# 17 The functional equation

In the previous lecture we proved that the Riemann zeta function $\zeta(s)$ has an Euler product and an analytic continuation to the right half-plane $\operatorname{Re}(s) > 0$. In this lecture we complete the picture by deriving a *functional equation* that relates the values of $\zeta(s)$ to those of $\zeta(1-s)$. This will then also allow us to extend $\zeta(s)$ to a meromorphic function on $\mathbb{C}$ that is holomorphic except for a simple pole at $s = 1$.

## 17.1 Fourier transforms and Poisson summation

A key tool we will use to derive the functional equation is the *Poisson summation formula*, a result from harmonic analysis that we now recall.

**Definition 17.1.** A *Schwartz function* on $\mathbb{R}$ is a complex-valued $C^\infty$ function $f\colon \mathbb{R} \to \mathbb{C}$ that decays rapidly to zero: for all $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$\sup_{x\in\mathbb{R}} \left| x^m f^{(n)}(x) \right| < \infty,$$

where $f^{(n)}$ denotes the $n$th derivative of $f$. The *Schwartz space* $\mathcal{S}(\mathbb{R})$ of all Schwartz functions on $\mathbb{R}$ is a $\mathbb{C}$-vector space (of infinite dimension).

**Remark 17.2.** For any $p \in \mathbb{R}_{\geq 1}$, the Schwartz space $\mathcal{S}(\mathbb{R})$ is contained in the space $L^p(\mathbb{R})$ of functions on $f\colon \mathbb{R} \to \mathbb{C}$ for which the Lebesgue integral $\int_{\mathbb{R}} |f(x)|^p dx$ exists. The space $L^p(\mathbb{R})$ is a complete normed $\mathbb{C}$-vector space under the $L^p$-norm $\|f\|_p := (\int_{\mathbb{R}} |f(x)|^p dx)^{1/p}$, and is thus a Banach space. The Swchartz space $\mathcal{S}(\mathbb{R})$ is not complete under the $L^p$ norm, but it is dense in $L^p(\mathbb{R})$ under the subspace topology. One can equip the Schwartz space with a translation-invariant metric of its own under which it is a complete metric space (and thus a Fréchet space, since it is also locally convex), but the topology of $\mathcal{S}(\mathbb{R})$ will not concern us here. Similar comments apply to $\mathcal{S}(\mathbb{R}^n)$.

It follows immediately from the definition and standard properties of the derivative that the Schwartz space $\mathcal{S}(\mathbb{R})$ is closed under differentiation, multiplication by polynomials, sums and products, and linear change of variable. It is also closed under *convolution*: for any $f, g \in \mathcal{S}(\mathbb{R})$ the function

$$(f * g)(x) := \int_{\mathbb{R}} f(y)g(x-y)dy$$

is also an element of $\mathcal{S}(\mathbb{R})$. Convolution is commutative, associative, and bilinear.

**Example 17.3.** All compactly supported functions $C^\infty$ functions are Schwartz functions, as is the Gaussian $g(x) := e^{-\pi x^2}$. Non-examples include any function that does not tend to zero as $x \to \pm\infty$ (so all nonzero polynomials), as well as functions like $(1 + x^{2n})^{-1}$.

**Definition 17.4.** The *Fourier transform* of a Schwartz function $f \in \mathcal{S}(\mathbb{R})$ is the function

$$\hat{f}(y) := \int_{\mathbb{R}} f(x)e^{-2\pi ixy}dx,$$

which is also a Schwartz function [1, Thm. 5.1.3]. We can recover $f(x)$ from $\hat{f}(y)$ via the inverse transform

$$f(x) := \int_{\mathbb{R}} \hat{f}(y)e^{+2\pi ixy}dy;$$

see [1, Thm. 5.1.9] for a proof of this fact. The maps $f \mapsto \hat{f}$ and $\hat{f} \mapsto f$ are thus inverse linear operators on $\mathcal{S}(\mathbb{R})$ (they are also continuous in the metric topology of $\mathcal{S}(\mathbb{R})$ and thus homeomorphisms).

**Remark 17.5.** The invertibility of the Fourier transform on the Schwartz space $\mathcal{S}(\mathbb{R})$ a key motivation for its definition. For functions in $L^1(\mathbb{R})$ (the largest space of functions for which our definition of the Fourier transform makes sense), the Fourier transform of a smooth function decays rapidly to zero, and the Fourier transform of a function that decays rapidly to zero is smooth; this leads one to consider the subspace $\mathcal{S}(\mathbb{R})$ of smooth functions that decay rapidly to zero. One can show that $\mathcal{S}(\mathbb{R})$ is the largest subspace of $L^1(\mathbb{R})$ closed under multiplication by polynomials on which the Fourier transform is invertible.[1]

The Fourier transform changes convolutions into products, and vice versa. We have

$$\widehat{f * g} = \hat{f}\hat{g} \qquad \text{and} \qquad \widehat{fg} = \hat{f} * \hat{g},$$

for all $f, g \in \mathcal{S}(\mathbb{R})$ (see Problem Set 8). One can thus view the Fourier transform as an isomorphism of (non-unital) $\mathbb{C}$-algebras that sends $(\mathcal{S}(\mathbb{R}), +, \times)$ to $(\mathcal{S}(\mathbb{R}), +, *)$.

**Lemma 17.6.** *For all $a \in \mathbb{R}_{>0}$ and $f \in \mathcal{S}(\mathbb{R})$, we have $\widehat{f(ax)}(y) = \frac{1}{a}\hat{f}(\frac{y}{a})$.*

*Proof.* Applying the substitution $t = ax$ yields

$$\widehat{f(ax)}(y) = \int_{\mathbb{R}} f(ax)e^{-2\pi ixy}dx = \frac{1}{a}\int_{\mathbb{R}} f(t)e^{-2\pi ity/a}dt = \frac{1}{a}\hat{f}\left(\frac{y}{a}\right). \qquad \square$$

**Lemma 17.7.** *For $f \in \mathcal{S}(\mathbb{R})$ we have $\frac{d}{dy}\hat{f}(y) = -2\pi i\,\widehat{xf(x)}(y)$ and $\widehat{\frac{d}{dx}f(x)}(y) = 2\pi iy\hat{f}(y)$.*

*Proof.* Noting that $xf \in \mathcal{S}(\mathbb{R})$, the first identity follows from

$$\tfrac{d}{dy}\hat{f}(y) = \tfrac{d}{dy}\left(\int_{\mathbb{R}} f(x)e^{-2\pi ixy}dx\right) = \int_{\mathbb{R}} f(x)(-2\pi ix)e^{-2\pi ixy}dx = -2\pi i\,\widehat{xf(x)}(y),$$

since we may differentiate under the integral via dominated convergence. For the second, we note that $\lim_{x\to\pm\infty} f(x) = 0$, so integration by parts yields

$$\widehat{\tfrac{d}{dx}f(x)}(y) = \int_{\mathbb{R}} f'(x)e^{-2\pi ixy}dx = 0 - \int_{\mathbb{R}} f(x)(-2\pi iy)e^{-2\pi ixy}dx = 2\pi iy\hat{f}(y). \qquad \square$$

The Fourier transform is compatible with the inner product $\langle f, g\rangle := \int_{\mathbb{R}} f(x)\overline{g(x)}dx$. Indeed, we can easily derive Parseval's identity:

$$\langle f, g\rangle = \int_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{\mathbb{R}}\int_{\mathbb{R}} \hat{f}(y)\overline{g(x)}e^{+2\pi ixy}dxdy = \int_{\mathbb{R}} \hat{f}(y)\overline{\hat{g}(y)}dy = \langle \hat{f}, \hat{g}\rangle,$$

which when applied to $g = f$ yields Plancherel's identity:

$$\|f\|_2 = \langle f, f\rangle = \langle \hat{f}, \hat{f}\rangle = \|\hat{f}\|_2,$$

where $\|f\|_2 = \int_{\mathbb{R}} |f(x)|^2dx = \int_{\mathbb{R}} f(x)\bar{f}(x)dx$ is the $L^2$-norm. For number-theoretic applications there is an analogous result due to Poisson.

[1] I thank Keith Conrad and Terry Tao for clarifying this point.

**Theorem 17.8** (Poisson Summation Formula)**.** *For all $f \in \mathcal{S}(\mathbb{R})$ we have the identity*

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n).$$

*Proof.* We first note that both sums are well defined; the rapid decay property of Schwartz functions guarantees absolute convergence. Let $F(x) := \sum_{n\in\mathbb{Z}} f(x+n)$. Then $F$ is a periodic $C^\infty$-function, so it has a Fourier series expansion

$$F(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi i n x},$$

with Fourier coefficients

$$c_n = \int_0^1 F(x)e^{-2\pi i n t}dt = \int_0^1 \sum_{m\in\mathbb{Z}} f(x+m)e^{-2\pi i n y}dy = \int_{\mathbb{R}} f(x)e^{-2\pi i n y}dy = \hat{f}(n).$$

We then note that

$$\sum_{n\in\mathbb{Z}} f(n) = F(0) = \sum_{n\in\mathbb{Z}} \hat{f}(n). \qquad \square$$

Finally, we note that the Gaussian function $e^{-\pi x^2}$ is its own Fourier transform.

**Lemma 17.9.** *Let $g(x) := e^{-\pi x^2}$. Then $\hat{g}(y) = g(y)$.*

*Proof.* The function $g(x)$ satisfies the first order ordinary differential equation

$$g' + 2\pi x g = 0, \tag{1}$$

with initial value $g(0) = 1$. Multiplying both sides by $i$ and taking Fourier transforms yields

$$i(\widehat{g'} + 2\pi\widehat{xg}) = i(2\pi i x\hat{g} - i\hat{g}') = \hat{g}' + 2\pi x\hat{g} = 0,$$

via Lemma 17.7. So $\hat{g}$ also satisfies (1), and $\hat{g}(0) = \int_{\mathbb{R}} e^{-\pi x^2}dx = 1$, so $\hat{g} = g$. $\square$

### 17.1.1 Jacobi's theta function

We now define the *theta function*[2]

$$\Theta(\tau) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau}.$$

The sum is absolutely convergent for $\operatorname{Im}\tau > 0$ and thus defines a holomorphic function on the upper half plane. It is easy to see that $\Theta(\tau)$ is periodic modulo 2, that is,

$$\Theta(\tau+2) = \Theta(\tau),$$

but it it also satisfies another functional equation.

**Lemma 17.10.** *For all $a \in \mathbb{R}_{>0}$ we have $\Theta(ia) = \Theta(i/a)/\sqrt{a}$.*

*Proof.* Put $g(x) := e^{-\pi x^2}$ and $h(x) := g(\sqrt{a}x) = e^{-\pi x^2 a}$. Lemmas 17.6 and 17.9 imply

$$\hat{h}(y) = \widehat{g(\sqrt{a}x)}(y) = \hat{g}\big(y/\sqrt{a}\big)/\sqrt{a} = g\big(y/\sqrt{a}\big)/\sqrt{a}.$$

Plugging $\tau = ia$ into $\Theta(\tau)$ and applying Poisson summation (Theorem 17.8) yields

$$\Theta(ia) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 a} = \sum_{n\in\mathbb{Z}} h(n) = \sum_{n\in\mathbb{Z}} \hat{h}(n) = \sum_{n\in\mathbb{Z}} g\big(n/\sqrt{a}\big)/\sqrt{a} = \Theta(i/a)/\sqrt{a} \qquad \square$$

[2]The function $\Theta(\tau)$ we define here is a special case of one of four parameterized families of theta functions $\Theta_i(z:\tau)$ originally defined by Jacobi for $i = 0,1,2,3$, which play an important role in the theory of elliptic functions and modular forms; in terms of Jacobi's notation, $\Theta(\tau) = \Theta_3(0;\tau)$.

### 17.1.2 Euler's gamma function

You are probably familiar with the gamma function $\Gamma(s)$, which plays a key role in the functional equation of not only the Riemann zeta function but many of the more general zeta functions and $L$-series we wish to consider. Here we recall some of its analytic properties. We begin with the definition of $\Gamma(s)$ as a Mellin transform.

**Definition 17.11.** The *Mellin transform* of a function $f\colon \mathbb{R}_{>0} \to \mathbb{C}$ is the complex function defined by

$$\mathcal{M}(f)(s) := \int_0^\infty f(t)t^{s-1}dt,$$

whenever this integral converges. It is holomorphic on $\operatorname{Re} s \in (a,b)$ for any interval $(a,b)$ in which the integral $\int_0^\infty |f(t)|t^{\sigma-1}dt$ converges for all $\sigma \in (a,b)$.

**Definition 17.12.** The *Gamma function*

$$\Gamma(s) := \mathcal{M}(e^{-t})(s) = \int_0^\infty e^{-t}t^{s-1}dt,$$

is the Mellin transform of $e^{-t}$. Since $\int_0^\infty |e^{-t}|t^{\sigma-1}dt$ converges for all $\sigma > 0$, the integral defines a holomorphic function on $\operatorname{Re}(s) > 0$.

Integration by parts yields

$$\Gamma(s) = \frac{t^se^{-t}}{s}\bigg|_0^\infty + \frac{1}{s}\int_0^\infty e^{-t}t^s dt = \frac{\Gamma(s+1)}{s},$$

thus $\Gamma(s)$ has a simple pole at $s = 0$ with residue 1 (since $\Gamma(1) = \int_0^\infty e^{-t}dt = 1$), and

$$\Gamma(s+1) = s\Gamma(s) \tag{2}$$

for $\operatorname{Re}(s) > 0$. Equation (2) allows us to extend $\Gamma(s)$ to a meromorphic function on $\mathbb{C}$ with simple poles at $s = 0, -1, -2, \ldots$, and no other poles.

An immediate consequence of (2) is that for integers $n > 0$ we have

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = n(n-1)(n-2)\cdots 2\cdot 1\cdot \Gamma(1) = n!,$$

thus the gamma function can be viewed as an extension of the factorial function. The gamma function satisfies many useful identities in addition to (2), including the following.

**Theorem 17.13** (Euler's Reflection Formula)**.** *We have*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

*as meromorphic functions on $\mathbb{C}$ with simple poles at each integer $s \in \mathbb{Z}$.*

*Proof.* Let $f(s) := \Gamma(s)\Gamma(1-s)\sin(\pi s)$. The function $\Gamma(s)\Gamma(1-s)$ has a simple pole at each $s \in \mathbb{Z}$ and no other poles, while $\sin(\pi s)$ has a zero at each $s \in \mathbb{Z}$ and no poles, so $f(s)$ is holomorphic on $\mathbb{C}$. We now note that

$$f(s+1) = \Gamma(s+1)\Gamma(-s)\sin(\pi s+\pi) = -s\Gamma(s)\Gamma(-s)\sin(\pi s) = \Gamma(s)\Gamma(1-s)\sin(\pi s) = f(s),$$

so $f$ is periodic (with period 1). Using the substitution $u = e^t$ we obtain

$$|\Gamma(s)| \le \int_0^\infty |e^{-t}t^{s-1}|dt = \int_{-\infty}^\infty |e^{-e^u}e^{u(s-1)}|e^u du = \int_{-\infty}^\infty e^{u\operatorname{Re}(s)-e^u}du.$$

This implies $|\Gamma(s)|$ is bounded on $\operatorname{Re}(s) \in [1,2]$, hence on $\operatorname{Re}(s) \in [0,1] \cap \operatorname{Im}(s) \ge 1$, via (2). It follows that in the strip $\operatorname{Re}(s) \in [0,1]$ we have

$$|f(s)| = |\Gamma(s)||\Gamma(1-s)||\sin(\pi s)| = O(e^{\operatorname{Im}(s)}),$$

as $\operatorname{Im}(s) \to \infty$, since $|\sin(\pi s)| = \frac{1}{2}|e^{is} - e^{i\bar{s}}| = O(e^{\operatorname{Im}(s)})$. By Lemma 17.14 below, $f(s)$ is constant. To determine the constant, as $s \to 0$ we have $\Gamma(s) \sim \frac{1}{s}$ and $\sin(\pi s) \sim \pi s$, thus

$$f(0) = \lim_{s\to 0} f(s) = \lim_{s\to 0} \Gamma(s)\Gamma(1-s)\sin(\pi s) = \lim_{s\to 0} \frac{1}{s} \cdot 1 \cdot \pi s = \pi,$$

and the theorem follows. $\square$

**Lemma 17.14.** *Let $f(s)$ be a holomorphic function on $\mathbb{C}$ such that $f(s+1) = f(s)$ and $|f(s)| = O(e^{\operatorname{Im}(s)})$ as $\operatorname{Im}(s) \to \infty$ in the vertical strip $\operatorname{Re}(s) \in [0,1]$. Then $f$ is constant.*

*Proof.* The function

$$g(s) = \frac{f(s) - f(a)}{\sin(\pi(s-a))}$$

is holomorphic on $\mathbb{C}$, since $f(s) - f(a)$ is holomorphic and vanishes at the zeros $a + \mathbb{Z}$ of $\sin(\pi(s-a))$ (all of which are simple). We also have $g(s+1) = g(s)$, and $|g(s)|$ is bounded on $\operatorname{Re}(s) \in [0,1]$, since as $\operatorname{Im}(s) \to \infty$ we have $|f(s) - f(a)| = O(e^{\operatorname{Im}(s)})$ and $|\sin(\pi(s-a))| \sim e^{\pi \operatorname{Im}(s)}$. It follows that $g(s)$ is bounded on $\mathbb{C}$, hence constant, by Liouville's theorem. We must have $g = 0$, since $|g(s)| = O(e^{(1-\pi)\operatorname{Im}(s)}) = o(1)$ as $\operatorname{Im}(s) \to \infty$, and this implies $f(s) = f(a)$ for all $s \in \mathbb{C}$. $\square$

**Example 17.15.** Putting $s = \frac{1}{2}$ in the reflection formula yields $\Gamma(\frac{1}{2})^2 = \pi$, so $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

**Corollary 17.16.** *The function $\Gamma(s)$ has no zeros on $\mathbb{C}$.*

*Proof.* Suppose $\Gamma(s_0) = 0$. The RHS of the reflection formula $\Gamma(s)\Gamma(1-s) = \pi/\sin(\pi s)$ is never zero, since $\sin(\pi s)$ has no poles, so $\Gamma(1-s)$ must have a pole at $s_0$. Therefore $1 - s_0 \in \mathbb{Z}_{\le 0}$, equivalently, $s_0 \in \mathbb{Z}_{\ge 1}$, but then $\Gamma(s_0) = (s_0 - 1)! \ne 0$, a contradiction. $\square$

### 17.1.3 Completing the zeta function

Let us now consider the function

$$F(s) := \pi^{-s}\Gamma(s)\zeta(2s),$$

which is a meromorphic on $\mathbb{C}$ and holomorphic on $\operatorname{Re}(s) > 1/2$. In the region $\operatorname{Re}(s) > 1/2$ we have an absolutely convergent sum

$$F(s) = \pi^{-s}\Gamma(s)\sum_{n\ge 1} n^{-2s} = \sum_{n\ge 1}(\pi n^2)^{-s}\Gamma(s) = \sum_{n\ge 1}\int_0^\infty (\pi n^2)^{-s}t^{s-1}e^{-t}dt,$$

and the substitution $t = \pi n^2 y$ with $dt = \pi n^2 dy$ yields

$$F(s) = \sum_{n\geq 1} \int_0^\infty (\pi n^2)^{-s} (\pi n^2 y)^{s-1} e^{-\pi n^2 y} \pi n^2 dy = \sum_{n\geq 1} \int_0^\infty y^{s-1} e^{-\pi n^2 y} dy.$$

By the Fubini-Tonelli theorem, we can swap the sum and the integral to obtain

$$F(s) = \int_0^\infty y^{s-1} \sum_{n\geq 1} e^{-\pi n^2 y} dy.$$

We have $\Theta(iy) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 y} = 1 + 2\sum_{n\geq 1} e^{-\pi n^2 y}$, thus

$$\begin{aligned} F(s) &= \frac{1}{2}\int_0^\infty y^{s-1}(\Theta(iy)-1)dy \\ &= \frac{1}{2}\left(\int_0^1 y^{s-1}\Theta(iy)dy - \frac{1}{s} + \int_1^\infty y^{s-1}(\Theta(iy)-1)dy\right) \end{aligned}$$

We now focus on the first integral on the RHS. The change of variable $t = \frac{1}{y}$ yields

$$\int_0^1 y^{s-1}\Theta(iy)dy = \int_\infty^1 t^{1-s}\Theta(i/t)(-t^{-2})dt = \int_1^\infty t^{-s-1}\Theta(i/t)dt.$$

By Lemma 17.10, $\Theta(i/t) = \sqrt{t}\Theta(it)$, and adding $-\int_1^\infty t^{-s-1/2}dt + \int_1^\infty t^{-s-1/2}dt = 0$ yields

$$\begin{aligned} &= \int_1^\infty t^{-s-1/2}\big(\Theta(it)-1\big)dt + \int_1^\infty t^{-s-1/2}dt \\ &= \int_1^\infty t^{-s-1/2}\big(\Theta(it)-1\big)dt - \frac{1}{1/2 - s}. \end{aligned}$$

Plugging this back into our equation for $F(s)$ we obtain the identity

$$F(s) = \frac{1}{2}\int_1^\infty \left(y^{s-1} + y^{-s-1/2}\right)\big(\Theta(iy)-1\big)dy - \frac{1}{2s} - \frac{1}{1-2s},$$

valid on $\mathrm{Re}(s) > 1/2$. We now observe that $F(s) = F(\frac{1}{2} - s)$ for $s \neq 0, \frac{1}{2}$, which allows us to analytically extend $F(s)$ to a meromorphic function on $\mathbb{C}$ with poles only at $s = 0, 1/2$. Replacing $s$ with $s/2$ leads us to define the *completed zeta function*

$$Z(s) := \pi^{-s/2}\Gamma(\tfrac{s}{2})\zeta(s), \tag{3}$$

which is meromorphic on $\mathbb{C}$ and satisfies the *functional equation*

$$Z(s) = Z(1-s). \tag{4}$$

It has simple poles at 0 and 1 (and no other poles). The only zeros of $Z(s)$ on $\mathrm{Re}(s) > 0$ are the zeros of $\zeta(s)$, since by Corollary 17.16, the gamma function $\Gamma(s)$ has no zeros (and neither does $\pi^{-s/2}$). Thus the zeros of $Z(s)$ on $\mathbb{C}$ all lie in the critical strip $0 < \mathrm{Re}(s) < 1$.

The functional equation also allows us to analytically extend $\zeta(s)$ to a meromorphic function on $\mathbb{C}$ whose only pole is a simple pole at $s = 1$; the pole of $Z(s)$ at $s = 0$ comes from the pole of $\Gamma(s/2)$ at $s = 0$. The function $\Gamma(s/2)$ also has poles at $-2, -4, \ldots$ where $Z(s)$ does not, so our extended $\zeta(s)$ must have zeros at $-2, -4, \ldots$. These are *trivial zeros*;

all the interesting zeros of $\zeta(s)$ lie in the critical strip and are conjectured to lie only on the critical line $\mathrm{Re}(s) = 1/2$ (this is the Riemann hypothesis).

We can compute $\zeta(0)$ using the functional equation. From (3) and (4) we have

$$\zeta(s) = \frac{Z(s)}{\pi^{-s/2}\Gamma(\frac{s}{2})} = \frac{Z(1-s)}{\pi^{-s/2}\Gamma(\frac{s}{2})} = \frac{\pi^{\frac{(s-1)}{2}}\Gamma(\frac{1-s}{2})}{\pi^{-s/2}\Gamma(\frac{s}{2})}\zeta(1-s) = \frac{\pi^{s-1/2}\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s), \tag{5}$$

We know that $\zeta(s)$ has a simple pole with residue 1 at $s = 1$, so

$$1 = \lim_{s\to 1^+}(s-1)\zeta(s) = \lim_{s\to 1^+}\frac{(s-1)\pi^{s-1/2}\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\zeta(1-s).$$

When $s = 1$, the denominator on the RHS is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, which cancels $\pi^{1-1/2} = \sqrt{\pi}$ in the numerator. Using $\Gamma(z) = \frac{1}{z}\Gamma(z+1)$ to shift $\Gamma(\frac{1-s}{2})$ in the numerator yields

$$1 = \lim_{s\to 1^+}(s-1)\tfrac{2}{1-s}\Gamma\left(\tfrac{3-s}{2}\right)\zeta(1-s) = -2\Gamma(1)\zeta(0) = -2\zeta(0).$$

Thus $\zeta(0) = -1/2$.

Using the reflection formula to replace $\Gamma(\frac{s}{2}) = \pi/(\Gamma(\frac{2-s}{2})\sin(\frac{\pi s}{2}))$ in (5), we have

$$\zeta(s) = \pi^{s-3/2}\Gamma(\tfrac{1-s}{2})\Gamma(\tfrac{2-s}{2})\sin(\tfrac{\pi s}{2})\zeta(1-s).$$

Applying the duplication formula $\Gamma(2z) = \pi^{-1/2}2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2})$ with $z = \frac{1-s}{2}$ then yields

$$\zeta(s) = 2^s\pi^{s-1}\sin(\tfrac{\pi s}{2})\Gamma(1-s)\zeta(1-s), \tag{6}$$

which is how one often sees the functional equation for $\zeta(s)$ written.

## 17.2 Gamma factors and a holomorphic zeta function

If we write out the Euler product for the completed zeta function, we have

$$Z(s) = \pi^{-s/2}\Gamma(\tfrac{s}{2})\cdot\prod_p(1-p^{-s})^{-1}.$$

One should think of this as a product over the places of the field $\mathbb{Q}$; the leading factor

$$\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(\tfrac{s}{2})$$

that distinguishes the completed zeta function $Z(s)$ from $\zeta(s)$ corresponds to the real archimedean place of $\mathbb{Q}$. When we discuss Dedekind zeta functions in a later lecture we will see that there are gamma factors $\Gamma_{\mathbb{R}}$ and $\Gamma_{\mathbb{C}}$ associated to each of the real and complex places of a number field.

If we insert an additional factor of $\binom{s}{2} := \frac{s(s-1)}{2}$ in $Z(s)$ we can remove the poles at 0 and 1, yielding a function $\xi(s)$ holomorphic on $\mathbb{C}$. This yields Riemann's seminal result.

**Theorem 17.17** (Analytic Continuation II)**.** *The function*

$$\xi(s) := \tbinom{s}{2}\Gamma_{\mathbb{R}}(s)\zeta(s)$$

*is holomorphic on $\mathbb{C}$ and satisfies the functional equation*

$$\xi(s) = \xi(1-s).$$

*The zeros of $\xi(s)$ all lie in the critical strip* $0 < \mathrm{Re}(s) < 1$.

**Remark 17.18.** We will usually work with $Z(s)$ and deal with the poles rather than making it holomorphic by introducing additional factors; some authors use $\xi(s)$ to denote our $Z(s)$.

## References

[1] Elias M. Stein and Rami Shakarchi, *Fourier analysis: an introduction*, Princeton University Press, 2007.