non-trivial differential is d_{2n}, and the spectral sequence degenerates into a long exact sequence (the *Gysin sequence*). Because the inductive hypothesis implies that $H^*(BU(n-1))$ is concentrated in even dimensions, we may deduce that the differential $d_{2n}(e_{2n-1})$ is non-trivial, and we define $c_n \in H^n(BU(n))$ to be the image of this differential

$$c_n := d_{2n}(e_{2n-1}).$$

The multiplicative structure implies that d_{2n} is given by cupping with c_n. Thus the E_∞-term is comprised of the elements of $H^*(BU(n))$ annihilated by c_n, and the quotient of $H^*(BU(n))$ by the ideal (c_n). By inducting over k, using the fact that $H^*(BU(n-1))$ is concentrated in even dimensions, we can argue that $H^k(BU(n))$ must be generated in even dimensions, implying that in fact there can be no elements annihilated by c_n. Thus $H^*(BU(n))$ is generated by $c_1, \ldots, c_n$, with

$$H^*(BU(n))/(c_n) = \mathbb{Z}[c_1, \ldots, c_{n-1}].$$

We deduce that there is an isomorphism $H^*(BU(n)) \cong \mathbb{Z}[c_1, \ldots, c_n]$. $\square$