LECTURE 8: PUPPE SEQUENCES

Let $f : X \to Y$ be a map of pointed spaces. Taking iterated cofibers, we obtain an infinite sequence

$$X \xrightarrow{f} Y \xrightarrow{j} C(f) \xrightarrow{\pi} \Sigma X \xrightarrow{-\Sigma f} \Sigma Y \xrightarrow{-\Sigma j} \Sigma C(f) \xrightarrow{-\Sigma\pi} \Sigma^2 X \xrightarrow{\Sigma^2 f} \cdots$$

The minus signs means take the degree -1 map on the suspension coordinate. This sequence $\{Z_i\}$ has the property that at each stage there is a homotopy commutative diagram

$$\begin{tikzcd}
Z_i \arrow[r, "g"] \arrow[d, equal] & Z_{i+1} \arrow[r] \arrow[d, equal] & C(g) \arrow[d, "\simeq"'] \\
Z_i \arrow[r, "g"'] & Z_{i+1} \arrow[r] & Z_{i+2}
\end{tikzcd}$$

whose vertical maps are homotopy equivalences.

We deduce that for a pointed space Z, there is a LES

$$[X, Z]_* \xleftarrow{f^*} [Y, Z]_* \leftarrow [C(f), Z]_* \leftarrow [\Sigma X, Z]_* \leftarrow \cdots$$

Remark 0.1. The set $[\Sigma^k X, Z]_*$ is a group for $k \geq 1$, and is an abelian group for $k \geq 2$.

Remark 0.2. Taking $Z = K(\pi, n)$ for n large, we recover the LES for a pair

$$\widetilde{H}^n(X, \pi) \xleftarrow{f^*} \widetilde{H}^n(Y, \pi) \leftarrow H^n(Y, X, \pi) \leftarrow \widetilde{H}^{n-1}(X, \pi) \leftarrow \cdots$$

Dually, taking iterated fibers, we obtain an infinite sequence

$$\cdots \xrightarrow{\Omega^2 f} \Omega^2 Y \xrightarrow{-\Omega\pi} \Omega F(f) \xrightarrow{-\Omega j} \Omega X \xrightarrow{-\Omega f} \Omega Y \xrightarrow{\pi} F(f) \xrightarrow{j} X \xrightarrow{f} Y$$

For a pointed space Z this induces a long exact sequence

$$\cdots \to [Z, \Omega Y]_* \to [Z, F(f)]_* \to [Z, X]_* \xrightarrow{f_*} [Z, Y]_*$$

Specializing to $Z = S^0$, we recover the long exact sequence for the homotopy fiber.

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