LECTURE 15: EILENBERG-MACLANE SPACES

1. Construction

Let π be a group. Recall, a pointed CW complex X is a $K(\pi, n)$ (Eilenberg-MacLane space) if

$$\pi_k(X) = \begin{cases} \pi & k = n \\ 0 & k \neq n \end{cases}$$

- A $K(\pi, 0)$ is a CW complex with $\pi_0 = \pi$ having contractible components. For example, π is a $K(\pi, 0)$ when viewed as a discrete space.
- π must be abelian for n > 1.

Let n be greater than or equal to 1, and let π be abelian. A pointed CW complex X is a $M(\pi, n)$ (Moore space) if it is (n - 1)-connected and has

$$\widetilde{H}_k(X) = \begin{cases} \pi & k = n \\ 0 & k \neq n \end{cases}$$

Construction of Moore spaces. We will assume for simplicity that $n \ge 2$. Form a short exact sequence of abelian groups.

$$0 \to \bigoplus_J \mathbb{Z} \xrightarrow{\alpha} \bigoplus_I \mathbb{Z} \to \pi \to 0$$

The Hurewicz homomorphism and the universal property of the wedge prove the following lemma.

Lemma 1.1. Assume $n \ge 2$. There is a canonical isomorphism

$$[\bigvee_{J} S^{n}, \bigvee_{I} S^{n}]_{*} \xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}\{J\}, \mathbb{Z}\{I\})$$

where $\mathbb{Z}{S}$ denotes the free abelian group generated by a set S.

There therefore exists a map

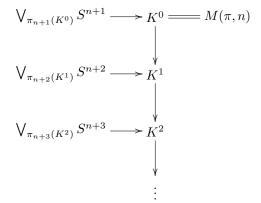
$$\widetilde{\alpha}:\bigvee_IS^n\to\bigvee_JS^n$$

which induces α on H_n . Define $M(\pi, n)$ to be the cofiber $C(\alpha)$. The long exact sequence of a cofiber implies that the homology of $M(\pi, n)$ has the desired properties.

Construction of Eilenberg-MacLane spaces. For simplicity, continue to assume that $n \ge 2$. The arguments have to be slightly altered for n = 1. We construct $K(\pi, n)$ as

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a colimit $\varinjlim K^t$ where we inductively kill the higher homotopy groups of $M(\pi, n)$. Inductively construct cofibers:



The properties of the cofiber imply that if X is another Eilenberg-MacLane space of type (π, n) , then there exists a weak equivalence

$$K(\pi, n) \xrightarrow{\cong} X.$$

Since X is a CW complex, we deduce that this must be a homotopy equivalence. We therefore have the following proposition.

Proposition 1.2. Any two Eilenberg-MacLane spaces of type (π, n) are homotopy equivalent.

Corollary 1.3. There are homotopy equivalences

$$K(\pi, n) \xrightarrow{\simeq} \Omega K(\pi, n+1).$$

In particular, $K(\pi, n)$'s are infinite loop spaces.

Corollary 1.4. A $K(\pi, n)$ is a homotopy commutative *H*-space.

2. Representing cohomology

We wish to prove the following theorem.

Theorem 2.1. There is a natural transformation of homotopy invariant functors $\text{Top}_* \rightarrow \text{AbelianGroups}$

$$[-, K(\pi, n)]_* \to H^n(-, \pi).$$

This natural transformation is an isomorphism when restricted to the subcategory of pointed CW complexes.

In this lecture, we merely construct the natural transformation. This is obtained by producing a "fundamental class"

$$[\iota_n] \in \widetilde{H}^n(K(\pi, n), \pi)$$

This should be thought of as the universal cohomology class. This trick is often used when dealing with representable functors. It induces a transformation (natural in X)

$$\eta_X : [X, K(\pi, n)]_* \to H^n(X, \pi)$$

by $\eta_X[f] = f^*[\iota_n].$

The fundamental class is obtained by the universal coefficient theorem. There is a short exact sequence

 $0 \to \operatorname{Ext}^1_{\mathbb{Z}}(\widetilde{H}_{n-1}(K(\pi,n)),\pi) \to \widetilde{H}^n(K(\pi,n),\pi) \to \operatorname{Hom}_{\mathbb{Z}}(\widetilde{H}_n(K(\pi,n)),\pi) \to 0$ which, using the Hurewicz theorem, gives an isomorphism

$$\widetilde{H}^n(K(\pi, n), \pi) \xrightarrow{\cong} \operatorname{Hom}_{\mathbb{Z}}(\pi, \pi).$$

The fundamental class $[\iota_n]$ corresponds to the identity map $Id:\pi\to\pi.$