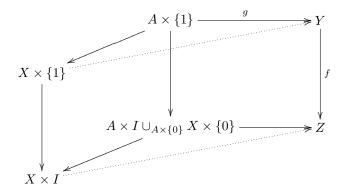
LECTURE 11: HELP! WHITEHEAD THEOREM AND CELLULAR APPROXIMATION

1. HELP

The proof of the Whitehead theorem last time did not extend to infinite CWcomplexes due to a technical difficulty with inverse limits. Rather that attempt to address this difficulty, we will instead use a more subtle homotopy extension property possessed by relative CW-complexes. This is the Homotopy Extension and Lifting Property (HELP):

Theorem 1.1 (HELP). Suppose that (X, A) is a relative CW-complex of dimension $\leq n$ and that $f: Y \to Z$ is an *n*-equivalence (where *n* may be chosen to be ∞). Then for each diagram as below there exist compatible extensions and lifts



In words: "for every map $g: A \to Y$ such that fg is homotopic to a map which extends to a map \tilde{g} over X, there is an extension \tilde{g}' of g over X, such that $f\tilde{g}'$ is homotopic to \tilde{g} , by a homotopy extending the original homotopy."

The proof of HELP is obtained by first considering the case $(X, A) = (D^n, S^{n-1})$ and then performing induction on the relative skeleta of (X, A).

Following May, the following Whitehead theorem may be deduced by clever application of HELP.

Theorem 1.2 (Whitehead theorem). Suppose that Z is a CW-complex of dimension $< n \le \infty$, and that $f: X \to Y$ is an *n*-equivalence. Then the induced map

$$[Z, X] \to [Z, Y]$$

is an isomorphism.

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2. Cellular Approximation

The Whitehead theorem implies that CW-complexes are good when dealing with weak equivalences. Not every space has the homotopy type of a CW-complex, however, every space is weakly equivalent to a CW-complex.

Theorem 2.1 (Cellular approximation). Suppose that X is a space. Then there exists a CW-complex \widetilde{X} together with a weak equivalence

 $\widetilde{X} \to X.$

Proof. Assume X is path connected. Start by mapping in a wedge of spheres $\widetilde{X}^{[0]}$, one for every generator of every homotopy group of X. Then, inductively by degree k, attach reduced cylinders to $\widetilde{X}^{[k-1]}$ for every pair of elements of $\pi_k(\widetilde{X}^{[k-1]})$ which map to the same element in $\pi_k(X)$. Use the homotopies to produce a map $\widetilde{X}^{[k]} \to X$. which is a k-equivalence.

Remark 2.2. The complex \widetilde{X} in the previous proof is technically not a CW complex, but rather a cell complex: the cell attachments are not ordered by dimension. This is OK: the attaching maps of k-cells factor up to homotopy through the (k-1)-skeleta.