# Problem Set 1

18.904 Spring 2011

**Instructions.** Write-up solutions in Latex, print them out and hand them in at the beginning of class on Tuesday, February 22nd. See the website for additional instructions.

**Problem 1.** Let $n \geq 1$ be an integer. Let $\mathbf{CP}^n$ denote the set of all lines in $\mathbf{C}^{n+1}$ passing through the origin. There is a natural map $\pi : \mathbf{C}^{n+1} \setminus \{0\} \to \mathbf{CP}^n$ taking a point to the line it spans. We give $\mathbf{CP}^n$ the quotient topology, so that a set $U$ in $\mathbf{CP}^n$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbf{C}^{n+1}$. Let $U_i \subset \mathbf{CP}^n$ denote the set of points of the form $\pi(x_0, \ldots, x_n)$ where $x_i \neq 0$.

(a) Show that the $U_i$ form an open cover of $\mathbf{CP}^n$.
(b) Show that an intersection of $k+1$ distinct elements of $\{U_0, \ldots, U_n\}$ is homeomorphic to $(\mathbf{C}^\times)^k \times \mathbf{C}^{n-k}$, for $0 \leq k \leq n$. (In particular, each $U_i$ is homeomorphic to $\mathbf{C}^n$.)
(c) Prove the following lemma. Let $X$ be a topological space and let $\mathscr{U}$ be a finite open cover of $X$. Suppose that each element of $\mathscr{U}$ is simply connected and any intersection of elements of $\mathscr{U}$ is non-empty and path-connected. Then $X$ is simply connected. [Hint: use van Kampen's theorem.]
(d) Conclude that $\mathbf{CP}^n$ is simply connected.

*Remark.* The space $\mathbf{CP}^n$ is called *complex projective space*. It is a very important space that shows up in all areas of mathematics. The space $\mathbf{CP}^1$ is called the *Riemann sphere*; it is homeomorphic to $S^2$ (convince yourself of this!).

**Problem 2.** Let $X$ be a topological space and let $x_1$ and $x_2$ be two points in $X$. Given a path $h$ between $x_1$ and $x_2$, we have seen that there is a canonical isomorphism

$$i_h : \pi_1(X, x_1) \to \pi_1(X, x_2).$$

Write $C(G)$ for the set of conjugacy classes in a group $G$, and let

$$\bar{i}_h : C(\pi_1(X, x_1)) \to C(\pi_1(X, x_2))$$

denote the map induced by $i_h$.

(a) Give an example (i.e., specify $X$, $x_1$, $x_2$, $h$ and $h'$) where $i_h \neq i_{h'}$, with proof.
(b) Show that $\bar{i}_h = \bar{i}_{h'}$ for any two paths $h$ and $h'$.
(c) Assume $\pi_1(X, x_1)$ is abelian. Show that $i_h = i_{h'}$ for any $h$ and $h'$.

*Remark.* Note the contrast between (a) and (b) — given two choices of basepoints $x_1$ and $x_2$, the sets $C(\pi_1(X, x_1))$ and $C(\pi_1(X, x_2))$ are in canonical bijection (assuming $X$ is path connected), while the groups $\pi_1(X, x_1)$ and $\pi_1(X, x_2)$ are not.

*Remark.* The fact that $i_h$ and $i_{h'}$ are not necessarily equal is why $\pi_1$ is only a functor for basepoint preserving maps.

**Problem 3.** Let $X$ be a metric (and thus topological) space. Fix a basepoint $x_0$ in $X$; the word "loop" will mean "loop based at $x_0$" in this problem. Let $\Omega X$ denote the set of all loops in $X$, i.e., the set of all continuous functions $p : [0, 1] \to X$ with $p(0) = p(1) = x_0$. Define a distance function on $\Omega X$ by $d(p_1, p_2) = \max_{x \in [0,1]} d(p_1(x), p_2(x))$.

(a) Show that concatentation of loops defines a continuous map $\Omega X \times \Omega X \to \Omega X$. Conclude that there is a natural map of sets $\pi_0(\Omega X) \times \pi_0(\Omega X) \to \pi_0(\Omega X)$. [Here $\pi_0$ denotes the set of path components.]
(b) Show that two loops in $X$ are homotopic if and only if the corresponding points of $\Omega X$ are in the same path component.

(c) Construct a canonical bijection of sets $\pi_0(\Omega X) \to \pi_1(X, x_0)$. Show that this map is a homorphism, in the sense that it respects the multiplications on the two sets (the one on $\pi_0(\Omega X)$ constructed in (a) and the usual group operation on $\pi_1(X, x_0)$).

*Remark.* In fact, the bijection from (c) is just the first in a sequence: there are natural group isomorphisms $\pi_{n-1}(\Omega X, x_0) \to \pi_n(X, x_0)$ for all $n \geq 1$. [Here $\pi_n$ denotes the $n$th homotopy group.]

*Remark.* The above theory does not at all require $X$ to be a metric space, it just simplifies the definition of the topology on $\Omega X$. When $X$ is a general topological space, the appropriate topology on $\Omega X$ is the "compact open topology."

**Problem 4.** In this problem, we will show that every finitely presented group occurs as a fundamental group.

(a) Let $G$ be a group, let $a$ be an element of $G$ and let $N$ be the normal closure of the subgroup generated by $a$. [Explicitly, $N$ is the subgroup of $G$ generated by all conjugates of $a$.] Let $\mathbf{Z} \to G$ be the map defined by $1 \mapsto a$. Show that the amalgamated free product $G *_{\mathbf{Z}} 1$ is isomorphic to $G/N$. [Here 1 denotes the trivial group.]

(b) Let $X$ be a topological space with base point $x_0$ and let $i : S^1 \to X$ be a loop based at $x_0$. Let $X'$ be the topological space obtained by attaching a 2-disc to $X$ via $i$; that is, $X'$ is the quotient of $X \amalg D^2$ where an element $x \in S^1 = \partial D^2$ is identified with $i(x) \in X$. Show that $\pi_1(X', x_0)$ is the quotient of $\pi_1(X, x_0)$ by the normal subgroup generated by the class of $i$. [Hint: use van Kampen's theorem.]

(c) Show that every finitely presented group occurs as a fundamental groups. [Hint: let $G$ be a finitely presented group. Pick a presentation. Start with a bouquet of circles, one for each generator. Attach a 2-disc for each relation and apply (b).]

*Remark.* The requirement that the group be finitely generated is completely unnecessary. The general case can be established by the same means.

*Remark.* There can be many very different homotopy types that have isomorphic fundamental groups; for instance, both $S^1$ and $S^1 \vee S^2$ have fundamental group $\mathbf{Z}$. However, given a group $G$ there is a *unique* homotopy type with fundamental group $G$ and contractible universal cover (or equivalently, with all other homotopy groups vanishing).

**Problem 5.** Let $G$ be a topological group; thus $G$ is simulateneously a group and a topological space, and the multiplication map $G \times G \to G$ and inversion map $G \to G$ are continuous.

(a) Show that there is a unique group structure on $\pi_0(G)$ such that the natural map $G \to \pi_0(G)$ is a group homomorphism.

(b) Show that $\pi_1(G, 1)$ is a commutative group. [Hint: if $c$ is a loop in $G$ based at 1 and $g$ is an element of $G$ then $t \mapsto gc(t)$ is a loop in $G$ based at $g$. Using this you can slide one loop along another to show that they commute in $\pi_1$.]

**Problem 6.** Let $G = \mathrm{SL}(2, \mathbf{R})$, the group of $2 \times 2$ real matrices with determinant 1. We can naturally regard $G$ as a closed subset of $\mathbf{R}^4$, and thus (after a few simple verifications) as a topological group. Let $B^\circ \subset G$ be the subgroup of matrices which are upper-triangular with positive entries on the diagonal. Let $K \subset G$ be the subgroup of rotations matrices. [An element of $G$ belongs to $K$ if and only if its two columns form an orthonormal basis of $\mathbf{R}^2$.]

(a) Show that $B^\circ$ is homeomorphic to $\mathbf{R}^2$, and is thus contractible.

(b) Show that $K$ is homeomorphic to $S^1$.

(c) Show that the map $B^\circ \times K \to G$ sending $(b, k)$ to $bk$ is a homeomorphism.

(d) Conclude that $G$ is homotopy equivalent to $S^1$, and thus has fundamental group $\mathbf{Z}$.

*Remark.* The group $\mathrm{SL}(2, \mathbf{R})$ is better than just a topological group: as a topological space it is actually a smooth manifold, and the group operations are smooth maps. Such topological groups are called *Lie groups.* They are among the most important objects in all of mathematics.

18.904 Seminar in Topology
Spring 2011