23 Tate cohomology

In this lecture we introduce a variant of group cohomology known as *Tate cohomology*, and we define the *Herbrand quotient* (a ratio of cardinalities of two Tate cohomology groups), which will play a key role in our proof of Artin reciprocity. We begin with a brief review of group cohomology, restricting our attention to the minimum we need to define the Tate cohomology groups we will use. At a number of points we will need to appeal to some standard results from homological algebra whose proofs can be found in Section 23.6. For those seeking a more thorough introduction to group cohomology, see [1]; for general background on homological algebra, we recommend [7].

23.1 Group cohomology

Definition 23.1. Let G be a group. A *G-module* is an abelian group A equipped with a G-action compatible with its group structure: $g(a+b) = ga + gb$ for all $g \in G, a, b \in A$.[1] This implies $|ga| = |a|$ (where $|a| \coloneqq \#\langle a\rangle$ is the order of a); in particular $ga = 0 \Leftrightarrow a = 0$.

A *trivial* G-module is an abelian group with trivial G-action: $ga = a$ for all $g \in G, a \in A$ (so every abelian group can be viewed as a trivial G-module). A morphism of G-modules is a morphism of abelian groups $\alpha\colon A \to B$ satisfying $\alpha(ga) = g\alpha(a)$. Kernels, images, quotients, and direct sums of G-modules are also G-modules.

Definition 23.2. Let A be a G-module. The *G-invariants* of A constitute the G-module

$$A^G \coloneqq \{a \in A : ga = a \text{ for all } g \in G\}$$

consisting of elements fixed by G. It is the largest trivial G-submodule of A.

Definition 23.3. Let A be a G-module and let $n \in \mathbb{Z}_{\geq 0}$. The group of *$n$-cochains* is the abelian group $C^n(G,A) \coloneqq \operatorname{Map}(G^n, A)$ of maps of sets $f\colon G^n \to A$ under pointwise addition. We have $C^0(G,A) \simeq A$, since $G^0 = \{1\}$ is a singleton set. The *nth coboundary map* $d^n\colon C^n(G,A) \to C^{n+1}(G,A)$ is the homomorphism of abelian groups defined by

$$\begin{aligned} d^n(f)(g_0,\ldots,g_n) \coloneqq{} & g_0 f(g_1,\ldots,g_n) - f(g_0g_1, g_2, \ldots, g_1) + f(g_0, g_1g_2, \ldots, g_n) \\ & \cdots + (-1)^n f(g_0,\ldots,g_{i-1}, g_{n-1}g_n) + (-1)^{n+1} f(g_0,\ldots,g_{n-1}). \end{aligned}$$

The group $C(G,A)$ contains subgroups of *n-cocycles* and *n-coboundaries* defined by

$$Z^n(G,A) \coloneqq \ker d^n \qquad \text{and} \qquad B^n(G,A) \coloneqq \operatorname{im} d^{n-1},$$

with $B^0(G,A) \coloneqq \{0\}$.

The coboundary map satisfies $d^{n+1} \circ d^n = 0$ for all $n \geq 0$ (this can be verified directly, but we will prove it in the next section), thus $B^n(G,A) \subseteq Z^n(G,A)$ for $n \geq 0$ and the groups $C^n(G,A)$ with connecting maps d^n form a *cochain complex*

$$0 \longrightarrow C^0(G,A) \xrightarrow{d^0} C^1(G,A) \xrightarrow{d^1} C^2(G,A) \longrightarrow \cdots$$

that we may denote $\mathcal{C}_A$. In general, a cochain complex (of abelian groups) is simply a sequence of homomorphisms d^n that satisfy $d^{n+1} \circ d^n = 0$. Cochain complexes form a category whose morphisms are commutative diagrams with cochain complexes as rows.

[1] Here we put the G-action on the left (one can also define right G-modules), and for the sake of readability we write A additively, even though we will be primarily interested in cases where A is a multiplicative group.

Definition 23.4. Let A be a G-module. The nth *cohomology group of G with coefficients in A* is the abelian group

$$H^n(G,A) \coloneqq Z^n(G,A)/B^n(G,A).$$

Example 23.5. We can work out the first few cohomology groups explicitly by writing out the coboundary maps and computing kernels and images:

- $d^0\colon C^0(G,A) \to C^1(G,A)$ is defined by $d^0(a)(g) \coloneqq ga - a$ (note $C^0(G,A) \simeq A$).
- $H^0(A,G) \simeq \ker d^0 = A^G$ (note $B^0(G,A) = \{0\}$).
- $\operatorname{im} d^0 = \{f\colon G \to A \mid \exists a \in A : f(g) = ga - a \text{ for all } g \in G\}$ (*principal crossed homomorphisms*).
- $d^1\colon C^1(G,A) \to C^2(G,A)$ is defined by $d^1(f)(g,h) \coloneqq gf(h) - f(gh) + f(g)$.
- $\ker d^1 = \{f\colon G \to A \mid f(gh) = f(g) + gf(h) \text{ for all } g \in G\}$ (*crossed homomorphisms*).
- $H^1(G,A) =$ *crossed homomorphisms* modulo *principal crossed homomorphisms*.
- If A is a trivial G-module then $H^1(G,A) \simeq \operatorname{Hom}(G,A)$.

Lemma 23.6. *Let $\alpha\colon A \to B$ be a morphism of G-modules. We have induced group homomorphisms $\alpha^n\colon C^n(G,A) \to C^n(G,B)$ defined by $f \mapsto \alpha \circ f$ that commute with the coboundary maps. In particular, α^n maps cocycles to cocycles and coboundaries to coboundaries and thus induces homomorphisms $\alpha^n\colon H^n(G,A) \to H^n(G,B)$ of cohomology groups, and we have a morphism of cochain complexes $\alpha\colon \mathcal{C}_A \to \mathcal{C}_B$:*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^0(G,A) & \xrightarrow{d^0} & C^1(G,A) & \xrightarrow{d^1} & C^2(G,A) & \xrightarrow{d^2} & \cdots \\
 & & \downarrow{\scriptstyle\alpha^0} & & \downarrow{\scriptstyle\alpha^1} & & \downarrow{\scriptstyle\alpha^2} & & \\
0 & \longrightarrow & C^0(G,B) & \xrightarrow{d^0} & C^1(G,B) & \xrightarrow{d^1} & C^2(G,B) & \xrightarrow{d^2} & \ldots
\end{array}$$

Proof. Consider any $n \geq 0$. For all $f \in C^n(G,A)$, and $g_0,\ldots,g_n \in G$ we have

$$\begin{aligned}
\alpha^{n+1}(d^n(f)(g_0,\ldots,g_n)) &= \alpha^{n+1}\big(g_0 f(g_1,\ldots,g_n) - \cdots + (-1)^{n+1} f(g_0,\ldots,g_{n-1})\big)\\
&= g_0(\alpha\circ f)(g_1,\ldots,g_n) - \cdots + (-1)^{n+1}(\alpha\circ f)(g_0,\ldots,g_{n-1})\\
&= d^n(\alpha\circ f)(g_0,\ldots,g_n) = d^n(\alpha^n(f))(g_0,\ldots,g_n),
\end{aligned}$$

thus $\alpha^{n+1}\circ d^n = d^n \circ \alpha^n$. The lemma follows. □

Lemma 23.6 implies that we have a family of functors $H^n(G,\bullet)$ from the category of G-modules to the category of abelian groups (note that $\mathrm{id}\circ f = f$ and $(\alpha\circ\beta)\circ f = \alpha\circ(\beta\circ f)$), and also a functor from the category of G-modules to the category of cochain complexes.

Lemma 23.7. *Suppose that we have a short exact sequence of G-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

Then for every $n \geq 0$ we have a corresponding exact sequence of n-cochains

$$0 \longrightarrow C^n(G,A) \xrightarrow{\alpha^n} C^n(G,B) \xrightarrow{\beta^n} C^n(G,C) \longrightarrow 0.$$

Proof. The injectivity of α^n follows from the injectivity of α. If $f \in \ker \beta^n$, then $\beta \circ f = 0$ and $\operatorname{im} f \subseteq \ker \beta = \operatorname{im} \alpha$. Via the bijection $\alpha^{-1}\colon \operatorname{im} \alpha \to A$ we can define $\alpha^{-1} \circ f \in C^n(G, A)$, thus $\operatorname{im} \alpha^n \subseteq \ker \beta^n$. We also have $\ker \beta^n \subseteq \alpha^n$, since $\beta \circ \alpha \circ f = 0 \circ f = 0$ for all $f \in C^n(G, B)$, and exactness at $C^n(G, B)$ follows. Every $f \in C^n(G, C)$ satisfies $\operatorname{im} f \subseteq C = \operatorname{im} \beta$, and we can define $h \in C^n(G, B)$ satisfying $\beta \circ h = f$: for each $g_0, \ldots, g_n$ let $h(g_0, \ldots, g_n)$ be any element of $\beta^{-1}(f(g_0, \ldots, g_n))$. Thus $f \in \operatorname{im} \beta^n$ and β^n is surjective. $\square$

Lemmas 23.6 and 23.7 together imply that we have an exact functor from the category of G-modules to the category of cochain complexes.

Theorem 23.8. *Every short exact sequence of G-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

induces a long exact sequence of cohomology groups

$$0 \to H^0(G, A) \xrightarrow{\alpha^0} H^0(G, B) \xrightarrow{\beta^0} H^0(G, C) \xrightarrow{\delta^0} H^1(G, A) \longrightarrow \cdots$$

and commutative diagrams of short exact sequences of G-modules induce corresponding commutative diagrams of long exact sequences of cohomology groups.

Proof. Lemmas 23.6 and 23.7 gives us the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C^n(G,A) \arrow[r, "\alpha^n"] \arrow[d, "d^n"] & C^n(G,B) \arrow[r, "\beta^n"] \arrow[d, "d^n"] & C^n(G,C) \arrow[r] \arrow[d, "d^n"] & 0 \\
0 \arrow[r] & C^{n+1}(G,A) \arrow[r, "\alpha^{n+1}"] & C^{n+1}(G,B) \arrow[r, "\beta^{n+1}"] & C^{n+1}(G,C) \arrow[r] & 0
\end{tikzcd}$$

We have $B^n(G, A) \subseteq Z^n(G, A) \subseteq C^n(G, A) \xrightarrow{d^n} B^{n+1}(G, A) \subseteq Z^{n+1}(G, A) \subseteq C^{n+1}$, thus d^n induces a homomorphism $d^n\colon C^n(G, A)/B^n(G, A) \to Z^{n+1}(G, A)$, and similarly for the G-modules B and C. The fact that α^n maps coboundaries to coboundaries and cocycles to cocycles implies that we have induced maps $C^n(G, A)/B^n(G, A) \to C^n(G, B)/B^n(G, B)$ and $Z^{n+1}(G, B) \to Z^{n+1}(G, B)$; similar comments apply to β^n.

We thus have the following commutative diagram:

$$\begin{tikzcd}
 & \frac{C^n(G,A)}{B^n(G,A)} \arrow[r, "\alpha^n"] \arrow[d, "d^n"] & \frac{C^n(G,B)}{B^n(G,B)} \arrow[r, "\beta^n"] \arrow[d, "d^n"] & \frac{C^n(G,C)}{B^n(G,C)} \arrow[r] \arrow[d, "d^n"] & 0 \\
0 \arrow[r] & Z^{n+1}(G,A) \arrow[r, "\alpha^{n+1}"] & Z^{n+1}(G,B) \arrow[r, "\beta^{n+1}"] & Z^{n+1}(G,C) &
\end{tikzcd}$$

The kernels of the vertical maps d^n are (by definition) the cohomology groups $H^n(G, A)$, $H^n(G, B)$, $H^n(G, C)$, and the cokernels are $H^{n+1}(G, A)$, $H^{n+1}(G, B)$, $H^{n+1}(G, C)$. Applying the snake lemma yields the exact sequence

$$H^n(G,A) \xrightarrow{\alpha^n} H^n(G,B) \xrightarrow{\beta^n} H^n(G,C) \xrightarrow{\delta^n} H^{n+1}(G,A) \xrightarrow{\alpha^{n+1}} H^{n+1}(G,B) \xrightarrow{\beta^{n+1}} H^{n+1}(G,C),$$

where α^n and β^n are the homomorphisms in cohomology induced by α and β (coming from α^n and β^n in the previous diagram via Lemma 23.6), and the connecting homomorphism δ^n given by the snake lemma can be explicitly described as

$$\begin{aligned} \delta^n\colon H^n(G,C) &\to H^{n+1}(G,A) \\ [f] &\mapsto [\alpha^{-1} \circ d^n(\hat{f})] \end{aligned}$$

where $[f]$ denotes the cohomology class of a cocycle $f \in C^n(G,C)$ and $\hat{f} \in C^n(G,B)$ is a cochain satisfying $\beta \circ \hat{f} = f$. Here α^{-1} denotes the inverse of the isomorphism $A \to \alpha(A)$. The fact that δ^n is well defined (independent of the choice of $\hat{f}$) is part of the snake lemma. The map $H^0(G,A) \to H^0(G,B)$ is the restriction of $\alpha\colon A \to B$ to A^G, and is thus injective (recall that $H^0(G,A) \simeq A^G$). This completes the first part of the proof.

For the second part, suppose we have the following commutative diagram of short exact sequences of G-modules

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\alpha"] \arrow[d, "\phi"] & B \arrow[r, "\beta"] \arrow[d, "\psi"] & C \arrow[r] \arrow[d, "\varphi"] & 0 \\ 0 \arrow[r] & A' \arrow[r, "\alpha'"] & B' \arrow[r, "\beta'"] & C' \arrow[r] & 0 \end{tikzcd}$$

By Lemma 23.6, to verify the commutativity of the corresponding diagram of long exact sequences in cohomology we only need to check commutativity at squares of the form

$$\begin{tikzcd} H^n(G,C) \arrow[r, "\delta^n"] \arrow[d, "\varphi^n"] & H^{n+1}(G,A) \arrow[d, "\phi^{n+1}"] \\ H^n(G,C') \arrow[r, "\delta'^n"] & H^{n+1}(G,A') \end{tikzcd} \tag{1}$$

Let $f\colon G^n \to C$ be a cocycle and choose $\hat{f} \in C^n(G,B)$ such that $\beta \circ \hat{f} = f$. We have

$$\phi^{n+1}(\delta^n([f])) = \phi^{n+1}([\alpha^{-1} \circ d^n(\hat{f})] = [\phi \circ \alpha^{-1} \circ d^n(\hat{f})].$$

Noting that $\varphi \circ f = \varphi \circ \beta \circ \hat{f} = \beta' \circ \psi \circ \hat{f}$ and $\phi \circ \alpha^{-1} = \alpha'^{-1} \circ \psi$ (as maps $\alpha(A) \to A'$) yields

$$\delta'^n(\varphi^n([f]) = \delta'^n([\beta' \circ \psi \circ f]) = [\alpha'^{-1} \circ d^n(\psi \circ \hat{f})] = [\alpha'^{-1} \circ \psi \circ d^n(\hat{f})] = [\phi \circ \alpha^{-1} \circ d^n(\hat{f})],$$

thus diagram (1) commutes as desired. $\square$

Definition 23.9. A family of functors F^n from the category of G-modules to the category of abelian groups that associates to each short exact sequence of G-modules a long exact sequence of abelian groups such that commutative diagrams of short exact sequences yield commutative diagrams of long exact sequences is called a δ-*functor*. A δ-functor is said to be *cohomological* if the connecting homomorphisms in long exact sequences are of the form $\delta^n\colon F^n(G,C) \to F^{n+1}(G,A)$. If we instead have $\delta^n\colon F^{n+1}(G,C) \to F^n(G,A)$ then the δ-functor is *homological*.

Theorem 23.54 implies that the family of functors $H^n(G,\bullet)$ is a cohomological δ-functor. In fact is the universal cohomological δ-functor (it satisfies a universal property that determines it up to a unique isomorphism of δ-functors), but we will not explore this further.

23.2 Cohomology via free resolutions

Recall that the *group ring* $\mathbb{Z}[G]$ consists of formal sums $\sum_g a_g g$ indexed by $g \in G$ with coefficients $a_g \in \mathbb{Z}$, all but finitely many zero. Multiplication is given by $\mathbb{Z}$-linearly extending the group operation in G; the ring $\mathbb{Z}[G]$ is commutative if and only if G is. As an abelian group under addition, $\mathbb{Z}[G]$ is the free $\mathbb{Z}$-module with basis G.

The notion of a G-module defined in the previous section is equivalent to that of a (left) $\mathbb{Z}[G]$-module: to define multiplication by $\mathbb{Z}[G]$ one must define a G-action, and the

G-action on a G-module extends $\mathbb{Z}$-linearly, since every G-module is also a $\mathbb{Z}$-module. The multiplicative identity 1 of the ring $\mathbb{Z}[G]$ is the identity element of G; the additive identity 0 is the empty sum, which acts on A by sending $a \in A$ to the identity element of A.[2]

For any $n \geq 0$ we view $\mathbb{Z}[G^n]$ as a G-module with G acting diagonally on the left: $g \cdot (g_1, \ldots, g_n) \coloneqq (gg_1, \ldots, gg_n)$. This makes $\mathbb{Z}[G^0] = \mathbb{Z}$ a trivial G-module (here we are viewing the empty tuple as the identity element of the trivial group G^0).

Definition 23.10. Let G be a group. The *standard resolution of $\mathbb{Z}$ by G-modules* is the exact sequence of G-module homomorphisms

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \xrightarrow{d_n} \mathbb{Z}[G^n] \longrightarrow \cdots \xrightarrow{d_1} \mathbb{Z}[G] \xrightarrow{d_0} \mathbb{Z} \longrightarrow 0,$$

where the boundary maps d_n are defined by

$$d_n(g_0, \ldots, g_n) \coloneqq \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n)$$

and extended $\mathbb{Z}$-linearly (the notation $\hat{g}_i$ means omit g_i from the tuple). The map d_0 sends each $g \in G$ to 1, and extends to the map $\sum_g a_g g \mapsto \sum_g a_g$, which is also known as the *augmentation map* and may be denoted ε.

Let us verify the exactness of the standard resolution.

Lemma 23.11. *The standard resolution of $\mathbb{Z}$ by G-modules is exact.*

Proof. The map d_0 is clearly surjective. To check $\operatorname{im} d_{n+1} \subseteq \ker d_n$ it suffices to note that for any $g_0, \ldots, g_n \in G$ we have

$$\begin{aligned} d_n(d_{n+1}(g_0, \ldots, g_n)) = \sum_{0 \leq i \leq n} \Bigg(& \sum_{0 \leq j < i} (-1)^{i+j} (g_0, \ldots, \hat{g}_j, \ldots, \hat{g}_i \ldots, g_n) + \\ & \sum_{i < j \leq n} (-1)^{i+j-1} (g_0, \ldots, \hat{g}_i, \ldots, \hat{g}_j, \ldots, g_n) \Bigg) = 0 \end{aligned}$$

Let G_1^{n+1} be the subgroup $1 \times G^n$ of G^{n+1}, and let $h\colon \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G_1^{n+2}] \subseteq \mathbb{Z}[G^{n+2}]$ be the $\mathbb{Z}$-linear map defined by $(g_0, \ldots, g_{n+1}) \mapsto (1, g_0, \ldots, g_{n+1})$. For $x \in \mathbb{Z}[G^{n+1}]$ we have $d_{n+1}(h(x)) \in x + \mathbb{Z}[G_1^{n+1}]$, and if $x \in \ker d_n$ then $x - d_{n+1}(h(x)) \in \ker d_n \cap \mathbb{Z}[G_1^{n+1}]$, since $\operatorname{im} d_{n+1} \subseteq \ker d_n$. To prove $\ker d_n \subseteq \operatorname{im} d_{n+1}$, it suffices to show $\ker d_n \cap \mathbb{Z}[G_1^{n+1}] \subseteq \operatorname{im} d_{n+1}$. For $n = 0$ we have $\ker d_0 \cap \mathbb{Z}[G_1^1] = \{0\}$, and we now proceed by induction on $n \geq 1$.

Let $G_{11}^{n+1} \coloneqq 1 \times 1 \times G^{n-1} \subseteq G_1^{n+1}$. We can write the free $\mathbb{Z}$-module $\mathbb{Z}[G_1^{n+1}]$ as the internal direct sum $\mathbb{Z}[G_{11}^{n+1}] + X$, where X is generated by elements of the form $(1, g_1, \ldots, g_n)$ with $g_1 \neq 1$ (if $G = \{1\}$ then $X = \{0\}$). We have $d(x)(g_1, \ldots, g_n) = x(1, g_1, \ldots, g_n)$ (here we use functional notation) for all $x \in X$, since $g_1 \neq 1$, and this implies $X \cap \ker d_n = \{0\}$. It thus suffices to show $\ker d_n \cap \mathbb{Z}[G_{11}^{n+1}] \subseteq \operatorname{im} d_{n+1}$.

Let $x \in \ker d_n \cap \mathbb{Z}[G_{11}^{n+1}]$. If $n = 1$ then $x = d_2(h(x)) \in \operatorname{im} d_{n+1}$. For $n \geq 2$, let $\pi\colon \mathbb{Z}[G^{n+1}] \to \mathbb{Z}[G^{n-1}]$ be the $\mathbb{Z}$-linear map defined by $(g_0, g_1, g_2, \ldots, g_n) \mapsto (g_2, \ldots, g_n)$. We have $\pi(x) \in \ker d_{n-2} \subset \operatorname{im} d_{n-1}$ (by the inductive hypothesis), and for any $y \in d_{n-1}^{-1}(\pi(x))$ we have $x = d_{n+1}(h_{11}(y)) \in \operatorname{im} d_{n+1}$, where $h_{11}\colon \mathbb{Z}[G^{n-1}] \to \mathbb{Z}[G^{n+1}]$ is the $\mathbb{Z}$-linear map defined by $(g_0, \ldots, g_{n-1}) \mapsto (g_0, \ldots, g_{n+1})$. Thus $\ker d_n \cap \mathbb{Z}[G_{11}^{n+1}] \subseteq \operatorname{im} d_{n+1}$ as desired. $\square$

[2] When A is written multiplicatively its identity is denoted 1 and one should think of 0 as acting via exponentiation (but for the moment we continue to use additive notation and view A as a left $\mathbb{Z}[G]$-module).

Definition 23.12. Let R be a (not necessarily commutative) ring. A *free resolution* P of a (left) R-module M is an exact sequence of free (left) R-modules P_n

$$\cdots \xrightarrow{d_{n+1}} P_{n+1} \xrightarrow{d_n} P_n \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_1} P_1 \xrightarrow{d_0} M \longrightarrow 0.$$

Free resolutions arise naturally as presentations of an R-module. Every R-module M admits a surjection from a free module (one can always take P_1 to be the free R-module with basis M). This yields an exact sequence $P_1 \to M \to 0$, and the kernel of the homomorphism on the left is itself an R-module that admits a surjection from a free R-module P_2; continuing in this fashion yields a free resolution.

Now let A be an abelian group. If we *truncate* the free resolution P by removing the R-module M and apply the contravariant left exact functor $\operatorname{Hom}_R(\bullet, A)$ we obtain a cochain complex of R-modules[3]

$$\cdots \xleftarrow{d_{n+1}^*} P_{n+1}^* \xleftarrow{d_n^*} P_n^* \xleftarrow{d_{n-1}^*} \cdots \xleftarrow{d_1^*} P_1^* \longleftarrow 0.$$

where $d_n^*(\varphi) := \varphi \circ d_n$. The maps d_n^* satisfy $d_{n+1}^* \circ d_n^* = 0$: for all $\varphi \in \operatorname{Hom}_R(P_n, A)$ we have

$$(d_{n+1}^* \circ d_n^*)(\varphi) = (d_n \circ d_{n+1})^*(\varphi) = \varphi \circ d_n \circ d_{n+1} = \varphi \circ 0 = 0.$$

This cochain complex need not be exact, because the functor $\operatorname{Hom}_R(\bullet, A)$ is not right-exact,[4] so we have potentially nontrivial cohomology groups $\ker d_{n+1}^*/\operatorname{im} d_n^*$, which are denoted $\operatorname{Ext}_R^n(M, A)$. A key result of homological algebra is that (up to isomorphism) these cohomology groups do not depend on the resolution P, only on A and M; see Theorem 23.71.

Recall that $\mathbb{Z}[G]$ is a free $\mathbb{Z}$-module (with basis G), and for all $n \geq 0$ we have

$$\mathbb{Z}[G^{n+1}] \simeq \bigoplus_{(g_1,\ldots,g_n)\in G^n} \mathbb{Z}[G](1, g_1, \ldots, g_n).$$

It follows that the standard resolution is a free resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules; note that $\mathbb{Z}$, like any abelian group, can always be viewed as a trivial G-module, hence a $\mathbb{Z}[G]$-module.

With a free resolution in hand, we now want to consider the cochain complex

$$0 \to \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \longrightarrow \cdots \longrightarrow \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^n], A) \xrightarrow{d_n^*} \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{n+1}], A) \longrightarrow \cdots$$

where d_n^* is defined by $\varphi \mapsto \varphi \circ d_n$. Let $\mathcal{S}_A$ denote this cochain complex.

Proposition 23.13. *Let A be a G-module. For every $n \geq 0$ we have an isomorphism of abelian groups*

$$\Phi^n\colon \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{n+1}], A) \xrightarrow{\sim} C^n(G, A)$$

that sends $\varphi\colon \mathbb{Z}[G^{n+1}] \to A$ to the function $f\colon G^n \to A$ defined by

$$f(g_1, \ldots, g_n) := \varphi(1,\, g_1,\, g_1g_2,\, \ldots,\, g_1g_2\cdots g_n).$$

The isomorphisms Φ^n satisfy $\Phi^{n+1} \circ d_{n+1}^* = d^n \circ \Phi^n$ for all $n \geq 0$ and thus define an isomorphism of cochain complexes $\Phi_A\colon \mathcal{S}_A \to \mathcal{C}_A$.

[3] The intuition here is that P contains a presentation of M that effectively serves as a replacement for M.
[4] Applying $\operatorname{Hom}_{\mathbb{Z}}(\bullet, \mathbb{Z})$ to $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ yields $0 \leftarrow \mathbb{Z} \leftarrow 0 \leftarrow 0 \leftarrow 0$, for example.

Proof. We first check that Φ^n is injective. Let $\varphi \in \ker \Phi^n$. Given $g_0, \ldots, g_n \in G$, let $h_i := g_{i-1}^{-1} g_i$ for $1 \le i \le n$ so that $h_1 \cdots h_i = g_0^{-1} g_i$ and observe that

$$\varphi(g_0, \ldots, g_n) = g_0 \varphi(1, g_0^{-1} g_1, \ldots, g_0^{-1} g_n) = g_0 \varphi(1, h_1, h_1 h_2, \ldots, h_1 \cdots h_n) = 0.$$

so $\varphi = 0$ as desired. For surjectivity, let $f \in C^n(G, A)$ and define $\varphi \in \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{n+1}], A)$ via $\varphi(g_0, \ldots, g_n) := g_0 f(g_0^{-1} g_1, g_1^{-1} g_2, \ldots, g_{n-1}^{-1} g_n)$. For any $g_1, \ldots, g_n \in G$ we have

$$\Phi^n(\varphi)(g_1, \ldots, g_n) = \varphi(1, g_1, g_1 g_2, \ldots, g_1 g_2 \cdots g_n) = f(g_1, \ldots, g_n),$$

so $f \in \operatorname{im} \Phi^n$ and Φ^n is surjective.

It is clear from the definition that $\Phi^n(\varphi_1 + \varphi_2) = \Phi^n(\varphi_1) + \Phi^n(\varphi_2)$, so Φ^n is a bijective group homomorphism, hence an isomorphism. Finally, for any $\varphi \in \operatorname{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^{n+1}], A)$ and $g_1, \ldots, g_{n+1} \in G$ we have

$$\begin{aligned}
(\Phi^{n+1}(d_{n+1}^*(\varphi))(g_1, \ldots, g_{n+1}) &= d_{n+1}^*(\varphi)(1, g_1,\, g_1 g_2,\, \ldots, g_1 \cdots g_{n+1}) \\
&= \varphi(d_{n+1}(1, g_1,\, g_1 g_2,\, \ldots, g_1 \cdots g_{n+1})) \\
&= \sum_{i=0}^{n+1} (-1)^i \varphi(1, g_1, \ldots, g_1 \cdots g_{i-i},\; g_1 \cdots g_{i+1}, \ldots, g_1 \cdots g_{n+1}) \\
&= g_1 \Phi^n(\varphi)(g_2, \ldots, g_{n+1}) \\
&\quad + \sum_{i=1}^{n} (-1)^i \Phi^n(\varphi)(g_1, \ldots, g_{i-2},\, g_{i-1} g_i,\, g_{i+1}, \ldots, g_{n+1}) \\
&\quad + (-1)^{n+1} \Phi^n(\varphi)(g_1, \ldots, g_n) \\
&= d^n(\Phi^n(\varphi))(g_1, \ldots, g_{n+1}),
\end{aligned}$$

which shows that $\Phi^{n+1} \circ d_{n+1}^* = d_n^* \circ \Phi^n$ as claimed. □

Corollary 23.14. *Let A be a G-module. The cochain complexes $\mathcal{S}_A$ and $\mathcal{C}_A$ have the same cohomology groups, in other words, $H^n(G, A) \simeq \operatorname{Ext}^n_{\mathbb{Z}[G]}(\mathbb{Z}, A)$ for all $n \ge 0$, and we can compute $H^n(G, A)$ using any free resolution of $\mathbb{Z}$ by G-modules.*

Proof. This follows immediately from Proposition 23.13 and Theorem 23.71. □

Corollary 23.15. *For any G-modules A and B we have*

$$H^n(G, A \oplus B) \simeq H^n(G, A) \oplus H^n(G, B)$$

for all $n \ge 0$, and the isomorphism commutes with the natural inclusion and projection maps for the direct sums on both sides.

Proof. By Lemma 23.73, the functor $\operatorname{Ext}_{\mathbb{Z}}[G]^n(\mathbb{Z}, \bullet)$ is an additive functor. □

Definition 23.16. A category containing finite coproducts (such as direct sums) in which each set of morphisms between objects has the structure of an abelian group whose addition distributes over composition (and vice versa) is called an *additive category*. A functor F between additive categories is an *additive functor* if it maps zero objects to zero objects and satisfies $F(X \oplus Y) \simeq F(X) \oplus F(Y)$, where the isomorphism commutes with the natural inclusion and projection maps for the direct sums on both sides.

Definition 23.17. Let G be a group and let A be an abelian group. The abelian group

$$\mathrm{CoInd}^G(A) := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$$

with G-action defined by $(g\varphi)(z) := \varphi(zg)$ is the *coinduced* G-module associated to A.

Warning 23.18. Some texts [3, 5] use $\mathrm{Ind}^G(A)$ instead of $\mathrm{CoInd}^G(A)$ to denote the G-module $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], A)$ and refer to is as "induced" rather than "coinduced". Here we follow [1, 4, 7] and reserve the notation $\mathrm{Ind}^G(A)$ for the induced G-module $\mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ defined below (see Definition 23.25). As shown by Lemma 23.27, this clash in terminology is fairly harmless when G is finite, since we then have $\mathrm{Ind}^G(A) \simeq \mathrm{CoInd}^G(A)$.

Lemma 23.19. *Let G be a group and A an abelian group. Then $H^0(G, \mathrm{CoInd}^G(A)) \simeq A$ and $H^n(G, \mathrm{CoInd}^G(A)) = 0$ for all $n \geq 1$.*

Proof. For all $n \geq 1$ we have an isomorphims of abelian groups

$$\begin{aligned}
\alpha\colon \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G^n], \mathrm{CoInd}^G(A)) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G^n], A) \\
\varphi &\mapsto (z \mapsto \varphi(z)(1)) \\
(z \mapsto (y \mapsto \phi(yz))) &\leftarrow\!\shortmid \phi
\end{aligned}$$

Indeed,

$$\begin{aligned}
&\alpha(\alpha^{-1}(\phi)) = \alpha(z \mapsto (y \mapsto \phi(yz)))) = (z \mapsto \phi(z)) = \phi, \\
&\alpha^{-1}(\alpha(\varphi)) = \alpha^{-1}(z \mapsto \varphi(z)(1)) = (z \mapsto (y \mapsto \varphi(yz)(1))) = (z \mapsto \varphi(z)) = \varphi.
\end{aligned}$$

Thus computing $H^n(G, \mathrm{CoInd}^G(A))$ using the standard resolution P of $\mathbb{Z}$ by G-modules is the same as computing $H^n(\{1\}, A)$ using the resolution P viewed as a resolution of $\mathbb{Z}$ by $\{1\}$-modules (abelian groups); note that $\mathbb{Z}[G^n]$ is also a free $\mathbb{Z}[\{1\}]$-module, and the G-module morphisms d_n in the standard resolution are also $\{1\}$-module morphisms (morphisms of abelian groups). Therefore $H^n(G, \mathrm{CoInd}^G(A)) \simeq H^n(\{1\}, A)$ for all $n \geq 0$.

But we can also compute $H^n(\{1\}, A)$ using the free resolution $\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$, which implies $H^n(\{1\}, A) = 0$ for $n \geq 1$ and $H^0(\{1\}, A) \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A) \simeq A$. $\square$

23.3 Homology via free resolutions

In the previous section we applied the contravariant functor $\mathrm{Hom}_{\mathbb{Z}[G]}(\bullet, A)$ to the truncation of the standard resolution of $\mathbb{Z}$ by G-modules to get a cochain complex with cohomology groups $H^n(G, A) \simeq \mathrm{Ext}^n_{\mathbb{Z}]G]}(\mathbb{Z}, A)$. If we do the same thing using the covariant functor $\bullet \otimes_{\mathbb{Z}[G]} A$ we get a *chain complex* (of $\mathbb{Z}$-modules)

$$\cdots \longrightarrow \mathbb{Z}[G^{n+1}] \otimes_{\mathbb{Z}[G]} A \xrightarrow{d_{n*}} \mathbb{Z}[G^n] \otimes_{\mathbb{Z}[G]} A \longrightarrow \cdots \longrightarrow \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A \longrightarrow 0,$$

where d_{n*} is defined by $(g_0, \ldots, g_n) \otimes a \mapsto d_n(g_0, \ldots, g_n) \otimes a$. One minor technical point: in order for these tensor products to make sense we need to view $\mathbb{Z}[G^n]$ as a *right* $\mathbb{Z}[G]$-module, so we define $(g_1, \ldots, g_n) \cdot g := (g_1 g, \ldots, g_n g)$; the corresponding G-module is isomorphic to the left $\mathbb{Z}[G]$-module defined above (right action by g corresponds to left action by g^{-1}).

We then have *homology groups* $\ker d_{n*} / \operatorname{im} d_{n+1*}$. As with the groups $\mathrm{Ext}_{\mathbb{Z}}[G]^n(\mathbb{Z}, A)$, we get the same homology groups using any free resolution of $\mathbb{Z}$ by right $\mathbb{Z}[G]$-modules, and they are generically denoted $\mathrm{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, A)$; see Theorem 23.75.

Definition 23.20. Let A be a G-module. The nth *homology group of G with coefficients in A* is the abelian group $H_n(G,A) \coloneqq \operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},A)$. If $\alpha\colon A \to B$ is a morphism of G-modules, the natural morphism $\alpha_n\colon H_n(G,A) \to H_n(G,B)$ is given by $x \otimes a \mapsto x \otimes \varphi(a)$. Each $H_n(G,\bullet)$ is a functor from the category of G-modules to the category of abelian groups.

The family of functors $H_n(G,\bullet)$ is a homological δ-functor.

Theorem 23.21. *Every short exact sequence of G-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

induces a long exact sequence of homology groups

$$\cdots \longrightarrow H_1(G,C) \xrightarrow{\delta_0} H_0(G,A) \xrightarrow{\alpha_0} H_0(G,B) \xrightarrow{\beta_0} H_0(G,C) \longrightarrow 0,$$

and commutative diagrams of short exact sequences of G-modules induce corresponding commutative diagrams of long exact sequences of homology groups.

Proof. The proof is directly analogous to that of Theorem 23.8 (or see Theorem 23.50). $\square$

As with $H^n(G,\bullet)$, the functors $H_n(G,\bullet)$ are additive functors.

Corollary 23.22. *For any G-modules A and B we have*

$$H_n(G, A \oplus B) \simeq H_(G,A) \oplus H_n(G,B)$$

for all $n \geq 0$, and the isomorphism commutes with the natural inclusion and projection maps for the direct sums on both sides.

Proof. By Lemma 23.77, the functor $\operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z},\bullet)$ is an additive functor. $\square$

For $n=0$ we have

$$H_0(G,A) \coloneqq \operatorname{Tor}_0^{\mathbb{Z}[G]}(\mathbb{Z},A) = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A,$$

where we are viewing $\mathbb{Z}$ as a (right) $\mathbb{Z}[G]$-module with G acting trivially; see Lemma 23.78 for a proof of the second equality. This means that $\sum a_g g \in \mathbb{Z}[G]$ acts on $\mathbb{Z}$ via multiplication by the integer $\sum a_g$. This motivates the following definition.

Definition 23.23. Let G be a group. The *augmentation map* $\varepsilon\colon \mathbb{Z}[G] \to \mathbb{Z}$ is the ring homomorphism $\sum a_g g \mapsto \sum a_g$.[5] The *augmentation ideal* I_G is the kernel of the augmentation map; it is a free $\mathbb{Z}$-module with basis $\{g-1 : g \ \in G\}$.

The augmentation ideal I_G is precisely the annihilator of the $\mathbb{Z}[G]$-module $\mathbb{Z}$; therefore

$$\mathbb{Z} \otimes_{\mathbb{Z}[G]} A \simeq A/I_G A.$$

Definition 23.24. Let A be a G-module. The group of *G-coinvariants* of A is the G-module

$$A_G \coloneqq A/I_G A;$$

it is the largest trivial G-module that is a quotient of A.

[5] The augmentation map is the boundary map d_0 in the standard resolution of $\mathbb{Z}$ by G-modules.

We thus have $H_0(G,A) \simeq A_G$ and $H^0(G,A) \simeq A^G$.

Definition 23.25. Let G be a group and let A be an abelian group. The abelian group

$$\operatorname{Ind}^G(A) := \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$$

with G-action defined by $g(z \otimes a) = (gz) \otimes a$ is the *induced* G-module associated to A.

Lemma 23.26. *Let G be a group and A an abelian group. Then $H_0(G, \operatorname{Ind}^G(A)) \simeq A$ and $H_n(G, \operatorname{Ind}^G(A)) = 0$ for all $n \geq 1$.*

Proof. Viewing $\mathbb{Z}[G^n]$ as a right $\mathbb{Z}[G]$-module and $\mathbb{Z}[G]$ as a left $\mathbb{Z}[G]$-module, for all $n \geq 1$,

$$\mathbb{Z}[G^n] \otimes_{\mathbb{Z}[G]} (\mathbb{Z}[G] \otimes_{\mathbb{Z}} A) \simeq (\mathbb{Z}[G^n] \otimes_{\mathbb{Z}[G]} \mathbb{Z}[G]) \otimes_{\mathbb{Z}} A \simeq \mathbb{Z}[G^n] \otimes_{\mathbb{Z}} A,$$

by associativity of the tensor product (and the fact that $M \otimes_R R \simeq M$ for any right R-module M). This implies that computing $H_n(G, \operatorname{Ind}^G(A))$ using the standard resolution P of $\mathbb{Z}$ by (right) G-modules is the same as computing $H_n(\{1\}, A)$ using the resolution P viewed as a resolution of $\mathbb{Z}$ by $\{1\}$-modules (abelian groups). Thus

$$H_n(G, \operatorname{Ind}^G(A)) = \operatorname{Tor}_n^{\mathbb{Z}[G]}(\mathbb{Z}, \operatorname{Ind}^G(A)) \simeq \operatorname{Tor}_n^{\mathbb{Z}}(\mathbb{Z}, A) = H_n(\{1\}, A).$$

But we can also compute $H_n(\{1\}, A)$ using the free resolution $\cdots \to 0 \to \mathbb{Z} \to \mathbb{Z} \to 0$, which implies $H_n(\{1\}, A) = 0$ for $n \geq 1$ and $H_0(\{1\}, A) \simeq \mathbb{Z} \otimes A \simeq A$. $\square$

Lemma 23.27. *Let G be a finite group and A an abelian group. The G-modules $\operatorname{Ind}^G(A)$ and $\operatorname{CoInd}^G(A)$ are isomorphic.*

Proof. We claim that we have a canonical G-module isomorphism given by

$$\begin{aligned} \alpha\colon\ \operatorname{CoInd}^G(A) &\xrightarrow{\sim} \operatorname{Ind}^G(A) \\ \varphi &\mapsto \sum_{g \in G} g^{-1} \otimes \varphi(g) \\ (g^{-1} \mapsto a) &\leftarrow\!\shortmid g \otimes a \end{aligned}$$

where $(g^{-1} \mapsto a)(h) = 0$ for $h \in G - \{g^{-1}\}$. It is obvious that α and α^{-1} are inverse homomorphisms of abelian groups, we just need to check that there are morphisms of G-modules. For any $h \in G$ and $\varphi \in \operatorname{CoInd}^G(A)$ we have

$$\alpha(h\varphi) = \sum_{g \in G} g^{-1} \otimes (h\varphi)(g) = h \sum_{g \in G} (gh)^{-1} \otimes \varphi(gh) = h \sum_{g \in G} g^{-1} \otimes \varphi(g) = h\alpha(\varphi),$$

and for any $h \in G$ and $g \otimes a \in \operatorname{Ind}^G(A)$ we have

$$\alpha^{-1}(h(g \otimes a)) = \alpha^{-1}(hg \otimes a) = ((hg)^{-1} \mapsto a) = h(g^{-1} \mapsto a) = h\alpha^{-1}(g \otimes a),$$

since for $\varphi = (g^{-1} \mapsto a)$ the identity $(h\varphi)(z) = \varphi(zh)$ implies $h\varphi = ((hg)^{-1} \mapsto a)$. $\square$

Corollary 23.28. *Let G be a finite group, A be an abelian group, and let B be $\operatorname{Ind}^G(A)$ or $\operatorname{CoInd}^G(A)$. Then $H_0(G,B) \simeq H^0(G,B) \simeq A$ and $H_n(G,B) = H^n(G,B) = 0$ for all $n \geq 1$.*

Proof. This follows immediately from Lemmas 23.19, 23.26, 23.27. $\square$

23.4 Tate cohomology

We now assume that G is a finite group.

Definition 23.29. The *norm element* of $\mathbb{Z}[G]$ is $N_G := \sum_{g\in G} g$.

Lemma 23.30. *Let A be a G-module and let $N_G\colon A \to A$ be the G-module endomorphism $a \mapsto N_G a$. We then have $I_G A \subseteq \ker N_G$ and $\operatorname{im} N_G \subseteq A^G$, thus N_G induces a morphism $\hat{N}_G\colon A_G \to A^G$ of trivial G-modules.*

Proof. We have $gN_G = N_G$ for all $g \in G$, so $\operatorname{im} N_G \subseteq A^G$, and $N_G(g-1) = 0$ for all $g \in G$, so N_G annihilates the augmentation ideal I_G and $I_G A \subseteq \ker N_G$. The lemma follows. $\square$

Definition 23.31. Let A be a G-module for a finite group G. For $n \geq 0$ the *Tate cohomology and homology groups* are defined by

$$\hat{H}^n(G,A) := \begin{cases} \operatorname{coker} \hat{N}_G & \text{for } n = 0 \\ H^n(G,A) & \text{for } n > 0 \end{cases} \qquad \hat{H}_n(G,A) := \begin{cases} \ker \hat{N}_G & \text{for } n = 0 \\ H_n(G,A) & \text{for } n > 0 \end{cases}$$

$$\hat{H}^{-n}(G,A) := \hat{H}_{n-1}(G,A) \qquad\qquad \hat{H}_{-n}(G,A) := \hat{H}^{n-1}(G,A).$$

Note that $\hat{H}^0(G,A)$ is a quotient of $H^0(G,A) \simeq A^G$ (the largest trivial G-module in A) and $\hat{H}_0(G,A)$ is a submodule of $H_0(G,A) \simeq A_G$ (the largest trivial G-module quotient of A).

Thus any morphism of G-modules induces natural morphisms of Tate cohomology and homology groups in degree $n = 0$ (and all other degrees, as we already know). We thus have functors $\hat{H}^n(G,\bullet)$ and $\hat{H}_n(G,\bullet)$ from the category of G-modules to the category of abelian groups.

Given that every Tate homology group is also a Tate cohomology group, in practice one usually refers only to the groups $\hat{H}^n(G,A)$, but the notation $\hat{H}_n(G,A)$ can be helpful to highlight symmetry.

Theorem 23.32. *Let G be a finite group. Every short exact sequence of G-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0.$$

induces a long exact sequence of Tate cohomology groups

$$\cdots \longrightarrow \hat{H}^n(G,A) \xrightarrow{\hat{\alpha}^n} \hat{H}^n(G,B) \xrightarrow{\hat{\beta}^n} \hat{H}^n(G,C) \xrightarrow{\hat{\delta}^n} \hat{H}^{n+1}(G,A) \longrightarrow \cdots,$$

equivalently, a long exact sequence of Tate homology groups

$$\cdots \longrightarrow \hat{H}_n(G,A) \xrightarrow{\hat{\alpha}_n} \hat{H}_n(G,B) \xrightarrow{\hat{\beta}_n} \hat{H}_n(G,C) \xrightarrow{\hat{\delta}_n} \hat{H}_{n-1}(G,A) \longrightarrow \cdots.$$

Commutative diagrams of short exact sequences of G-modules induce commutative diagrams of long exact sequences of Tate cohomology groups (equivalently, Tate homology groups).

Proof. It follows from Theorems 23.8 and 23.21 that it is enough to prove exactness at the terms $\hat{H}^0(G,\bullet) = \hat{H}_{-1}(G,\bullet)$ and $\hat{H}_0(G,\bullet) = \hat{H}^{-1}(G,\bullet)$. We thus consider the diagram

$$\begin{tikzcd}
H_1(C,G) \arrow[r, "\delta_0"] & A_G \arrow[r, "\alpha_0"] \arrow[d, "\hat{N}_G"] & B_G \arrow[r, "\beta_0"] \arrow[d, "\hat{N}_G"] & C_G \arrow[r] \arrow[d, "\hat{N}_G"] & 0 \\
0 \arrow[r] & A^G \arrow[r, "\alpha^0"] & B^G \arrow[r, "\beta^0"] & C^G \arrow[r, "\delta^0"] & H^1(A,G)
\end{tikzcd}$$

whose top and bottom rows are the end and beginning of the long exact sequences in homology and cohomology given by Theorems 23.21 and 23.8, respectively; here we are using $H_0(G,\bullet)\simeq \bullet_G$ and $H^0(G,\bullet)\simeq \bullet^G$.

For any $[a]\in A_G = A/I_GA$ we have $\hat{N}_G(\alpha_0([a])) = N_G\alpha(a) = \alpha(N_Ga) = \alpha^0(\hat{N}_G([a]))$, so the first square commutes, as does the second (by the same argument). Applying the snake lemma yields an exact sequence of kernels and cokernels of $\hat{N}_G$

$$\hat{H}_0(G,A)\overset{\hat{\alpha}_0}{\to}\hat{H}_0(G,B)\overset{\hat{\beta}_0}{\to}\hat{H}_0(G,C)\overset{\hat{\delta}}{\to}\hat{H}^0(G,A)\overset{\hat{\alpha}^0}{\to}\hat{H}^0(G,B)\overset{\hat{\beta}^0}{\to}\hat{H}^0(G,C),$$

where $\hat{\delta}([c]) = [a]$ for any $a\in A$, $b\in B$, $c\in C$ with $\alpha(a) = N_Gb$ and $\beta(b) = c\in\ker N_G$ (that this uniquely defines the connecting homomorphism $\hat{\delta}$ is part of the snake lemma). Note that $\operatorname{im}\delta_0 = \ker\alpha_0 = \ker\hat{\alpha}_0\subseteq\ker\hat{N}_G$, since α^0 is injective, so δ_0 gives a well-defined map $\hat{\delta}_0\colon\hat{H}_1(G,C)\to\hat{H}_0(G,A)$ that makes the sequence is exact at $\hat{H}_0(G,A)$. Similarly, $\operatorname{im}\hat{N}_G\subseteq\operatorname{im}\beta^0 = \ker\delta^0$, since β_0 is surjective, so δ^0 induces a well-defined map $\hat{\delta}^0\colon\hat{H}^0(G,C)\to H^1(A,G)$ that makes the sequence exact at $\hat{H}^0(G,C)$.

For the last statement of the theorem, suppose we have the following commutative diagram of exact sequences of G-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\varphi} & & \\
0 & \longrightarrow & A' & \xrightarrow{\alpha'} & B' & \xrightarrow{\beta'} & C' & \longrightarrow & 0
\end{array}$$

By Theorems 23.21 and 23.8, we only need to verify the commutativity of the square

$$\begin{array}{ccc}
\hat{H}_0(G,C) & \xrightarrow{\hat{\delta}} & \hat{H}^0(G,A)\\
\downarrow{\scriptstyle\varphi_0} & & \downarrow{\scriptstyle\phi^0}\\
\hat{H}_0(G,C') & \xrightarrow{\hat{\delta}'} & \hat{H}^0(G,A')
\end{array}$$

Let $a\in A$, $b\in B$, $c\in C$ satisfy $\alpha(a) = N_Gb$ and $\beta(b) = c\in\ker N_G$ as in the definition of $\hat{\delta}$ above, so that $\hat{\delta}([c]) = [a]$. Now let $a' = \phi(a)$, $b' = \psi(b)$, $c = \varphi(c)$. Then

$$\begin{aligned}
\alpha'(a') &= \alpha'(\phi(a)) = \psi(\alpha(a)) = \psi(N_Gb) = N_G\psi(b) = N_Gb'\\
\beta'(b') &= \beta'(\psi(b)) = \varphi(\beta(b)) = \varphi(c) = c'\in\ker N_G,
\end{aligned}$$

where we have used $N_Gc' = N_G\varphi(c) = \varphi(N_Gc) = \varphi(0) = 0$. Thus $\hat{\delta}'([c'] = [a']$ and

$$\phi^0(\hat{\delta}([c])) = \phi^0([a]) = [\phi(a)] = [a'] = \hat{\delta}'([c']) = \hat{\delta}'([\varphi(c)] = \hat{\delta}'(\varphi_0([c])),$$

so $\phi^0\circ\hat{\delta} = \hat{\delta}'\circ\varphi_0$ as desired. □

Theorem 23.32 implies that the family $\hat{H}^n(G,\bullet)$ is a cohomological δ-functor, and that the family $\hat{H}_n(G,\bullet)$ is a homological δ-functor.

Corollary 23.33. *Let G be a finite group. For any G-modules A and B we have*

$$\hat{H}^n(G,A\oplus B)\simeq\hat{H}^n(G,A)\oplus\hat{H}^n(G,B),$$

for all $n\in\mathbb{Z}$, and the isomorphisms commute with the natural inclusion and projection maps for the direct sums on both sides.

Proof. For $n \neq 0, -1$ this follows from Corollaries 23.15 and 23.22. For $n = 0, -1$ it suffices to note that N_G acts on $A \oplus B$ component-wise, and the induced morphism $\hat{N}_G$ thus acts on $(A \oplus B)_G = A_G \oplus B_G$ component-wise. $\square$

Theorem 23.34. *Let G be a finite group and let B be an induced or co-induced G-module associated to some abelian group A. Then $\hat{H}^n(G, B) = \hat{H}_n(G, B) = 0$ for all $n \in \mathbb{Z}$.*

Proof. By Corollary 23.28, we only need to show $\hat{H}_0(G, B) = \hat{H}^0(G, B) = 0$, and by Lemma 23.27 it suffices to consider the case $B = \mathrm{Ind}^G(A) = \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$. Equivalently, we need to show that $N_G \colon B \to B$ has kernel $I_G B$ and image B^G. By definition, the $\mathbb{Z}[G]$-action on $B = \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ only affects the factor $\mathbb{Z}[G]$, so this amounts to showing that, as an endomorphism of $\mathbb{Z}[G]$, we have $\ker N_G = I_G$ and $\operatorname{im} N_G = \mathbb{Z}[G]^G$. But this is clear: the action of N_G on $\mathbb{Z}[G]$ is $\sum_{g \in G} a_g g \mapsto (\sum_{g \in G} a_g) N_G$. The kernel of this action is the augmentation ideal I_G, and its image is $\mathbb{Z}[G]^G = \{\sum_{g \in G} a_g g : \text{all } a_g \in \mathbb{Z} \text{ equal}\} = N_G \mathbb{Z}$. $\square$

Remark 23.35. Theorem 23.34 explains a major motivation for using Tate cohomology. It is the minimal modification needed to ensure that induced (and co-induced) G-modules have trivial homology and cohomology in all degrees.

Corollary 23.36. *Let G be a finite group and let A be a free $\mathbb{Z}[G]$-module. Then $\hat{H}_n(G, A) = \hat{H}^n(G, A) = 0$ for all $n \in \mathbb{Z}$.*

Proof. Let S be a $\mathbb{Z}[G]$-basis for A and let B be the free $\mathbb{Z}$-module with basis S. Then $A \simeq \mathrm{Ind}^G(B)$ and the corollary follows from Theorem 23.34. $\square$

23.5 Tate cohomology of cyclic groups

We now assume that G is a cyclic group $\langle g \rangle$ of finite order. In this case the augmentation ideal I_G is principal, generated by $g - 1$ (as an ideal in the ring $\mathbb{Z}[G]$, not as a $\mathbb{Z}$-module). For any G-module A we have a free resolution

$$\cdots \longrightarrow \mathbb{Z}[G] \xrightarrow{N_G} \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{N_G} \mathbb{Z}[G] \xrightarrow{g-1} \mathbb{Z}[G] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0. \tag{2}$$

The fact that augmentation ideal $I_G = (g - 1)$ is principal (because G is cyclic) ensures $\operatorname{im} N_G = \ker(g - 1)$, making the sequence exact.

The group ring $\mathbb{Z}[G]$ is commutative, since G is abelian, so we need not distinguish left and right $\mathbb{Z}[G]$-modules. For any G-module A we can view $\mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A$ as a G-module via $g(h \otimes a) = gh \otimes a = h \otimes ga$ and view $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A)$ as a G-module via $(g\varphi)(h) := \varphi(gh)$.[6]

Theorem 23.37. *Let $G = \langle g \rangle$ be a finite cyclic group and let A be a G-module. For all $n \in \mathbb{Z}$ we have $\hat{H}^{2n}(G, A) \simeq \hat{H}_{2n-1}(G, A) \simeq \hat{H}^0(G, A)$ and $\hat{H}_{2n}(G, A) \simeq \hat{H}^{2n-1}(G, A) \simeq \hat{H}_0(G, A)$.*

Proof. We have canonical G-module isomorphisms $\mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}[G], A) \simeq A \simeq \mathbb{Z}[G] \otimes_{\mathbb{Z}[G]} A$ induced by $\varphi \mapsto \varphi(1)$ and $a \mapsto 1 \otimes a$, respectively, and these isomorphisms preserve the multiplication-by-g endomorphisms (that is, $(g\varphi)(1) = g\varphi(1)$ and $1 \otimes ga = g(1 \otimes a)$). Using the free resolution in (2), we can thus compute $H^n(G, A)$ using the cochain complex

$$0 \longrightarrow A \xrightarrow{g-1} A \xrightarrow{N_G} A \xrightarrow{g-1} A \xrightarrow{N_G} A \cdots,$$

[6] Note that we must have $g_1 g_2 \varphi(h) = g_1(g_2\varphi)(h) = (g_2\varphi)(g_1 h) = \varphi(g_2 g_1 h) = g_2 g_1 \varphi(h)$ in order for φ to be both a $\mathbb{Z}[G]$-module morphism and an element of a $\mathbb{Z}[G]$-module, so this will not work if G is not abelian.

and we can compute $H_n(G,A)$ using the chain complex

$$\cdots \longrightarrow A \xrightarrow{N_G} A \xrightarrow{g-1} A \xrightarrow{N_G} A \xrightarrow{g-1} A \longrightarrow 0.$$

We now observe that $A^G = \ker(g-1)$, so for all $n \geq 1$ we have

$$H^{2n}(G,A) = H_{2n-1}(G,A) = \ker(g-1)/\operatorname{im} N_G = \operatorname{coker} \hat{N}_G = \hat{H}^0(G,A),$$

so $\hat{H}^{2n}(G,A) = \hat{H}_{2n-1}(G,A) = \hat{H}^0(G,A)$ for all $n \in \mathbb{Z}$, since $\hat{H}^{-2n}(G,A) = \hat{H}_{2n-1}(G,A)$ and $\hat{H}_{-2n+1} = \hat{H}^{2n}$ for all $n \geq 0$.

We also note that $\operatorname{im}(g-1) = I_G A$, so for all $n \geq 1$ we have

$$H_{2n}(G,A) = H^{2n-1}(G,A) = \ker N_G/\operatorname{im}(g-1) = \ker \hat{N}_G = \hat{H}_0(G,A),$$

so $\hat{H}_{2n}(G,A) = \hat{H}^{2n-1}(G,A) = \hat{H}_0(G,A)$ for all $n \in \mathbb{Z}$, since $\hat{H}_{-2n}(G,A) = \hat{H}^{2n-1}(G,A)$ and $\hat{H}^{-2n+1} = \hat{H}_{2n}$ for all $n \geq 0$. ☐

It follows from Theorem 23.37 that when G is a finite cyclic group, all of the Tate homology/cohomology groups are determined by $\hat{H}_0(G,A) = \ker \hat{N}_G = \ker N_G/\operatorname{im}(g-1)$ and $\hat{H}^0(G,A) = \operatorname{coker} \hat{N}_G = \ker(g-1)/\operatorname{im} N_G$. This motivates the following definition.

Definition 23.38. Let G be a finite cyclic group and let A be a G-module. We define $h^n(A) := h^n(G,A) := \#\hat{H}^n(G,A)$ and $h_n(A) := h_n(G,A) := \#\hat{H}_n(G,A)$. Whenever $h^0(A)$ and $h_0(A)$ are both finite, we also define the *Herbrand quotient* $h(A) := h^0(A)/h_0(A) \in \mathbb{Q}$.

Remark 23.39. Some authors define the Herbrand quotient via $h(A) := h^0(A)/h^1(A)$ or $h(A) := h^0(A)/h^{-1}(A)$ or $h(A) := h^2(A)/h^1(A)$, but Theorem 23.37 implies that these definitions are all the same as ours. The notation $q(A)$ is often used instead of $h(A)$, and one occasionally sees the Herbrand quotient defined as the reciprocal of our definition (as in [2], for example), but this is less standard.

Corollary 23.40. *Let G be a finite cyclic group. Given an exact sequence of G-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

we have a corresponding exact hexagon

$$\begin{tikzcd}
& \hat{H}^0(G,A) \arrow[r, "\hat{\alpha}^0"] & \hat{H}^0(G,B) \arrow[dr, "\hat{\beta}^0"] & \\
\hat{H}_0(G,C) \arrow[ur, "\hat{\delta}_0"] & & & \hat{H}^0(G,C) \arrow[dl, "\hat{\delta}_0"] \\
& \hat{H}_0(G,B) \arrow[ul, "\hat{\beta}_0"] & \hat{H}_0(G,A) \arrow[l, "\hat{\alpha}_0"] &
\end{tikzcd}$$

Proof. This follows immediately from Theorems 23.32 and 23.37. ☐

Corollary 23.41. *Let G be a finite cyclic group. For any exact sequence of G-modules*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0,$$

if any two of $h(A), h(B), h(C)$ are defined then so is the third and $h(B) = h(A)h(C)$.

Proof. Using the exact hexagon given by Corollary 23.40 we can compute the cardinality

$$h^0(A) = \#\hat{H}^0(G,A) = \#\ker\hat{\alpha}^0\#\operatorname{im}\hat{\alpha}^0 = \#\ker\alpha^0\#\ker\beta^0.$$

Applying a similar calculation to $\hat{H}^0(G,C)$ and $\hat{H}^1(G,B)$ yields

$$h^0(A)h^0(C)h_0(B) = \#\ker\hat{\alpha}^0\#\ker\hat{\beta}^0\#\ker\hat{\delta}^0\#\ker\hat{\alpha}_0\#\ker\hat{\beta}_0\#\ker\hat{\delta}_0.$$

Doing the same for $\hat{H}^0(G,B)$, $\hat{H}_0(G,A)$, $\hat{H}_0(G,C)$ yields

$$h^0(B)h_0(A)h_0(C) = \#\ker\hat{\beta}^0\#\ker\hat{\delta}^0\#\ker\hat{\alpha}_0\#\ker\hat{\beta}_0\#\ker\hat{\delta}_0\#\ker\hat{\alpha}^0 = h^0(A)h^0(C)h_0(B).$$

If any two of $h(A), h(B), h(C)$ are defined then at least four of the groups in the exact hexagon are finite, and the remaining two are non-adjacent, but these two must then also be finite. In this case we can rearrange the identity above to obtain $h(B) = h(A)h(C)$. $\square$

Corollary 23.42. *Let G be a finite cyclic group, and let A and B be G-modules. If $h(A)$ and $h(B)$ are defined then so is $h(A\oplus B) = h(A)h(B)$.*

Proof. Apply Corollary 23.41 to the split exact sequence $0\to A\to A\oplus B\to B\to 0$. $\square$

Lemma 23.43. *Let $G=\langle g\rangle$ be a finite cyclic group. If A is an induced or finite G-module then $h(A) = 1$.*

Proof. If A is an induced G-module then $h_0(A) = h^0(A) = h(A) = 1$, by Theorem 23.34. If A is finite, then the exact sequence

$$0 \longrightarrow A^G \longrightarrow A \xrightarrow{g-1} A \longrightarrow A_G \longrightarrow 0$$

implies $\#A^G = \#\ker(g-1) = \#\operatorname{coker}(g-1) = \#A_G$, and therefore

$$h_0(A) = \#\ker\hat{N}_G = \#\operatorname{coker}\hat{N}_G = h^0(A),$$

so $h(A) = h^0(A)/h_0(A) = 1$. $\square$

Corollary 23.44. *Let G be a finite cyclic group and let A be a G-module that is a finitely generated abelian group. Then $h(A) = h(A/A_{\mathrm{tor}})$ whenever either is defined.*

Proof. Apply Corollary 23.41 and Lemma 23.43 to $0\to A_{\mathrm{tor}}\to A\to A/A_{\mathrm{tor}}\to 0$. $\square$

Remark 23.45. The hypothesis of Corollary 23.44 actually guarantees that $h(A)$ is defined, but we won't prove this here.

Corollary 23.46. *Let G be a finite cyclic group and let A be a trivial G-module that is a finitely generated abelian group. Then $h(A) = (\#G)^r$, where r is the rank of A.*

Proof. We have $A/A_{\mathrm{tor}} \simeq \mathbb{Z}^r$, where $\mathbb{Z}$ is a trivial G-module. Then $\mathbb{Z}_G = \mathbb{Z} = \mathbb{Z}^G$, and $\hat{N}_G\colon \mathbb{Z}_G\to\mathbb{Z}^G$ is multiplication by $\#G$, so $h(\mathbb{Z}) = \#\operatorname{coker}\hat{N}_G/\#\ker\hat{N}_G = \#G$. Now apply Corollaries 23.42 and 23.44. $\square$

Lemma 23.47. *Let G be a finite cyclic group and let $\alpha\colon A\to B$ be a morphism of G-modules with finite kernel and cokernel. If either $h(A)$ or $h(B)$ is defined then $h(A) = h(B)$.*

Proof. Applying Corollary 23.41 to the exact sequences

$$0 \to \ker \alpha \to A \to \operatorname{im} \alpha \to 0$$
$$0 \to \operatorname{im} \alpha \to B \to \operatorname{coker} \alpha \to 0$$

yields $h(A) = h(\ker \alpha)h(\operatorname{im} \alpha) = h(\operatorname{im} \alpha) = h(\operatorname{im} \alpha)h(\operatorname{coker} \alpha) = h(B)$, by Lemma 23.43, since $\ker \alpha$ and $\operatorname{coker} \alpha$ are finite. The lemma follows. □

Corollary 23.48. *Let G be a finite cyclic group and let A be a G-module containing a sub-G-module B of finite index. Then $h(A) = h(B)$ whenever either is defined.*

Proof. Apply Lemma 23.47 to the inclusion $B \to A$. □

23.6 A little homological algebra

In an effort to keep these notes self-contained, in this final section we present proofs of the homological results that were used above. For the sake of concreteness we restrict our attention to modules, but everything in this section generalizes to suitable abelian categories. We use R to denote an arbitrary (not necessarily commutative) ring (in previous section R was the group ring $\mathbb{Z}[G]$). Statements that use the term R-module without qualification are understood to apply in both the category of left R-modules and the category of right R-modules.

23.6.1 Complexes

Definition 23.49. A *chain complex* C is a sequence of R-module morphisms

$$\cdots \xrightarrow{d_2} C_2 \xrightarrow{d_1} C_1 \xrightarrow{d_0} C_0 \longrightarrow 0,$$

with $d_n \circ d_{n+1} = 0$; the d_n are *boundary maps*. The *nth homology group* of C is the R-module $H_n(C) \coloneqq Z_n(C)/B_n(C)$, where $Z_n(C) \coloneqq \ker d_{n-1}$ and $B_n(C) \coloneqq \operatorname{im} d_n$ are the R-modules of *cycles* and *boundaries*, respectively; for $n < 0$ we define $C_n = 0$ and d_n is the zero map.

A morphism of chain complexes $f\colon C \to D$ is a sequence of R-module morphisms $f_n\colon C_n \to D_n$ that commute with boundary maps (so $f_n \circ d_n = d_n \circ f_{n+1}$).[7] Such a morphism necessarily maps cycles to cycles and boundaries to boundaries, yielding natural morphisms $H_n(f)\colon H_n(C) \to H_n(D)$ of homology groups.[8] We thus have a family of functors $H_n(\bullet)$ from the category of chain complexes to the category of abelian groups. The category of chain complexes has kernels and cokernels (and thus exact sequences). The set $\operatorname{Hom}(C, D)$ of morphisms of chain complexes $C \to D$ is an abelian group under addition: $(f+g)_n = f_n+g_n$.

The category of chain complexes of R-modules contains direct sums and direct products: if A and B are chain complexes of R-modules then $(A \oplus B)_n \coloneqq A_n \oplus B_n$ and the boundary maps $d_n\colon (A \oplus B)_{n+1} \to (A \oplus B)_n$ are defined component-wise: $d_n(a \oplus b) \coloneqq d_n(a) \oplus d_n(b)$. Because the boundary maps are defined component-wise, the kernel of the boundary map of

[7] We use the symbols d_n to denote boundary maps of both C and D; in general, the domain and codomain of any boundary or coboundary map should be inferred from context.

[8] In fact $H_n(f)\colon H_n(C) \to H_n(D)$ is a morphism of R-modules, but in all the cases of interest to us, either the homology groups are all trivial (as occurs for exact chain complexes, such as the standard resolution of $\mathbb{Z}$ by $\mathbb{Z}[G]$-modules), or $R = \mathbb{Z}$ (as in the chain complexes used to define the Ext and Tor groups below), so we will generally refer to homology groups rather than homology modules.

a direct sum is the direct sum of the kernels of the boundary maps on the components, and similarly for images. It follows that $H_n(A \oplus B) \simeq H_n(A) \oplus H_n(B)$, and this isomorphism commutes with the natural inclusion and projection maps in to and out of the direct sums on both sides. In other words, $H_n(\bullet)$ is an additive functor (see Definition 23.16). This extends to arbitrary (possibly infinite) direct sums, and also to arbitrary direct products, although we will only be concerned with finite direct sums/products.[9]

Theorem 23.50. *Associated to each short exact sequence of chain complexes*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is a long exact sequence of homology groups

$$\cdots \longrightarrow H_{n+1}(A) \xrightarrow{H_{n+1}(\alpha)} H_{n+1}(B) \xrightarrow{H_{n+1}(\beta)} H_{n+1}(C) \xrightarrow{\delta_n} H_n(A) \xrightarrow{H_n(\alpha)} H_n(B) \xrightarrow{H_n(\beta)} H_n(C) \longrightarrow \cdots$$

and this association maps morphisms of short exact sequences to morphisms of long exact sequences. In other words, the family of functors $H_n(\bullet)$ is a homological δ-functor.

For $n < 0$ we have $H_n(\bullet) = 0$, by definition, so this sequence ends at $H_0(C) \to 0$.

Proof. For any chain complex C, let $Y_n(C) := C_n/B_n(C)$. Applying the snake lemma to

$$\begin{tikzcd} & Y_{n+1}(A) \arrow[r, "\alpha_{n+1}"] \arrow[d, "d_n"] & Y_{n+1}(B) \arrow[r, "\beta_{n+1}"] \arrow[d, "d_n"] & Y_{n+1}(C) \arrow[r] \arrow[d, "d_n"] & 0 \\ 0 \arrow[r] & Z_n(A) \arrow[r, "\alpha_n"] & Z_n(B) \arrow[r, "\beta_n"] & Z_n(C) & \end{tikzcd}$$

(where α_n, β_n, d_n denote obviously induced maps) yields the exact sequence

$$H_{n+1}(A) \xrightarrow{\alpha_{n+1}} H_{n+1}(B) \xrightarrow{\beta_{n+1}} H_{n+1}(C) \xrightarrow{\delta_n} H_n(A) \xrightarrow{\alpha_n} H_n(B) \xrightarrow{\beta_n} H_n(G).$$

The verification of the commutativity of diagrams of long exact sequences of homology groups associated to commutative diagrams of short exact sequences of chain complexes is as in the proof of Theorem 23.8, *mutatis mutandi.* □

Definition 23.51. Two morphisms $f, g\colon C \to D$ of chain complexes are *homotopic* if there exist morphisms $h_n\colon C_n \to D_{n+1}$ such that $f_n - g_n = d_n \circ h_n + h_{n-1} \circ d_{n-1}$ for all $n \geq 0$ (where $h_{-1} := 0$); this defines an equivalence relation $f \sim g$, since (a) $f \sim f$ (take $h = 0$), (b) if $f \sim g$ via h then $g \sim f$ via $-h$, and (c) if $f_1 \sim f_2$ via h_1 and $f_2 \sim f_3$ via h_2 then $f_1 \sim f_3$ via $h_1 + h_2$.

Lemma 23.52. *Homotopic morphisms of chain complexes $f, g\colon C \to D$ induce they some morphisms of homology groups $H_n(C) \to H_n(D)$; we have $H_n(f) = H_n(g)$ for all $n \geq 0$.*

Proof. Let $[z] \in H_n(C) = Z_n(C)/B_n(C)$ denote the homology class $z \in Z_n(C)$. We have

$$f_n(z) - g_n(z) = d_n(h_n(z)) + h_{n-1}(d_{n-1}(z)) = d_n(h_n(z)) + 0 \in B_n(D),$$

thus $H_n(f)([z]) - H_n(g)([z]) = 0$. It follows that $H_n(f) = H_n(g)$ for all $n \geq 0$. □

[9]This does not imply that the Ext and Tor functors defined below commute with arbitrary direct sums and direct products; see Remarks 23.62 and 23.66.

Definition 23.53. A *cochain complex* C is a sequence of R-module morphisms

$$0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \cdots$$

with $d^{n+1} \circ d^n = 0$. The *nth cohomology group* of C is the R-module $H^n(C) := Z^n(C)/B^n(C)$, where $Z^n(C) := \ker d^n$ and $B^n(C) := \operatorname{im} d^{n-1}$ are the R-modules of *cocycles* and *coboundaries*;; for $n < 0$ we define $C^n = 0$ and d^n is the zero map. A morphism of cochain complexes $f\colon C \to D$ consists of R-module morphisms $f^n\colon C^n \to D^n$ that commute with coboundary maps, yielding natural morphisms $H^n(f)\colon H^n(C) \to H^n(D)$ and a functors $H^n(\bullet)$ from the category of cochain complexes to the category of abelian groups. Cochain complexes form a category with kernels and cokernels, as well as direct sums and direct products (coboundary maps are defined component-wise). Like $H_n(\bullet)$, the functor $H^n(\bullet)$ is additive and commutes with arbitrary direct sums and direct products.

The set $\operatorname{Hom}(C, D)$ of morphisms of cochain complexes $C \to D$ forms an abelian group under addition: $(f + g)^n = f^n + g^n$. Morphisms of cochain complexes $f, g\colon C \to D$ are *homotopic* if there are morphisms $h^n\colon C^{n+1} \to D^n$ such that $f^n - g^n = h^n \circ d^n + d^{n-1} \circ h^{n-1}$ for all $n \geq 0$ (where $h^{-1} := 0$); this defines an equivalence relation $f \sim g$.[10]

Theorem 23.54. *Associated to every short exact sequence of cochain complexes*

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is a long exact sequence of homology groups

$$\cdots \longrightarrow H^n(A) \xrightarrow{H^n(\alpha)} H^n(B) \xrightarrow{H^n(\beta)} H^n(C) \xrightarrow{\delta^n} H^{n+1}(A) \xrightarrow{H^{n+1}(\alpha)} H^{n+1}(B) \xrightarrow{H^{n+1}(\beta)} H^{n+1}(C) \longrightarrow \cdots$$

and this association maps morphisms of short exact sequences ot morphisms of long exact sequences, that is, the family of functors $H^n(\bullet)$ is a cohomological δ-functor.

For $n < 0$ we have $H_n(\bullet) = 0$, by definition, so this sequence begins with $0 \to H^0(A)$.

Proof. Adapt the proof of Theorem 23.50. □

Lemma 23.55. *Homotopic morphisms of cochain complexes $f, g\colon C \to D$ induce they same morphisms of cohomology groups $H^n(C) \to H^n(D)$; we have $H^n(f) = H^n(g)$ for all $n \geq 0$.*

Proof. Adapt the proof of Lemma 23.52. □

23.6.2 Projective resolutions

Recall that a *projective* R-module is an R-module P with the property that if $\pi\colon M \twoheadrightarrow N$ is a surjective morphism of R-modules, every R-module morphism $\varphi\colon P \to N$ factors through π:

$$\begin{tikzcd} & P \arrow[dl, dotted, "\exists\phi"'] \arrow[d, "\varphi"] \\ M \arrow[r, two heads, "\pi"] & N \end{tikzcd}$$

Free modules are projective, since we can then fix an R-basis $\{e_i\}$ for P and define $\phi(e_i)$ by picking any element of $\pi^{-1}(\varphi(e_i))$ (note that the ϕ so constructed is in no way canonical).

[10] Note the the order of composition in the homotopy relations for morphisms of chain/cochain complexes.

Definition 23.56. Let M be an R-module. A *projective resolution* of M is an exact chain complex P with $P_0 = M$ and P_n projective for all $n > 0$.

Every R-module has a projective resolution, since (as noted earlier), every R-module M has a free resolution (we can always construct $d_0\colon P_1 \twoheadrightarrow M$ by taking P_1 to be free module with basis M, then similarly construct $d_1\colon P_2 \twoheadrightarrow \ker d_0$, and so on).

Proposition 23.57. *Let M and N be R-modules with projective resolutions P and Q, respectively. Every R-module morphism $\alpha_0\colon M \to N$ extends to a morphism $\alpha\colon P \to Q$ of chain complexes that is unique up to homotopy.*

Proof. We inductively construct α_n for $n \geq 1$ (the base case is given). Suppose we have constructed a commutative diagram of exact sequences

$$\begin{tikzcd}
\cdots \arrow[r, "d_{n+1}"] & P_{n+1} \arrow[r, "d_n"] & P_n \arrow[r, "d_{n-1}"] \arrow[d, "\alpha_n"] & P_{n-1} \arrow[r, "d_{n-2}"] \arrow[d, "\alpha_{n-1}"] & \cdots \arrow[r, "d_1"] \arrow[d, "\cdots"] & P_1 \arrow[r, "d_0"] \arrow[d, "\alpha_1"] & M \arrow[r] \arrow[d, "\alpha_0"] & 0 \arrow[d] \\
\cdots \arrow[r, "d_{n+1}"] & Q_{n+1} \arrow[r, "d_n"] & Q_n \arrow[r, "d_{n-1}"] & P_{n-1} \arrow[r, "d_{n-2}"] & \cdots \arrow[r, "d_1"] & Q_1 \arrow[r, "d_0"] & N \arrow[r] & 0
\end{tikzcd}$$

Then $d_{n-1} \circ \alpha_n \circ d_n = \alpha_{n-1} \circ d_{n-1} \circ d_n = 0$, so $\operatorname{im}(\alpha_n \circ d_n) \subseteq \ker d_{n-1} = \operatorname{im} d_n$. We now define $\alpha_{n+1}\colon P_{n+1} \to Q_{n+1}$ as a pullback of the morphism $\alpha_n \circ d_n\colon P_{n+1} \to \operatorname{im} d_n$ along the surjection $d_n\colon Q_{n+1} \to \operatorname{im} d_n$ such that $d_n \circ \alpha_{n+1} = \alpha_n \circ d_n$.

Now suppose $\beta\colon P \to Q$ is another morphism of projective resolutions with $\beta_0 = \alpha_0$, and let $\gamma = \alpha - \beta$. To show that α and β are homotopic it suffices to construct maps $h_n\colon P_n \to Q_{n+1}$ such that $d_n \circ h_n = \gamma_n - h_{n-1} \circ d_{n-1}$ (where $h_{-1} = d_{-1} = 0$). We have $\gamma_0 = \alpha_0 - \beta_0 = 0$, so let $h_0 \coloneqq 0$ and inductively assume $d_n \circ h_n = \gamma_n - h_{n-1} \circ d_{n-1}$. Then

$$d_n \circ (\gamma_{n+1} - h_n \circ d_n) = d_n \circ \gamma_{n+1} - (d_n \circ h_n) \circ d_n = \gamma_n \circ d_n - (\gamma_n - h_{n-1} \circ d_{n-1}) \circ d_n = 0,$$

so $\operatorname{im}(\gamma_{n+1} - h_n \circ d_n) \subseteq B_{n+1}(Q)$. The R-module P_{n+1} is projective, so we can pullback the morphism $(\gamma_{n+1} - h_n \circ d_n)\colon P_{n+1} \to B_{n+1}(Q)$ along the surjection $d_{n+1}\colon Q_{n+1} \to B_{n+1}(Q)$ to obtain h_{n+1} satisfying $d_{n+1} \circ h_{n+1} = \gamma_{n+1} - h_n \circ d_n$ as desired. $\square$

23.6.3 Hom and Tensor

If M and N are R-modules, the set $\operatorname{Hom}_R(M, N)$ of R-module morphisms $M \to N$ forms an abelian group under pointwise addition (so $(f + g)(m) \coloneqq f(m) + g(m)$) that we may view as a $\mathbb{Z}$-module. For each R-module A we have a contravariant functor $\operatorname{Hom}_R(\bullet, A)$ that sends each R-module M to the $\mathbb{Z}$-module

$$M^* \coloneqq \operatorname{Hom}_R(M, A)$$

and each R-module morphism $\varphi\colon M \to N$ to the $\mathbb{Z}$-module morphism

$$\begin{aligned} \varphi^*\colon \operatorname{Hom}_R(N, A) &\to \operatorname{Hom}_R(M, A) \\ f &\mapsto f \circ \varphi. \end{aligned}$$

To check this, note that

$$\varphi^*(f + g) = (f + g) \circ \varphi = f \circ \varphi + g \circ \varphi = \varphi^*(f) + \varphi^*(g),$$

so φ^* is a morphism of $\mathbb{Z}$=modules (homomorphism of abelian groups), and

$$\begin{aligned}\mathrm{id}_M^* &= (f \mapsto f \circ \mathrm{id}_M) = (f \mapsto f) = \mathrm{id}_{M^*},\\ (\phi \circ \varphi)^* &= (f \mapsto f \circ \phi \circ \varphi) = (f \mapsto f \circ \varphi) \circ (f \mapsto f \circ \phi) = \varphi^* \circ \phi^*,\end{aligned}$$

thus $\mathrm{Hom}_R(\bullet, A)$ is a contravariant functor.

Lemma 23.58. *Let $\varphi\colon M \to N$ and $\phi\colon N \to P$ be morphisms of R-modules. The sequence*

$$M \xrightarrow{\varphi} N \xrightarrow{\phi} P \longrightarrow 0$$

is exact if and only if for every R-module A the sequence

$$0 \longrightarrow \mathrm{Hom}_R(P, A) \xrightarrow{\phi^*} \mathrm{Hom}_R(N, A) \xrightarrow{\varphi^*} \mathrm{Hom}_R(M, A)$$

is exact.

Proof. **(⇒)**: If $\phi^*(f) = f \circ \phi = 0$ then $f = 0$, since ϕ is surjective, so ϕ^* is injective. We have $\varphi^* \circ \phi^* = (\varphi \circ \phi)^* = 0^* = 0$, so $\mathrm{im}\,\phi^* \subseteq \ker \varphi^*$. Let $\phi^{-1}\colon P \xrightarrow{\sim} N/\ker\phi$. Each $g \in \ker \varphi^*$ vanishes on $\mathrm{im}\,\varphi = \ker \phi$ inducing $\bar{g}\colon N/\ker\phi \to A$ with $g = \bar{g} \circ \phi^{-1} \circ \phi \in \mathrm{im}\,\phi^*$.

(⇐): For $A = P/\mathrm{im}\,\phi$ and $\pi\colon P \to P/\mathrm{im}\,\phi$ the projective map, we have $\phi^*(\pi) = 0$ and therefore $\pi = 0$, since ϕ^* is injective, so $P = \mathrm{im}\,\phi$ and ϕ is surjective. For $A = P$ we have $0 = (\varphi^* \circ \phi^*)(\mathrm{id}_P) = \mathrm{id}_P \circ \phi \circ \varphi = \phi \circ \varphi$, so $\mathrm{im}\,\varphi \subseteq \ker \phi$. For $A = N/\mathrm{im}\,\varphi$, and $\pi\colon N \to N/\mathrm{im}\,\varphi$ the projection map, we have $\pi \in \ker \varphi^* = \mathrm{im}\,\phi^*$, thus $\pi = \phi^*(\sigma) = \sigma \circ \phi$ for some $\sigma \in \mathrm{Hom}(P, A)$. Now $\pi(\ker\phi) = \sigma(\phi(\ker\phi)) = 0$ implies $\ker\phi \subseteq \ker\pi = \mathrm{im}\,\varphi$. □

Definition 23.59. A sequence of morphisms $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is *left exact* if it is exact at A and B ($\ker f = 0$ and $\mathrm{im}\, f = \ker g$), and *right exact* if it is exact at B and C ($\mathrm{im}\, f = \ker g$ and $\mathrm{im}\, g = C$). A functor that takes exact sequences to left (resp. right) exact sequences is said to be *left exact* (resp. *right exact*).

Corollary 23.60. *For any R-module A the functor $\mathrm{Hom}_R(\bullet, A)$ is left exact.*

Proof. This follows immediately from the forward implication in Lemma 23.58. □

Corollary 23.61. *For any R-module A, the functor $\mathrm{Hom}_R(\bullet, A)$ is an additive functor.*

Proof. See [6, Lemma 12.7.2] for a proof that this follows from left exactness; it is easy to check directly in any case. □

Remark 23.62. Corollary 23.61 implies that $\mathrm{Hom}_R(\bullet, A)$ commutes with finite direct sums, but it does *not* commute with infinite direct sums (direct products are fine).

Remark 23.63. The covariant functor $\mathrm{Hom}_R(A, \bullet)$ that sends $\varphi\colon M \to N$ to $(f \mapsto \varphi \circ f)$ is also left exact.

If M is a right R-module and A is a left R-module, the tensor product $M \otimes_R A$ is an abelian group consisting of sums of pure tensors $m \otimes a$ with $m \in M$ and $a \in A$ satisfying:

- $m \otimes (a + b) = m \otimes b + m \otimes b$;
- $(m + n) \otimes a = m \otimes a + m \otimes a$;
- $mr \otimes a = m \otimes ra$.

For each left R-module A we have a covariant functor $\bullet \otimes_R A$ that sends each right R-module M to the $\mathbb{Z}$-module
$$M_* := M \otimes_R A,$$
and each right R-module morphism $\varphi\colon M \to N$ to the $\mathbb{Z}$-module morphism
$$\begin{aligned}\varphi_*\colon M \otimes_R A &\to N \otimes_R A\\ m \otimes a &\mapsto \varphi(m) \otimes a\end{aligned}$$

For each left R-module A we also have a covariant functor $\operatorname{Hom}_{\mathbb{Z}}(A, \bullet)$ that sends each $\mathbb{Z}$-module B to the right R-module $\operatorname{Hom}_{\mathbb{Z}}(A, B)$ with $\varphi(a)r := \varphi(ra)$ and each $\mathbb{Z}$-module morphism $\varphi\colon B \to C$ to the right R-module morphism $\operatorname{Hom}(A, B) \to \operatorname{Hom}(A, C)$ defined by $f \mapsto \varphi \circ f$. Note that $(\varphi rs)(a) = \varphi(rsa) = (\varphi r)(sa) = ((\varphi r)s)(a)$, so $\operatorname{Hom}_{\mathbb{Z}}(A, B)$ is indeed a *right* R-module.

For any abelian group B there is a natural isomorphism of $\mathbb{Z}$-modules
$$\begin{aligned}\operatorname{Hom}_{\mathbb{Z}}(M \otimes_R A, B) &\xrightarrow{\sim} \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbb{Z}}(A, B))\\ \varphi &\mapsto (m \mapsto (a \mapsto \varphi(m \otimes a)))\\ (m \otimes a \mapsto \phi(m)(a) &\leftarrow\!\shortmid \phi\end{aligned} \tag{3}$$

The functors $\bullet \otimes_R A$ and $\operatorname{Hom}_{\mathbb{Z}}(A, \bullet)$ are thus *adjoint functors*. One can view (3) as a universal property that determines $M \otimes_R A$ up to a unique isomorphism.

Lemma 23.64. *For any left R-module the functor $\bullet \otimes_R A$ is right exact.*

Proof. Let
$$0 \longrightarrow M \xrightarrow{\varphi} N \xrightarrow{\phi} P \longrightarrow 0,$$
be an exact sequence of right R-modules. For any $\sum_i p_i \otimes a_i \in P_*$ we can pick $n_i \in N$ such that $\phi(n_i) = p_i$ and then $\phi(\sum_i n_i \otimes a) = \sum_i p_i \otimes a$, thus ϕ_* is surjective. For any $\sum_i m_i \otimes a_i \in M \otimes_R A$ we have $\phi_*(\varphi_*(\sum_i m_i \otimes a_i)) = \sum_i \phi(\varphi(m_i)) \otimes a_i = \sum_i 0 \otimes a_i = 0$, so $\operatorname{im} \varphi_* \subseteq \ker \phi_*$. To prove $\operatorname{im} \varphi_* = \ker \phi_*$ it suffices to show that $N_*/\operatorname{im} \varphi_* \simeq P_*$, since the surjectivity of ϕ_* implies $N^*/\ker \varphi_* \simeq P_*$. For every abelian group B the sequence
$$0 \longrightarrow \operatorname{Hom}_R(P, \operatorname{Hom}_{\mathbb{Z}}(A, B)) \xrightarrow{\phi^*} \operatorname{Hom}_R(N, \operatorname{Hom}_{\mathbb{Z}}(A, B)) \xrightarrow{\phi^*} \operatorname{Hom}_R(M, \operatorname{Hom}_{\mathbb{Z}}(A, B))$$
is left exact (by applying Corollary 23.60 to the right R-module $\operatorname{Hom}_{\mathbb{Z}}(A, B)$; note that the corollary applies to both left and right R-modules). Equivalently, by (3),
$$0 \longrightarrow \operatorname{Hom}_{\mathbb{Z}}(P_*, B) \xrightarrow{\phi_*^*} \operatorname{Hom}_{\mathbb{Z}}(N_*, B) \xrightarrow{\varphi_*^*} \operatorname{Hom}_{\mathbb{Z}}(M_*, B),$$
Applying Lemma 23.58 and the surjectivity of ϕ_* yields the desired right exact sequence
$$M_* \xrightarrow{\varphi_*} N_* \xrightarrow{\phi} P_* \longrightarrow 0.$$
□

Corollary 23.65. *For any left R-module A, the functor $\bullet \otimes_R A$ is an additive functor.*

Proof. See [6, Lemma 12.7.2] for a proof that this follows from right exactness; it is easy to check directly in any case. □

Remark 23.66. Corollary 23.65 implies that $\bullet \otimes_R A$ commutes with finite direct sums, and in fact it commutes with arbitrary direct sums (but not direct products).

Remark 23.67. For any right R-module A the functor $A \otimes_R \bullet$ is also right exact.

If A is an R-module and C is a chain complex of R-modules, applying the functor $\mathrm{Hom}(\bullet, A)$ to the R-modules C_n and boundary maps $d_n\colon C_{n+1} \to C_n$ yields a cochain complex C^* of $\mathbb{Z}$-modules $C^n \coloneqq C_n^*$ and coboundary maps $d^n \coloneqq d_n^*$,[11] and morphisms $f\colon C \to D$ of chain complexes become morphisms $f^*\colon C^* \to D^*$ of cochain complexes. We thus also have a contravariant left exact functor from the category of chain complexes to the category of cochain complexes.

Proposition 23.68. *Let A be an R-module and let $\bullet^*$ denote the application of the functor $\mathrm{Hom}(\bullet, A)$. Let $f, g\colon C \to D$ be homotopic morphisms of chain complexes of R-modules. Then $f^*, g^*\colon D^* \to C^*$ are homotopic morphisms of cochain complexes of $\mathbb{Z}$-modules.*

Proof. The morphisms f and g are homotopic, so their exist morphisms $h_n\colon C_n \to D_{n+1}$ such that $f_n - g_n = d_n \circ h_n + h_{n-1} \circ d_{n-1}$ for all $n \geq 0$. Applying the contravariant functor $\mathrm{Hom}(\bullet, A)$ yields

$$f_n^* - g_n^* = h_n^* \circ d_n^* + d_{n-1}^* \circ h_{n-1}^*,$$

where $h_n^*\colon D_{n+1} \to C_n$ for all $n \geq 0$, with $h_{-1} = 0$. Thus f^* and g^* are homotopic. $\square$

Proposition 23.69. *Let A be a left R-module and let $\bullet_*$ denote the application of the functor $\bullet \otimes_R A$. Let $f, g\colon C \to D$ be homotopic morphisms of chain complexes of right R-modules. Then $f_*, g_*\colon C_* \to D_*$ are homotopic morphisms of chain complexes of $\mathbb{Z}$-modules.*

Proof. The morphisms f and g are homotopic, so their exist morphisms $h_n\colon C_n \to D_{n+1}$ such that $f_n - g_n = d_n \circ h_n + h_{n-1} \circ d_{n-1}$ for all $n \geq 0$. Applying the covariant functor $\bullet \otimes_R A$ yields

$$f_{n*} - g_{n*} = d_{n*} \circ h_{n*} + h_{n-1_*} \circ d_{n-1_*},$$

where $h_{n*}\colon C_{n+1} \to D_n$ for all $n \geq 0$, with $h_{-1} = 0$. Thus f_* and g_* are homotopic. $\square$

23.6.4 Ext and Tor functors

Definition 23.70. Let P be a projective resolution of an R-module M. The *truncation* of P is the chain complex $\overline{P}$ with $\overline{P}_0 \coloneqq P_1$ and $\overline{P}_n \coloneqq P_{n+1}$ for all $n > 0$ (which need not be exact at $\overline{P}_0$).[12] Any morphism of projective resolutions $f\colon P \to Q$ induces a morphism $\bar{f}\colon \overline{P} \to \overline{Q}$ of their truncations with $\bar{f}_n \coloneqq f_{n+1}$.

Theorem 23.71. *Let P, Q be projective resolutions of an R-module M, let A be an R-module, and let $\bullet_A^*$ denote application of $\mathrm{Hom}_R(\bullet, A)$. Then $H^n(\overline{P}_A^*) \simeq H^n(\overline{Q}_A^*)$ for $n \geq 0$.*

Proof. We will drop the subscript A in the proof to ease the notation.

Let $f\colon P \to Q$ and $g\colon Q \to P$ be extensions of the identity morphism id_M given by Proposition 23.57. The composition $g \circ f\colon P \to P$ is an extension of id_M, as is id_P, so $g \circ f$ is homotopic to id_P, by Proposition 23.57. We have $(g \circ f)_0 = \mathrm{id}_M = (\mathrm{id}\, P)_0$, which implies that $\overline{g \circ f} = \bar{g} \circ \bar{f}$ and $\overline{\mathrm{id}_P} = \mathrm{id}_{\overline{P}}$ are also homotopic (via the same homotopy; note $h_0 = 0$ in the proof of Proposition 23.57). Similarly, $\bar{f} \circ \bar{g}$ and $\mathrm{id}_{\overline{Q}}$ are homotopic.

Applying $\mathrm{Hom}_R(\bullet, A)$ yields homotopic morphisms $\bar{f}^*\colon \overline{Q}^* \to \overline{P}^*$ and $\bar{g}^*\colon \overline{P}^* \to \overline{Q}^*$, with $\bar{f}^* \circ \bar{g}^*$ homotopic to $\mathrm{id}_{\overline{P}}^* = \mathrm{id}_{\overline{P}^*}$ and $\bar{g}^* \circ \bar{f}^*$ homotopic to $\mathrm{id}_{\overline{Q}}^* = \mathrm{id}_{\overline{Q}^*}$, by Proposition 23.68. By Lemma 23.55, $\bar{f}^*$ and $\bar{g}^*$ induce isomorphims $H^n(\overline{P}_A^*) \simeq H^n(\overline{Q}_A^*)$ for all $n \geq 0$. $\square$

[11] This justifies our indexing the boundary maps $d_n\colon C_{n+1} \to C_n$ rather than $d_n\colon C_n \to C_{n-1}$.

[12] The intuition is that the truncation of projective resolution of M can serve as a replacement for M.

Definition 23.72. Let A and M be R-modules. $\operatorname{Ext}_R^n(M,A)$ is the abelian group $H^n(\overline{P}_A^*)$ uniquely determined by Theorem 23.71 using any projective resolution P of M. If $\alpha\colon A\to B$ is a morphism of R-modules the map $\varphi\mapsto\alpha\circ\varphi$ induces a morphism of cochain complexes $\overline{P}^*, A\to\overline{P}_B^*$ and morphisms $\operatorname{Ext}_R^n(M,\alpha)\colon \operatorname{Ext}_R^n(M,A)\to\operatorname{Ext}_R^n(M,B)$ for each $n\ge 0$.

We thus have a family of functors $\operatorname{Ext}_R^n(M,\bullet)$ from the category of R-modules to the category of abelian groups that is a cohomological δ-functor (by Theorem 23.54).

Lemma 23.73. *Let M be an R-module. The functors $\operatorname{Ext}_R^n(M,\bullet)$ are additive functors and thus commute with finite direct sums and products.*

Proof. This follows from Corollary 23.61 and the fact $H^n(\bullet)$ is an additive functor. □

Lemma 23.74. *For any two R-modules M and A we have $\operatorname{Ext}_R^0(M,A)\simeq\operatorname{Hom}_R(M,A)$.*

Proof. Let $\cdots\to P_2\to P_1\to M\to 0$ be a projective resolution of M. Applying $\operatorname{Hom}_R(\bullet,A)$ yields an exact sequence $0\to M^*\to P_1^*\to P_2^*\to\cdots$, and we observe that

$$\operatorname{Ext}_R^0(M,A)=H^0(\overline{P}^*)=Z^0(\overline{P}^*)/B^0(\overline{P}^*)=\ker(P_1^*\to P_2^*)/\operatorname{im}(0\to P_1^*)\simeq M^*. \qquad \square$$

Theorem 23.75. *Let P, Q be projective resolutions of a right R-module M. Let A be a left R-module, and let $\bullet_*^A$ denote application of $\bullet\otimes_R A$. Then $H_n(\overline{P}_*^A)\simeq H_n(\overline{Q}_*^A)$ for $n\ge 0$.*

Proof. We drop the superscript A in the proof to ease the notation.

Let $f\colon P\to Q$ and $g\colon Q\to P$ be extensions of the identity morphism id_M given by Proposition 23.57. As in the proof of Theorem 23.71, $\bar g\circ\bar f$ and $\mathrm{id}_{\overline{P}}$ are homotopic, as are $\bar f\circ\bar g$ and $\mathrm{id}_{\overline{Q}}$.

Applying $\bullet\otimes_R A$ yields homotopic morphisms $\bar f_*\colon\overline{P}_*\to\overline{Q}_*$ and $\bar g_*\colon\overline{Q}_*\to\overline{P}_*$, with $\bar f_*\circ\bar g_*$ homotopic to $\mathrm{id}_{\overline{P}_*}$ and $\bar f_*\circ\bar g_*$ homotopic to $\mathrm{id}_{\overline{Q}_*}$. By Lemma 23.52, $\bar f_*$ and $\bar g_*$ induce isomorphisms $H_n(\overline{P}_*)\simeq H_n(\overline{Q}_*)$ for all $n\ge 0$. □

Definition 23.76. Let A a left R-module and let M be a right R-module. $\operatorname{Tor}_n^R(M,A)$ is the abelian group $H_n(\overline{P}_*^A)$ uniquely determined by Theorem 23.75 using any projective resolution P of M. If $\alpha\colon A\to B$ is a morphism of left R-modules the map $x\otimes a\mapsto x\otimes\varphi(a)$ induces a morphism $\overline{P}_*^A\to\overline{P}_*^B$ and morphisms $\operatorname{Tor}_n^R(M,\alpha)\colon\operatorname{Tor}_n^R(M,A)\to\operatorname{Ext}_n^R(M,B)$ for each $n\ge 0$. This yields a family of functors $\operatorname{Tor}_n^R(M,\bullet)$ from the category of left R-modules to the category of abelian groups that is a homological δ-functor (by Theorem 23.50).

Lemma 23.77. *Let M be a right R-module. The functors $\operatorname{Tor}_n^R(M,\bullet)$ are additive functors and thus commute with finite direct sums and products.*

Proof. This follows from Corollary 23.65 and the fact $H_n(\bullet)$ is an additive functor. □

Lemma 23.78. *For any two R-modules M and A we have $\operatorname{Tor}_0^R(M,A)\simeq M\otimes_R A$.*

Proof. Let $\cdots\to P_2\to P_1\to M\to 0$ be a projective resolution of M. Applying $\bullet\otimes_R A$ yields the exact sequence $\cdots P_{2*}\to P_{1*}\to M_*\to 0$, and we observe that

$$\operatorname{Tor}_0^R(M,A)=H_0(\overline{P}_*)=Z_0(\overline{P}_*)/B_0(\overline{P}_*)=\ker(P_{1*}\to 0)/\operatorname{im}(P_{2*}\to P_{1*})\simeq M_*, \qquad \square$$

Remark 23.79. One can also define $\operatorname{Ext}_R^n(M,A)$ and $\operatorname{Tor}_n^R(M,A)$ using injective resolutions; see [7, §2.7] for a proof that this yields the same results.

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