18.786 supplement: linear disjointness

Let me try to alleviate a bit of the confusion about the concept of linear disjointness (thanks to Dani Kane for helping straighten some of this out). This corrects some assertions from the March 14 lecture.

Let L_1, L_2 be finite extensions of a field K. We say that L_1 and L_2 are *linearly disjoint* if the K-algebra $L_1 \otimes_K L_2$ is a field (note that it is enough for it to be an integral domain, since we proved in class that any integral domain which is integral over a field is also a field). If so, it is then isomorphic to the compositum L_1L_2 within any overfield L_3 containing both L_1 and L_2 (because the multiplication map from $L_1 \otimes_K L_2$ to L_3 will be injective and its image will be a field containing both L_1 and L_2, so L_1L_2 can be no larger).

It is not true in general that just knowing that $L_1 \cap L_2 = K$ inside some overfield L_3 is enough to say that L_1 and L_2 are linearly disjoint. (For instance, look at two different copies of $\mathbb{Q}[x]/(x^3 - 2)$ inside $\mathbb{C}$; their compositum has degree 6, not 9, over $\mathbb{Q}$.)

However, if L_1 and L_2 are *Galois* over K, then it is true that the equality $L_1 \cap L_2 = K$ in any overfield implies that L_1 and L_2 are linearly disjoint. Proof: we first check that the compositum of two Galois extensions is Galois (without any extra hypothesis). Let L_3 be the compositum in some overfield. Since L_1/K is separable, it is generated by a root of a separable polynomial; that root also generates L_3 over L_2, so L_3/L_2 is separable. Since L_3/L_2 and L_2/K are separable, so is L_3/K. Since L_1 and L_2 are Galois, they are normal: they are splitting fields for some polynomials P_1 and P_2 over K. Then L_3 is a splitting field for P_1P_2, so it is also normal over K. Since L_3/K is normal and separable, it is Galois (but this, or more precisely the fact that normal plus separable implies that the automorphism group is as big as the extension degree, is nontrivial Galois theory!).

Put $G_i = \mathrm{Gal}(L_i/K)$; then G_3 surjects onto G_1 and G_2 via the Galois correspondence. Let H_i be the kernel of $G_3 \to G_i$ for $i = 1, 2$; then $H_1 \cap H_2$ fixes both L_1 and L_2, and since L_3 is the compositum of those, $H_1 \cap H_2$ must fix L_3. Hence $H_1 \cap H_2 = \{e\}$.

Finally, the hypothesis $L_1 \cap L_2 = K$ implies, via the Galois correspondence, that there is no normal subgroup of G_3 containing both H_1 and H_2; that is, $H_1H_2 = G_3$. We have maps $G_3 \to G_1$ and $G_3 \to G_2$ giving a map $G_3 \to G_1 \times G_2$; by elementary group theory, this map must now be an isomorphism. In particular, $[L_3 : K] = [L_1 : K][L_2 : K]$, so the surjection $L_1 \otimes_K L_2 \to L_3$ must be an isomorphism.