## **HOMEWORK 12**

DUE: 5/15/06

All base spaces are assumed to be paracompact.

1. (a) Show that there is a homeomorphism

$$(X \times Y)^{V \boxplus W} \approx X^V \wedge Y^W.$$

(b) Deduce that there is a homeomorphism

 $X^{V \oplus \mathbb{R}^k} \approx \Sigma^k X^V$ 

where  $\mathbb{R}^k$  is the trivial bundle over X.

2. Show that if V is a vector bundle with a non-vanishing section, then the Euler class e(V) must vanish. (Note: if X were a manifold, then this would be what you would expect from the geometric description I gave you in class.)

## Gysin maps

The next two problems investigate a map which goes the "wrong way" in cohomology called the Gysin map. From now on we *always* work with homology with mod 2 coefficients to avoid having to discuss orientations, and manifolds are assumed to be smooth, connected, closed, and compact.

Let  $i: N \hookrightarrow M$  be the inclusion of a submanifold of a manifold M, with dim N = nand dim M = m. Give the tangent bundle TM a metric, and define  $\nu = TN^{\perp}$ to be the normal bundle of N in TM. The "tubular neighborhood theorem" of differential topology asserts that there is a tubular neighbor Tube(N) of N in Mwhose closure Tube(N) is diffeomorphic to the disk bundle  $D(\nu)$ . Let

$$P: M \to \text{Tube}(N)/\partial \text{Tube}(N) \approx N^{*}$$

be the map which sends all points outside of Tube(N) to the basepoint. This map is called the *Pontryagin-Thom* collapse map. It induces, via the Thom isomorphism, a map going in the wrong way called a *Gysin* map:

$$i_!: H^*(N) \cong \widetilde{H}^{*+m-n}(N^{\nu}) \xrightarrow{P^*} H^{*+m-n}(M).$$

In particular, we get a (mod 2) cohomology class [N] whose dimension is the codimension of N in M:

$$[N] := i_!(1) \in H^{m-n}(M).$$

3. Verify that for the inclusion of a point  $* \hookrightarrow M$ , the class  $[*] \in H^m(M)$  is dual to the fundamental class  $[M] \in H_m(M)$ .

4. A pair of submanifolds  $N_1$  and  $N_2$  of dimensions  $n_1$  and  $n_2$ , respectively, are said to be *transverse* in M if for each point  $x \in N_1 \cap N_2$ , the tangent space  $TM_x$ 

is spanned by the subspaces  $(TN_1)_x$  and  $(TN_2)_x$ . The implicit function theorem then may be used to show that  $N_1 \cap N_2$  is a submanifold of dimension  $n_1 + n_2 - m$ , with tangent bundle  $TN_1 \cap TN_2 \hookrightarrow TM$ .

Verify the formula

$$[N_1] \cup [N_2] = [N_1 \cap N_2] \in H^{2m - n_1 - n_2}(M).$$

In other words, for geometric cocycles in general position, the cup product is given by intersection.