LECTURE NOTES ON CHEREDNIK ALGEBRAS

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Contents

1. Introduction

Double affine Hecke algebras, also called Cherednik algebras, were introduced by Cherednik in 1993 as a tool in his proof of Macdonald's conjectures about orthogonal polynomials for root systems. Since then, it has been realized that Cherednik algebras are of great independent interest; they appeared in many different mathematical contexts and found numerous applications.

The present notes are based on a course on Cherednik algebras given by the first author at MIT in the Fall of 2009. Their goal is to give an introduction to Cherednik algebras, and to review the web of connections between them and other mathematical objects. For this reason, the notes consist of many parts that are relatively independent of each other. Also, to keep the notes within the bounds of a one-semester course, we had to limit the discussion of many important topics to a very brief outline, or to skip them altogether. For a more in-depth discussion of Cherednik algebras, we refer the reader to research articles dedicated to this subject.

The notes do not contain any original material. In each section, the sources of the exposition are listed in the notes at the end of the section.

The organization of the notes is as follows.

In Section 2, we define the classical and quantum Calogero-Moser systems, and their analogs for any Coxeter groups introduced by Olshanetsky and Perelomov. Then we introduce Dunkl operators, prove the fundamental result of their commutativity, and use them to establish integrability of the Calogero-Moser and Olshanetsky-Perelomov systems. We also prove the uniqueness of the first integrals for these systems.

In Section 3, we conceptualize the commutation relations between Dunkl operators and coordinate operators by introducing the main object of these notes - the rational Cherednik algebra. We develop the basic theory of rational Cherednik algebras (proving the PBW theorem), and then pass to the representation theory of rational Cherednik algebras, more precisely, study the structure of category $\mathcal{O}$. After developing the basic theory (parallel to the case of semisimple Lie algebras), we completely work out the representations in the rank 1 case, and prove a number of results about finite dimensional representations and about representations of the rational Cherednik algebra attached to the symmetric group.

In Section 4, we evaluate the Macdonald-Mehta integral, and then use it to find the supports of irrieducible modules over the rational Cherednik algebras with the trivial lowest weight, in particular giving a simple proof of the theorem of Varagnolo and Vasserot, classifying such representations which are finite dimensional.

In Section 5, we describe the theory of parabolic induction and restriction functors for rational Cherednik algebras, developed in [BE], and give some applications of this theory, such as the description of the category of Whittaker modules and of possible supports of modules lying in category $\mathcal{O}$.

In Section 6, we define Hecke algebras of complex reflection groups, and the Knizhnik-Zamolodchikov (KZ) functor from the category $\mathcal{O}$ of a rational Cherednik algebra to the category of finite dimensional representations of the corresponding Hecke algebra. We use this functor to prove the formal flatness of Hecke algebras of complex reflection groups (a theorem of Broué, Malle, and Rouquier), and state the theorem of Ginzburg-Guay-Opdam-Rouquier that the KZ functor is an equivalence from the category $\mathcal{O}$ modulo its torsion part to the category of representations of the Hecke algebra.

In Section 7, we define rational Cherednik algebras for orbifolds. We also define the corresponding Hecke algebras, and define the KZ functor from the category of modules over the former to that over the latter. This generalizes to the “curved” case the KZ functor for rational Cherednik algebras of complex reflection groups, defined in Section 6. We then apply the KZ functor to showing that if the universal cover of the orbifold in question has a trivial H^2 (with complex coefficients), then the orbifold Hecke algebra is formally flat, and explain why the condition of trivial H^2 cannot be dropped. Next, we list examples of orbifold Hecke algebras which satisfy the condition of vanishing H^2 (and hence are formally flat). These include usual, affine, and double affine Hecke algebras, as well as Hecke algebras attached to Fuchsian groups, which include quantizations of del Pezzo surfaces and their Hilbert schemes; we work these examples out in some detail, highlighting connections with other subjects. Finally, we discuss the issue of algebraic flatness, and prove it in the case of algebras of rank 1 attached to Fuchsian groups, using the theory of deformations of group algebras of Coxeter groups developed in [ER].

In Section 8, we define symplectic reflection algebras (which inlude rational Cherednik algebras as a special case), and generalize to them some of the theory of Section 3. Namely, we use the theory of deformations of Koszul algebras to prove the PBW theorem for symplectic reflection algebras. We also determine the center of symplectic reflection algebras, showing that it is trivial when the parameter t is nonzero, and is isomorphic to the shperical subalgebra if $t = 0$. Next, we give a deformation-theoretic interpretation of symplectic reflection algebras as universal deformations of Weyl algebras smashed with finite groups. Finally, we discuss finite dimensional representations of symplectic reflection algebras for $t = 0$, showing that the Azumaya locus on the space of such representations coincides with the smooth locus. This uses the theory of Cohen-Macaulay modules and of homological dimension in commutative algebra. In particular, we show that for Cherednik algebras of type A_{n-1}, the whole representation space is smooth and coincides with the spectrum of the center.

In Section 9, we give another description of the spectrum of the center of the rational Cherednik algebra of type A_{n-1} (for $t = 0$), as a certain space of conjugacy classes of pairs of matrices, introduced by Kazhdan, Kostant, and Sternberg, and called the Calogero-Moser space (this space is obtained by classical hamiltonian reduction, and is a special case of a quiver variety). This yields a new construction of the Calogero-Moser integrable system. We also sketch a proof of the Gan-Ginzburg theorem claiming that the quotient of the commuting scheme by conjugation is reduced, and hence isomorphic to $\mathbb{C}^{2n}/\mathfrak{S}_n$. Finally, we explain that the Calogero-Moser space is a topologically trivial deformation of the Hilbert scheme of the plane, we use the theory of Cherednik algebras to compute the cohomology ring of this space.

In Section 10, we generalize the results of Section 9 to the quantum case. Namely, we prove the quantum analog of the Gan-Ginzburg theorem (the Harish Chandra-Levasseur-Stafford theorem), and explain how to quantize the Calogero-Moser space using quantum Hamiltonian reduction. Not surprisingly, this gives the same quantization as was constructed in the previous sections, namely, the spherical subalgebra of the rational Cherednik algebra.

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2. Classical and quantum Olshanetsky-Perelomov systems for finite Coxeter groups

2.1. **The rational quantum Calogero-Moser system.** Consider the differential operator

$$H = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - c(c+1)\sum_{i\neq j}\frac{1}{(x_i - x_j)^2}.$$

This is the quantum Hamiltonian for a system of n particles on the line of unit mass and the interaction potential (between particle 1 and 2) $c(c+1)/(x_1 - x_2)^2$. This system is called *the rational quantum Calogero-Moser system.*

It turns out that the rational quantum Calogero-Moser system is completely integrable. Namely, we have the following theorem.

Theorem 2.1. *There exist differential operators L_j with rational coefficients of the form*

$$L_j = \sum_{i=1}^{n} (\frac{\partial}{\partial x_i})^j + \text{ lower order terms}, \quad j = 1, \ldots, n,$$

which are invariant under the symmetric group $\mathfrak{S}_n$, homogeneous of degree $-j$, and such that $L_2 = H$ and $[L_j, L_k] = 0, \forall j, k = 1, \ldots, n$.

We will prove this theorem later.

Remark 2.2. $L_1 = \sum_i \frac{\partial}{\partial x_i}$.

2.2. **Complex reflection groups.** Theorem 2.1 can be generalized to the case of any finite Coxeter group. To describe this generalization, let us recall the basic theory of finite Coxeter groups and, more generally, complex reflection groups.

Let $\mathfrak{h}$ be a finite-dimensional complex vector space. We say that a semisimple element $s \in \mathrm{GL}(\mathfrak{h})$ is a *(complex) reflection* if $\mathrm{rank}\,(1 - s) = 1$. This means that s is conjugate to the diagonal matrix $\mathrm{diag}(\lambda, 1, \ldots, 1)$ where $\lambda \neq 1$.

Now assume $\mathfrak{h}$ carries a nondegenerate inner product $(\cdot, \cdot)$. We say that a semisimple element $s \in \mathrm{O}(\mathfrak{h})$ is a *real reflection* if $\mathrm{rank}\,(1 - s) = 1$; equivalently, s is conjugate to $\mathrm{diag}(-1, 1, \ldots, 1)$.

Now let $G \subset \mathrm{GL}(\mathfrak{h})$ be a finite subgroup.

Definition 2.3. (i) We say that G is a *complex reflection group* if it is generated by complex reflections.

(ii) If $\mathfrak{h}$ carries an inner product, then a finite subgroup $G \subset \mathrm{O}(\mathfrak{h})$ is a *real reflection group* (or a *finite Coxeter group*) if G is generated by real reflections.

For the complex reflection groups, we have the following important theorem.

Theorem 2.4 (The Chevalley-Shepard-Todd theorem, [Che])**.** *A finite subgroup G of* $\mathrm{GL}(\mathfrak{h})$ *is a complex reflection group if and only if the algebra $(S\mathfrak{h})^G$ is a polynomial (i.e., free) algebra.*

By the Chevalley-Shepard-Todd theorem, the algebra $(S\mathfrak{h})^G$ has algebraically independent generators P_i, homogeneous of some degrees d_i for $i = 1, \ldots, \dim \mathfrak{h}$. The numbers d_i are uniquely determined, and are called *the degrees* of G.

Example 2.5. If $G = \mathfrak{S}_n$, $\mathfrak{h} = \mathbb{C}^{n-1}$ (the space of vectors in $\mathbb{C}^n$ with zero sum of coordinates), then one can take $P_i(p_1, \ldots, p_n) = p_1^{i+1} + \cdots + p_n^{i+1}$, $i = 1, \ldots, n-1$ (where $\sum_i p_i = 0$).

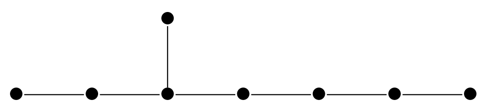
2.3. **Parabolic subgroups.** Let $G \subset \mathrm{GL}(\mathfrak{h})$ be a finite subgroup.

Definition 2.6. A parabolic subgroup of G is the stabilizer G_a of a point $a \in \mathfrak{h}$.

Note that by Chevalley's theorem, a parabolic subgroup of a complex (respectively, real) reflection group is itself a complex (respectively, real) reflection group.

Also, if W is a real reflection group, then it can be shown that a subgroup $W' \subset W$ is parabolic if and only if it is conjugate to a subgroup generated by a subset of simple reflections of W. In this case, the rank of W', i.e. the number of generating simple reflections, equals the codimension of the space $\mathfrak{h}^{W'}$.

Example 2.7. Consider the Coxeter group of type E_8. It has the Dynkin diagram:



The parabolic subgroups will be Coxeter groups whose Dynkin diagrams are obtained by deleting vertices from the above graph. In particular, the maximal parabolic subgroups are $D_7, A_7, A_1 \times A_6, A_2 \times A_1 \times A_4, A_4 \times A_3, D_5 \times A_2, E_6 \times A_1, E_7$.

Suppose $G' \subset G$ is a parabolic subgroup, and $b \in \mathfrak{h}$ is such that $G_b = G'$. In this case, we have a natural G'-invariant decomposition $\mathfrak{h} = \mathfrak{h}^{G'} \oplus (\mathfrak{h}^{*G'})^{\perp}$, and $b \in \mathfrak{h}^{G'}$. Thus we have a nonempty open set $\mathfrak{h}^{G'}_{\mathrm{reg}}$ of all $a \in \mathfrak{h}^{G'}$ for which $G_a = G'$; this set is nonempty because it contains b. We also have a G'-invariant decomposition $\mathfrak{h}^* = \mathfrak{h}^{*G'} \oplus (\mathfrak{h}^{G'})^{\perp}$, and we can define the open set $\mathfrak{h}^{*G'}_{\mathrm{reg}}$ of all $\lambda \in \mathfrak{h}^{G'}$ for which $G_\lambda = G'$. It is clear that this set is nonempty. This implies, in particular, that one can make an alternative definition of a parabolic subgroup of G as the stabilizer of a point in $\mathfrak{h}^*$.

2.4. **Olshanetsky-Perelomov operators.** Let $s \subset \mathrm{GL}(\mathfrak{h})$ be a complex reflection. Denote by $\alpha_s \in \mathfrak{h}^*$ an eigenvector in $\mathfrak{h}^*$ of s with nontrivial eigenvalue.

Let $W \subset \mathrm{O}(\mathfrak{h})$ be a real reflection group and $\mathcal{S} \subset W$ the set of reflections. Clearly, W acts on $\mathcal{S}$ by conjugation. Let $c : \mathcal{S} \to \mathbb{C}$ be a conjugation invariant function.

Definition 2.8. [OP] The quantum Olshanetsky-Perelomov Hamiltonian attached to W is the second order differential operator

$$H := \Delta_{\mathfrak{h}} - \sum_{s \in \mathcal{S}} \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\alpha_s^2},$$

where $\Delta_{\mathfrak{h}}$ is the Laplace operator on $\mathfrak{h}$.

Here we use the inner product on $\mathfrak{h}^*$ which is dual to the inner product on $\mathfrak{h}$.

Let us assume that $\mathfrak{h}$ is an irreducible representation of W (i.e. W is an irreducible finite Coxeter group, and $\mathfrak{h}$ is its reflection representation.) In this case, we can take $P_1(\mathbf{p}) = \mathbf{p}^2$.

Theorem 2.9. *The system defined by the Olshanetsky-Perelomov operator H is completely integrable. Namely, there exist differential operators L_j on $\mathfrak{h}$ with rational coefficients and symbols P_j, such that L_j are homogeneous (of degree $-d_j$), $L_1 = H$, and $[L_j, L_k] = 0$, $\forall j, k$.*

This theorem is obviously a generalization of Theorem 2.1 about $W = \mathfrak{S}_n$.

To prove Theorem 2.9, one needs to develop the theory of Dunkl operators.

Remark 2.10. 1. We will show later that the operators L_j are unique.

2. Theorem 2.9 for classical root systems was proved by Olshanetsky and Perelomov (see [OP]), following earlier work of Calogero, Sutherland, and Moser in type A. For a general Weyl group, this theorem (in fact, its stronger trigonometric version) was proved by analytic methods in the series of papers [HO],[He3],[Op3],[Op4]. A few years later, a simple algebraic proof using Dunkl operators, which works for any finite Coxeter group, was found by Heckman, [He1]; this is the proof we will give below.

For the trigonometric version, Heckman also gave an algebraic proof in [He2], which used non-commuting trigonometric counterparts of Dunkl operators. This proof was later improved by Cherednik ([Ch1]), who defined commuting (although not Weyl group invariant) versions of Heckman's trigonometric Dunkl operators, now called Dunkl-Cherednik operators.

2.5. **Dunkl operators.** Let $G \subset \mathrm{GL}(\mathfrak{h})$ be a finite subgroup. Let $\mathcal{S}$ be the set of reflections in G. For any reflection $s \in \mathcal{S}$, let λ_s be the eigenvalue of s on $\alpha_s \in \mathfrak{h}^*$ (i.e. $s\alpha_s = \lambda_s \alpha_s$), and let $\alpha_s^\vee \in \mathfrak{h}$ be an eigenvector such that $s\alpha_s^\vee = \lambda_s^{-1}\alpha_s^\vee$. We normalize them in such a way that $\langle \alpha_s, \alpha_s^\vee \rangle = 2$.

Let $c : \mathcal{S} \to \mathbb{C}$ be a function invariant with respect to conjugation. Let $a \in \mathfrak{h}$.

The following definition was made by Dunkl for real reflection groups, and by Dunkl and Opdam for complex reflection groups.

Definition 2.11. The Dunkl operator $D_a = D_a(c)$ on $\mathbb{C}(\mathfrak{h})$ is defined by the formula

$$D_a = D_a(c) := \partial_a - \sum_{s \in \mathcal{S}} \frac{2c_s \alpha_s(a)}{(1-\lambda_s)\alpha_s}(1-s).$$

Clearly, $D_a \in \mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$, where $\mathfrak{h}_{\mathrm{reg}}$ is the set of regular points of $\mathfrak{h}$ (i.e. not preserved by any reflection), and $\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$ denotes the algebra of differential operators on $\mathfrak{h}_{\mathrm{reg}}$.

Example 2.12. Let $G = \mathbb{Z}_2$, $\mathfrak{h} = \mathbb{C}$. Then there is only one Dunkl operator up to scaling, and it equals to

$$D = \partial_x - \frac{c}{x}(1-s),$$

where the operator s is given by the formula $(sf)(x) = f(-x)$.

Remark 2.13. The Dunkl operators D_a map the space of polynomials $\mathbb{C}[\mathfrak{h}]$ to itself.

Proposition 2.14. (i) *For any $x \in \mathfrak{h}^*$, one has*

$$[D_a, x] = (a, x) - \sum_{s \in \mathcal{S}} c_s (a, \alpha_s)(x, \alpha_s^\vee) s.$$

(ii) *If $g \in G$ then $gD_a g^{-1} = D_{ga}$.*

Proof. (i) The proof follows immediately from the identity

$$x - sx = \frac{1-\lambda_s}{2}(x, \alpha_s^\vee)\alpha_s.$$

(ii) The identity is obvious from the invariance of the function c. ∎

The main result about Dunkl operators, on which all their applications are based, is the following theorem.

Theorem 2.15 (C. Dunkl, [Du1])**.** *The Dunkl operators commute:*

$$[D_a, D_b] = 0 \text{ for any } a, b \in \mathfrak{h}.$$

Proof. Let $x \in \mathfrak{h}^*$. We have

$$[[D_a, D_b], x] = [[D_a, x], D_b] - [[D_b, x], D_a].$$

Now, using Proposition 2.14, we obtain:

$$\begin{aligned} [[D_a, x], D_b] &= -[\sum_{s \in \mathcal{S}} c_s(a, \alpha_s)(x, \alpha_s^\vee)s, D_b] \\ &= -\sum_{s \in \mathcal{S}} c_s(a, \alpha_s)(x, \alpha_s^\vee)(b, \alpha_s) s D_{\alpha_s^\vee} \cdot \frac{1 - \lambda_s^{-1}}{2}. \end{aligned}$$

Since a and b occur symmetrically, we obtain that $[[D_a, D_b], x] = 0$. This means that for any $f \in \mathbb{C}[\mathfrak{h}]$, $[D_a, D_b]f = f[D_a, D_b]1 = 0$. So for $f, g \in \mathbb{C}[\mathfrak{h}]$, $g \cdot [D_a, D_b]\frac{f}{g} = [D_a, D_b]f = 0$. Thus $[D_a, D_b]\frac{f}{g} = 0$ which implies $[D_a, D_b] = 0$ in the algebra $\mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\text{reg}})$ (since this algebra acts faithfully on $\mathbb{C}(\mathfrak{h})$). ☐

2.6. **Proof of Theorem 2.9.** For any element $B \in \mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\text{reg}})$, define $m(B)$ to be the differential operator $\mathbb{C}(\mathfrak{h})^W \to \mathbb{C}(\mathfrak{h})$, defined by B. That is, if $B = \sum_{g \in W} B_g g$, $B_g \in \mathcal{D}(\mathfrak{h}_{\text{reg}})$, then $m(B) = \sum_{g \in W} B_g$. It is clear that if B is W-invariant, then $\forall A \in \mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\text{reg}})$,

$$m(AB) = m(A)m(B).$$

Proposition 2.16 ([Du1], [He1])**.** *Let $\{y_1, \ldots, y_r\}$ be an orthonormal basis of $\mathfrak{h}$. Then we have*

$$m(\sum_{i=1}^{r} D_{y_i}^2) = \overline{H},$$

where $\overline{H} = \Delta_{\mathfrak{h}} - \sum_{s \in \mathcal{S}} \frac{c_s(\alpha_s, \alpha_s)}{\alpha_s} \partial_{\alpha_s^\vee}$.

Proof. For any $y \in \mathfrak{h}$, we have $m(D_y^2) = m(D_y \partial_y)$. A simple computation shows that

$$\begin{aligned} D_y \partial_y &= \partial_y^2 - \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(y)}{\alpha_s}(1 - s)\partial_y \\ &= \partial_y^2 - \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(y)}{\alpha_s}(\partial_y(1 - s) + \alpha_s(y)\partial_{\alpha_s^\vee} s). \end{aligned}$$

This means that

$$m(D_y^2) = \partial_y^2 - \sum_{s \in \mathcal{S}} \frac{c_s \alpha_s(y)^2}{\alpha_s} \partial_{\alpha_s^\vee}.$$

So we get

$$m(\sum_{i=1}^{r} D_{y_i}^2) = \sum_{i=1}^{r} \partial_{y_i}^2 - \sum_{s\in\mathcal{S}} c_s \sum_{i=1}^{r} \frac{\alpha_s(y_i)^2}{\alpha_s}\partial_{\alpha_s^\vee} = \overline{H},$$

since $\sum_{i=1}^{r} \alpha_s(y_i)^2 = (\alpha_s, \alpha_s)$. □

Recall that by the Chevalley-Shepard-Todd theorem, the algebra $(S\mathfrak{h})^W$ is free. Let $P_1 = \mathbf{p}^2, P_2, \ldots, P_r$ be homogeneous generators of $(S\mathfrak{h})^W$.

Corollary 2.17. *The differential operators $\overline{L}_j = m(P_j(D_{y_1}, \ldots, D_{y_r}))$ are pairwise commutative, have symbols P_j, homogeneity degree $-d_j$, and $\overline{L}_1 = \overline{H}$.*

Proof. Since Dunkl operators commute, the operators L_j are well defined. Since $m(AB) = m(A)m(B)$ when B is invariant, the operators L_j are pairwise commutative. The rest is clear. □

Now to prove Theorem 2.9, we will show that the operators H and $\overline{H}$ are conjugate to each other by a certain function; this will complete the proof.

Proposition 2.18. *Let $\delta_c(\mathbf{x}) := \prod_{s\in\mathcal{S}} \alpha_s(\mathbf{x})^{c_s}$. Then we have*

$$\delta_c^{-1} \circ \overline{H} \circ \delta_c = H.$$

Remark 2.19. The function $\delta_c(\mathbf{x})$ is not rational. It is a multivalued analytic function. Nevertheless, it is easy to see that for any differential operator L with rational coefficients, $\delta_c^{-1} \circ L \circ \delta_c$ also has rational coefficients.

Proof of Proposition 2.18. We have

$$\sum_{i=1}^{r} \partial_{y_i}(\log \delta_c)\partial_{y_i} = \sum_{s\in\mathcal{S}} \frac{c_s(\alpha_s, \alpha_s)}{2\alpha_s}\partial_{\alpha_s^\vee}.$$

Therefore, we have

$$\delta_c \circ H \circ \delta_c^{-1} = \Delta_{\mathfrak{h}} - \sum_{s\in\mathcal{S}} \frac{c_s(\alpha_s, \alpha_s)}{\alpha_s}\partial_{\alpha_s^\vee} + U,$$

where

$$U = \delta_c(\Delta_{\mathfrak{h}}\delta_c^{-1}) - \sum_{s\in\mathcal{S}} \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\alpha_s^2}.$$

Let us compute U. We have

$$\delta_c(\Delta_{\mathfrak{h}}\delta_c^{-1}) = \sum_{s\in\mathcal{S}} \frac{c_s(c_s+1)(\alpha_s, \alpha_s)}{\alpha_s^2} + \sum_{s\neq u\in\mathcal{S}} \frac{c_s c_u(\alpha_s, \alpha_u)}{\alpha_s\alpha_u}.$$

We claim that the last sum Σ is actually zero. Indeed, this sum is invariant under the Coxeter group, so $\prod_{s\in\mathcal{S}} \alpha_s \cdot \Sigma$ is a regular anti-invariant function of degree $|\mathcal{S}| - 2$. But the smallest degree of a nonzero anti-invariant is $|\mathcal{S}|$, so $\Sigma = 0$, $U = 0$, and we are done (Proposition 2.18 and Theorem 2.9 are proved). □

Remark 2.20. A similar method works for any complex reflection group G. Namely, the operators $L_i = m(P_i(D_{y_1}, \ldots, D_{y_r}))$ form a quantum integrable system. However, if G is not a real reflection group, this system does not have a quadratic Hamiltonian in momentum variables (so it does not have a physical meaning).

2.7. **Uniqueness of the operators L_j.**

Proposition 2.21. *The operators L_j are unique.*

Proof. Assume that we have two choices for L_j: L_j and L'_j. Denote $L_j - L'_j$ by M.

Assume $M \neq 0$. We have

(i) M is a differential operator on $\mathfrak{h}$ with rational coefficients, of order smaller than d_j and homogeneity degree $-d_j$;
(ii) $[M, H] = 0$.

Let M_0 be the symbol of M. Then M_0 is a polynomial of $\mathbf{p} \in \mathfrak{h}^*$ with coefficients in $\mathbb{C}(\mathfrak{h})$. We have, from (ii),

$$\{M_0, \mathbf{p}^2\} = 0, \quad \forall \mathbf{p} \in \mathfrak{h}^*,$$

and from (i) we see that the coefficients of M_0 are not polynomial (as they have negative degree).

However, we have the following lemma.

Lemma 2.22. *Let $\mathfrak{h}$ be a finite dimensional vector space. Let $\psi : (\mathbf{x}, \mathbf{p}) \mapsto \psi(\mathbf{x}, \mathbf{p})$ be a rational function on $\mathfrak{h} \oplus \mathfrak{h}^*$ which is a polynomial in $\mathbf{p} \in \mathfrak{h}^*$. Let $f : \mathfrak{h}^* \to \mathbb{C}$ be a polynomial such that the differentials $\mathrm{d}f(\mathbf{p})$ for $\mathbf{p} \in \mathfrak{h}^*$ span $\mathfrak{h}$ (e.g., $f(\mathbf{p}) = \mathbf{p}^2$). Suppose that the Poisson bracket of f and ψ vanishes: $\{\psi, f\} = 0$. Then ψ is a polynomial.*

Proof. (R. Raj) Let $\mathfrak{Z} \subset \mathfrak{h}$ be the pole divisor of ψ. Let $\mathbf{x}_0 \in \mathfrak{h}$ be a generic point in $\mathfrak{Z}$. Then ψ^{-1} is regular and vanishes at $(\mathbf{x}_0, \mathbf{p})$ for generic $\mathbf{p} \in \mathfrak{h}^*$. Also from $\{\psi^{-1}, f\} = 0$, we have ψ^{-1} vanishes along the entire flowline of the Hamiltonian flow defined by f and starting at $\mathbf{x}_0$. This flowline is defined by the formula

$$\mathbf{x}(t) = \mathbf{x}_0 + t\mathrm{d}f(\mathbf{p}), \quad \mathbf{p}(t) = \mathbf{p},$$

and it must be contained in the pole divisor of ψ near $\mathbf{x}_0$. This implies that $\mathrm{d}f(\mathbf{p})$ must be in $T_{\mathbf{x}_0}\mathfrak{Z}$ for almost every, hence for every $\mathbf{p} \in \mathfrak{h}^*$. This is a contradiction with the assumption on f, which implies that in fact ψ has no poles. □

□

2.8. **Classical Dunkl operators and Olshanetsky-Perelomov Hamiltonians.** We continue to use the notations in Section 2.4.

Definition 2.23. The classical Olshanetsky-Perelomov Hamiltonian corresponding to W is the following classical Hamiltonian on $\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^* = T^*\mathfrak{h}_{\mathrm{reg}}$:

$$H_0(\mathbf{x}, \mathbf{p}) = \mathbf{p}^2 - \sum_{s \in \mathcal{S}} \frac{c_s^2(\alpha_s, \alpha_s)}{\alpha_s^2(\mathbf{x})}.$$

Theorem 2.24 ([OP],[HO, He3, Op3, Op4],[He1])**.** *The Hamiltonian H_0 defines a classical integrable system. Namely, there exist unique regular functions L_j^0 on $\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*$, where highest terms in $\mathbf{p}$ are P_j, such that L_j^0 are homogeneous of degree $-d_j$ (under $\mathbf{x} \mapsto \lambda\mathbf{x}, \mathbf{x} \in \mathfrak{h}^*, \mathbf{p} \mapsto \lambda^{-1}\mathbf{p}, \mathbf{p} \in \mathfrak{h}$), and such that $L_1^0 = H_0$ and $\{L_j^0, L_k^0\} = 0, \forall j, k$.*

Proof. The proof is given in the next subsection. □

Example 2.25. Let $W = \mathfrak{S}_n$, $\mathfrak{h} = \mathbb{C}^{n-1}$. Then

$$H_0 = \sum_{i=1}^{n} p_i^2 - c^2 \sum_{i \neq j} \frac{1}{(x_i - x_j)^2} \quad \text{(the classical Calogero-Moser Hamiltonian).}$$

So the theorem says that there are functions $L_j^0, j = 1, \dots, n-1$,

$$L_j^0 = \sum_i p_i^{j+1} + \text{ lower terms,}$$

homogeneous of degree zero, such that $L_1^0 = H_0$ and $\{L_j^0, L_k^0\} = 0$.

2.9. **Rees algebras.** Let $\overline{A}$ be a filtered algebra over a field k: $k = F^0\overline{A} \subset F^1\overline{A} \subset \cdots$, $\cup_i F^i\overline{A} = \overline{A}$. Then the *Rees algebra* $A = \mathrm{Rees}(\overline{A})$ is defined by the formula $A = \oplus_{n=0}^{\infty} F^n\overline{A}$. This is an algebra over $k[\hbar]$, where $\hbar$ is the element 1 of the summand $F^1\overline{A}$.

2.10. **Proof of Theorem 2.24.** The proof of Theorem 2.24 is similar to the proof of its quantum analog. Namely, to construct the functions L_j^0, we need to introduce classical Dunkl operators. To do so, we introduce a parameter $\hbar$ (Planck's constant) and define Dunkl operators $D_a(\hbar) = D_a(c, \hbar)$ with $\hbar$:

$$D_a(c, \hbar) = \hbar D_a(c/\hbar) = \hbar \partial_a - \sum_{s \in \mathcal{S}} \frac{2c_s \alpha_s(a)}{(1 - \lambda_s)\alpha_s}(1 - s), \text{ where } a \in \mathfrak{h}.$$

These operators can be regarded as elements of the Rees algebra $A = \mathrm{Rees}(\mathbb{C}W \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}))$, where the filtration is by order of differential operators (and W sits in degree 0). Reducing these operators modulo $\hbar$, we get classical Dunkl operators $D_a^0(c) \in A_0 := A/\hbar A = \mathbb{C}W \ltimes \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$. They are given by the formula

$$D_a^0(c) = p_a - \sum_{s \in \mathcal{S}} \frac{2c_s \alpha_s(a)}{(1 - \lambda_s)\alpha_s}(1 - s),$$

where p_a is the classical momentum (the linear function on $\mathfrak{h}^*$ corresponding to $a \in \mathfrak{h}$).

It follows from the commutativity of the quantum Dunkl operators $D_a(c)$ that the Dunkl operators $D_a(c, \hbar)$ commute. Hence, so do the classical Dunkl operators D_a^0:

$$[D_a^0, D_b^0] = 0.$$

We also have the following analog of Proposition 2.14:

Proposition 2.26. (i) *For any $x \in \mathfrak{h}^*$, one has*

$$[D_a^0, x] = -\sum_{s \in \mathcal{S}} c_s (a, \alpha_s)(x, \alpha_s^\vee) s.$$

(ii) *If $g \in W$ then $g D_a^0 g^{-1} = D_{ga}^0$.*

Now let us construct the classical Olshanetsky-Perelomov Hamiltonians. As in the quantum case, we have the operation $m(\cdot)$, which is given by the formula $\sum_{g \in W} B_g \cdot g \mapsto \sum B_g$, $B \in \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$. We define the Hamiltonian

$$\overline{H}_0 := m(\sum_{i=1}^{r} (D_{y_i}^0)^2).$$

By taking the limit of quantum situation, we find

$$\overline{H}_0 = \mathbf{p}^2 - \sum_{s\in\mathcal{S}} \frac{c_s(\alpha_s,\alpha_s)}{\alpha_s(\mathbf{x})} p_{\alpha_s^\vee}.$$

Unfortunately, this is no longer conjugate to H_0. However, consider the (outer) automorphism θ_c of the algebra $\mathbb{C}W \ltimes \mathcal{O}(T^*\mathfrak{h}_{\mathrm{reg}})$ defined by the formulas

$$\theta_c(x) = x,\ \theta_c(s) = s,\ \theta_c(p_a) = p_a + \partial_a \log \delta_c,$$

for $x \in \mathfrak{h}^*$, $a \in \mathfrak{h}$, $s \in W$. It is easy to see that if $b_0 \in A_0$ and $b \in A$ is a deformation of b_0 then $\theta_c(b_0) = \lim_{\hbar\to 0} \delta_{c/\hbar}^{-1} b \delta_{c/\hbar}$. Therefore, taking the limit $\hbar \to 0$ in Proposition 2.16, we find that $H_0 = \theta_c(\overline{H}_0)$.

Now set $L_j^0 = m(\theta_c(P_j(D_{y_1}^0, \ldots, D_{y_r}^0)))$. These functions are well defined since D_a^0 commute, are homogeneous of degree zero, and $L_1^0 = H_0$.

Moreover, we can define the operators $L_j(\hbar)$ in $\mathrm{Rees}(\mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W)$ in the same way as L_j, but using the Dunkl operators $D_{y_i}(\hbar)$ instead of D_{y_i}. Then $[L_j(\hbar), L_k(\hbar)] = 0$, and $L_j(\hbar)|_{\hbar=0} = L_j^0$. This implies that L_j^0 Poisson commute: $\{L_j^0, L_k^0\} = 0$.

Theorem 2.24 is proved.

Remark 2.27. As in the quantum situation, Theorem 2.24 can be generalized to complex reflection groups, giving integrable systems with Hamiltonians which are non-quadratic in momentum variables.

2.11. **Notes.** Section 2.1 follows Section 5.4 of [E4]; the definition of complex reflection groups and their basic properties can be found in [GM]; the definition of parabolic subgroups and the notations are borrowed from Section 3.1 of [BE]; the remaining parts of this section follow Section 6 of [E4].

3. The rational Cherednik algebra

3.1. **Definition and examples.** Above we have made essential use of the commutation relations between operators $x \in \mathfrak{h}^*, g \in G$, and $D_a, a \in \mathfrak{h}$. This makes it natural to consider the algebra generated by these operators.

Definition 3.1. The rational Cherednik algebra associated to $(G, \mathfrak{h})$ is the algebra $H_c(G, \mathfrak{h})$ generated inside $A = \mathrm{Rees}(\mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}))$ by the elements $x \in \mathfrak{h}^*, g \in G$, and $D_a(c, \hbar), a \in \mathfrak{h}$. If $t \in \mathbb{C}$, then the algebra $H_{t,c}(G, \mathfrak{h})$ is the specialization of $H_c(G, \mathfrak{h})$ at $\hbar = t$.

Proposition 3.2. *The algebra H_c is the quotient of the algebra $\mathbb{C}G \ltimes \mathbf{T}(\mathfrak{h} \oplus \mathfrak{h}^*)[\hbar]$ (where $\mathbf{T}$ denotes the tensor algebra) by the ideal generated by the relations*

$$[x, x'] = 0, \ [y, y'] = 0, \ [y, x] = \hbar(y, x) - \sum_{s \in \mathcal{S}} c_s(y, \alpha_s)(x, \alpha_s^\vee)s,$$

where $x, x' \in \mathfrak{h}^*$, $y, y' \in \mathfrak{h}$.

Proof. Let us denote the algebra defined in the proposition by $H_c' = H_c'(G, \mathfrak{h})$. Then according to the results of the previous sections, we have a surjective homomorphism $\phi : H_c' \to H_c$ defined by the formula $\phi(x) = x$, $\phi(g) = g$, $\phi(y) = D_y(c, \hbar)$.

Let us show that this homomorphism is injective. For this purpose assume that y_i is a basis of $\mathfrak{h}$, and x_i is the dual basis of $\mathfrak{h}^*$. Then it is clear from the relations of H_c' that H_c' is spanned over $\mathbb{C}[\hbar]$ by the elements

$$g \prod_{i=1}^{r} y_i^{m_i} \prod_{i=1}^{r} x_i^{n_i}. \tag{3.1}$$

Thus it remains to show that the images of the elements (3.1) under the map ϕ, i.e. the elements

$$g \prod_{i=1}^{r} D_{y_i}(c, \hbar)^{m_i} \prod_{i=1}^{r} x_i^{n_i}.$$

are linearly independent. But this follows from the obvious fact that the symbols of these elements in $\mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}][\hbar]$ are linearly independent. The proposition is proved. $\square$

Remark 3.3. 1. Similarly, one can define the universal algebra $H(G, \mathfrak{h})$, in which both $\hbar$ and c are variables. (So this is an algebra over $\mathbb{C}[\hbar, c]$.) It has two equivalent definitions similar to the above.

2. It is more convenient to work with algebras defined by generators and relations than with subalgebras of a given algebra generated by a given set of elements. Therefore, from now on we will use the statement of Proposition 3.2 as a definition of the rational Cherednik algebra H_c. According to Proposition 3.2, this algebra comes with a natural embedding $\Theta_c : H_c \to \mathrm{Rees}(\mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}))$, defined by the formula $x \to x$, $g \to g$, $y \to D_y(c, \hbar)$. This embedding is called *the Dunkl operator embedding*.

Example 3.4. 1. Let $G = \mathbb{Z}_2$, $\mathfrak{h} = \mathbb{C}$. In this case c reduces to one parameter, and the algebra $H_{t,c}$ is generated by elements x, y, s with defining relations

$$s^2 = 1, \ sx = -xs, \ sy = -ys, \ [y, x] = t - 2cs.$$

2. Let $G = \mathfrak{S}_n$, $\mathfrak{h} = \mathbb{C}^n$. In this case there is also only one complex parameter c, and the algebra $H_{t,c}$ is the quotient of $\mathfrak{S}_n \ltimes \mathbb{C}\langle x_1, \dots, x_n, y_1, \dots, y_n\rangle$ by the relations

$$[x_i, x_j] = [y_i, y_j] = 0,\ [y_i, x_j] = cs_{ij},\ [y_i, x_i] = t - c\sum_{j\neq i} s_{ij}.$$

Here $\mathbb{C}\langle E\rangle$ denotes the free algebra on a set E, and s_{ij} is the transposition of i and j.

3.2. **The PBW theorem for the rational Cherednik algebra.** Let us put a filtration on H_c by setting $\deg y = 1$ for $y \in \mathfrak{h}$ and $\deg x = \deg g = 0$ for $x \in \mathfrak{h}^*, g \in G$. Let $\mathrm{gr}(H_c)$ denote the associated graded algebra of H_c under this filtration, and similarly for $H_{t,c}$. We have a natural surjective homomorphism

$$\xi : \mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*][\hbar] \to \mathrm{gr}(H_c).$$

For $t \in \mathbb{C}$, it specializes to surjective homomorphisms

$$\xi_t : \mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \to \mathrm{gr}(H_{t,c}).$$

Proposition 3.5 (The PBW theorem for rational Cherednik algebras)**.** *The maps ξ and ξ_t are isomorphisms.*

Proof. The statement is equivalent to the claim that the elements (3.1) are a basis of $H_{t,c}$, which follows from the proof of Proposition 3.2. □

Remark 3.6. 1. We have

$$H_{0,0} = \mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] \text{ and } H_{1,0} = \mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}).$$

2. For any $\lambda \in \mathbb{C}^*$, the algebra $H_{t,c}$ is naturally isomorphic to $H_{\lambda t,\lambda c}$.
3. The Dunkl operator embedding Θ_c specializes to embeddings

$$\Theta_{0,c} : H_{0,c} \hookrightarrow \mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}],$$

given by $x \mapsto x$, $g \mapsto g$, $y \mapsto D_a^0$, and

$$\Theta_{1,c} : H_{1,c} \hookrightarrow \mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}}),$$

given by $x \mapsto x$, $g \mapsto g$, $y \mapsto D_a$. So $H_{0,c}$ is generated by x, g, D_a^0, and $H_{1,c}$ is generated by x, g, D_a.

Since Dunkl operators map polynomials to polynomials, the map $\Theta_{1,c}$ defines a representation of $H_{1,c}$ on $\mathbb{C}[\mathfrak{h}]$. This representation is called the *polynomial representation* of $H_{1,c}$.

3.3. **The spherical subalgebra.** Let $\mathsf{e} \in \mathbb{C}G$ be the symmetrizer, $\mathsf{e} = |G|^{-1}\sum_{g\in G} g$. We have $\mathsf{e}^2 = \mathsf{e}$.

Definition 3.7. $B_c := \mathsf{e}H_c\mathsf{e}$ is called *the spherical subalgebra* of H_c. The spherical subalgebra of $H_{t,c}$ is $B_{t,c} := B_c/(\hbar - t) = \mathsf{e}H_{t,c}\mathsf{e}$.

Note that

$$\mathsf{e}\left(\mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})\right)\mathsf{e} = \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^G, \quad \mathsf{e}\left(\mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*]\right)\mathsf{e} = \mathbb{C}[\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*]^G.$$

Therefore, the restriction gives the embeddings: $\Theta_{1,c} : B_{1,c} \hookrightarrow \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^G$, and $\Theta_{0,c} : B_{0,c} \hookrightarrow \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}]^G$. In particular, we have

Proposition 3.8. *The spherical subalgebra $B_{0,c}$ is commutative and does not have zero divisors. Also $B_{0,c}$ is finitely generated.*

Proof. The first statement is clear from the above. The second statement follows from the fact that $\mathrm{gr}(B_{0,c}) = B_{0,0} = \mathbb{C}[\mathfrak{h} \times \mathfrak{h}^*]^G$, which is finitely generated by Hilbert's theorem. $\square$

Corollary 3.9. $M_c = \mathrm{Spec} B_{0,c}$ *is an irreducible affine algebraic variety.*

Proof. Directly from the definition and the proposition. $\square$

We also obtain

Proposition 3.10. B_c *is a flat quantization (non-commutative deformation) of* $B_{0,c}$ *over* $\mathbb{C}[\hbar]$.

So $B_{0,c}$ carries a Poisson bracket $\{\cdot,\cdot\}$(thus M_c is a Poisson variety), and B_c is a quantization of the Poisson bracket, i.e. if $a, b \in B_c$ and a_0, b_0 are the corresponding elements in $B_{0,c}$, then

$$[a, b]/\hbar \equiv \{a_0, b_0\} \pmod{\hbar}.$$

Definition 3.11. The Poisson variety M_c is called *the Calogero-Moser space* of $G, \mathfrak{h}$ with parameter c.

3.4. **The localization lemma.** Let $H_{t,c}^{\mathrm{loc}} = H_{t,c}[\delta^{-1}]$ be the localization of $H_{t,c}$ as a module over $\mathbb{C}[\mathfrak{h}]$ with respect to the discriminant δ (a polynomial vanishing to the first order on each reflection plane). Define also $B_{t,c}^{\mathrm{loc}} = \mathsf{e} H_{t,c}^{\mathrm{loc}} \mathsf{e}$.

Proposition 3.12. (i) *For* $t \neq 0$ *the map* $\Theta_{t,c}$ *induces an isomorphism of algebras* $H_{t,c}^{\mathrm{loc}} \to \mathbb{C}G \ltimes \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})$, *which restricts to an isomorphism* $B_{t,c}^{\mathrm{loc}} \to \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^G$.

(ii) *The map* $\Theta_{0,c}$ *induces an isomorphism of algebras* $H_{0,c}^{\mathrm{loc}} \to \mathbb{C}G \ltimes \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}]$, *which restricts to an isomorphism* $B_{0,c}^{\mathrm{loc}} \to \mathbb{C}[\mathfrak{h}^* \times \mathfrak{h}_{\mathrm{reg}}]^G$.

Proof. This follows immediately from the fact that the Dunkl operators have poles only on the reflection hyperplanes. $\square$

Since $\mathrm{gr}(B_{0,c}) = B_{0,0} = \mathbb{C}[\mathfrak{h}^* \oplus \mathfrak{h}]^G$, we get the following geometric corollary.

Corollary 3.13. (i) *The family of Poisson varieties* M_c *is a flat deformation of the Poisson variety* $M_0 := (\mathfrak{h} \times \mathfrak{h}^*)/G$. *In particular,* M_c *is smooth outside of a subset of codimension* 2.

(ii) *We have a natural map* $\beta_c : M_c \to \mathfrak{h}/G$, *such that* $\beta_c^{-1}(\mathfrak{h}_{\mathrm{reg}}/G)$ *is isomorphic to* $(\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*)/G$. *The Poisson structure on* M_c *is obtained by extension of the symplectic Poisson structure on* $(\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*)/G$.

Example 3.14. Let $W = \mathbb{Z}_2$, $\mathfrak{h} = \mathbb{C}$. Then $B_{0,c} = \langle x^2, xp, p^2 - c^2/x^2 \rangle$. Let $X := x^2, Z := xp$ and $Y := p^2 - c^2/x^2$. Then $Z^2 - XY = c^2$. So M_c is isomorphic to the quadric $Z^2 - XY = c^2$ in the 3-dimensional space and it is smooth for $c \neq 0$.

3.5. **Category $\mathcal{O}$ for rational Cherednik algebras.** From the PBW theorem, we see that $H_{1,c} = S\mathfrak{h}^* \otimes \mathbb{C}G \otimes S\mathfrak{h}$. It is similar to the structure of the universal enveloping algebra of a simple Lie algebra: $U(\mathfrak{g}) = U(\mathfrak{n}_-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{n}_+)$. Namely, the subalgebra $\mathbb{C}G$ plays the role of the Cartan subalgebra, and the subalgebras $S\mathfrak{h}^*$ and $S\mathfrak{h}$ play the role of the positive and negative nilpotent subalgebras. This similarity allows one to define and study the category $\mathcal{O}$ analogous to the Bernstein-Gelfand-Gelfand category $\mathcal{O}$ for simple Lie algebras.

Definition 3.15. The category $\mathcal{O}_c(G,\mathfrak{h})$ is the category of modules over $H_{1,c}(G,\mathfrak{h})$ which are finitely generated over $S\mathfrak{h}^*$ and locally finite under $S\mathfrak{h}$ (i.e., for $M \in \mathcal{O}_c(G,\mathfrak{h})$, $\forall v \in M$, $(S\mathfrak{h})v$ is finite dimensional).

If M is a locally finite $(S\mathfrak{h})^G$-module, then

$$M = \oplus_{\lambda\in\mathfrak{h}^*/G} M_\lambda,$$

where

$$M_\lambda = \{v \in M | \forall p \in (S\mathfrak{h})^G, \exists N\, s.t.\, (p-\lambda(p))^N v = 0\},$$

(notice that $\mathfrak{h}^*/G = \mathrm{Specm}(S\mathfrak{h})^G$).

Proposition 3.16. *M_λ are $H_{1,c}$-submodules.*

Proof. Note first that we have an isomorphism $\mu : H_{1,c}(G,\mathfrak{h}) \cong H_{1,c}(G,\mathfrak{h}^*)$, which is given by $x_a \mapsto y_a, y_b \mapsto -x_b, g \mapsto g$. Now let $x_1,\dots,x_r$ be a basis of $\mathfrak{h}^*$ and $y_1,\dots,y_r$ a basis of $\mathfrak{h}$. Suppose $P = P(x_1,\dots,x_r) \in (S\mathfrak{h}^*)^G$. Then we have

$$[y,P] = \frac{\partial}{\partial y} P \in S\mathfrak{h}^*, \text{ where } y \in \mathfrak{h},$$

(this follows from the fact that both sides act in the same way in the polynomial representation, which is faithful). So using the isomorphism μ, we conclude that if $Q \in (S\mathfrak{h})^G, Q = Q(y_1,\dots,y_r)$, then $[x,Q] = -\partial_x Q$ for $x \in \mathfrak{h}^*$.

Now, to prove the proposition, the only thing we need to check is that M_λ is invariant under $x \in \mathfrak{h}^*$. For any $v \in M_\lambda$, we have $(Q-\lambda(Q))^N v = 0$ for some N. Then

$$(Q-\lambda(Q))^{N+1} xv = (N+1)\partial_x Q \cdot (Q-\lambda(Q))^N v = 0.$$

So $xv \in M_\lambda$. ☐

Corollary 3.17. *We have the following decomposition:*

$$\mathcal{O}_c(G,\mathfrak{h}) = \bigoplus_{\lambda\in\mathfrak{h}^*/G} \mathcal{O}_c(G,\mathfrak{h})_\lambda,$$

where $\mathcal{O}_c(G,\mathfrak{h})_\lambda$ is the subcategory of modules where $(S\mathfrak{h})^G$ acts with generalized eigenvalue λ.

Proof. Directly from the definition and the proposition. ☐

Note that $\mathcal{O}_c(G,\mathfrak{h})_\lambda$ is an abelian category closed under taking subquotients and extensions.

3.6. **The grading element.** Let

$$\mathbf{h} = \sum_i x_i y_i + \frac{1}{2}\dim\mathfrak{h} - \sum_{s\in\mathcal{S}} \frac{2c_s}{1-\lambda_s} s. \tag{3.2}$$

Proposition 3.18. *We have*

$$[\mathbf{h},x] = x,\ x \in \mathfrak{h}^*, \quad [\mathbf{h},y] = -y,\ y \in \mathfrak{h}.$$

Proof. Let us prove the first relation; the second one is proved similarly. We have

$$\begin{aligned}[\mathbf{h}, x] &= \sum_i x_i[y_i, x] - \sum_{s\in\mathcal{S}} \frac{2c_s}{1-\lambda_s} \cdot \frac{\lambda_s - 1}{2}(\alpha_s^\vee, x)\alpha_s \cdot s \\ &= \sum_i x_i(y_i, x) - \sum_i x_i \sum_{s\in\mathcal{S}} c_s(\alpha_s^\vee, x)(\alpha_s, y_i)s + \sum_{s\in\mathcal{S}} c_s(\alpha_s^\vee, x)\alpha_s \cdot s.\end{aligned}$$

The last two terms cancel since $\sum_i x_i(\alpha_s, y_i) = \alpha_s$, so we get $\sum_i x_i(y_i, x) = x$. □

Proposition 3.19. *Let $G = W$ be a real reflection group. Let*

$$\mathbf{h} = \sum_i x_i y_i + \frac{1}{2}\dim\mathfrak{h} - \sum_{s\in\mathcal{S}} c_s s, \quad \mathbf{E} = -\frac{1}{2}\sum_i x_i^2, \quad \mathbf{F} = \frac{1}{2}\sum_i y_i^2.$$

Then

(i) $\mathbf{h} = \sum_i (x_i y_i + y_i x_i)/2$;
(ii) $\mathbf{h}, \mathbf{E}, \mathbf{F}$ *form an* $\mathfrak{sl}_2$*-triple.*

Proof. A direct calculation. □

Theorem 3.20. *Let M be a module over $H_{1,c}(G, \mathfrak{h})$.*

(i) *If $\mathfrak{h}$ acts locally nilpotently on M, then* $\mathbf{h}$ *acts locally finitely on M.*
(ii) *If M is finitely generated over $S\mathfrak{h}^*$, then $M \in \mathcal{O}_c(G, \mathfrak{h})_0$ if and only if* $\mathbf{h}$ *acts locally finitely on M.*

Proof. (i) Assume that $S\mathfrak{h}$ acts locally nilpotently on M. Let $v \in M$. Then $S\mathfrak{h} \cdot v$ is a finite dimensional vector space and let $d = \dim S\mathfrak{h} \cdot v$. We prove that v is $\mathbf{h}$-finite by induction in dimension d. We can use $d = 0$ as base, so only need to do the induction step. The space $S\mathfrak{h} \cdot v$ must contain a nonzero vector u such that $y \cdot u = 0$, $\forall y \in \mathfrak{h}$. Let $U \subset M$ be the subspace of vectors with this property. $\mathbf{h}$ acts on U by an element of $\mathbb{C}G$, hence locally finitely. So it is sufficient to show that the image of v in $M/\langle U\rangle$ is $\mathbf{h}$-finite (where $\langle U\rangle$ is the submodule generated by U). But this is true by the induction assumption, as $u = 0$ in $M/\langle U\rangle$.

(ii) We need to show that if $\mathbf{h}$ acts locally finitely on M, then $\mathfrak{h}$ acts locally nilpotently on M. Assume $\mathbf{h}$ acts locally finitely on M. Then $M = \oplus_{\beta\in B} M[\beta]$, where $B \subset \mathbb{C}$. Since M is finitely generated over $S\mathfrak{h}^*$, B is a finite union of sets of the form $z + \mathbb{Z}_{\geq 0}$, $z \in \mathbb{C}$. So $S\mathfrak{h}$ must act locally nilpotently on M. □

We can obtain the following corollary easily.

Corollary 3.21. *Any finite dimensional $H_{1,c}(G, \mathfrak{h})$-module is in $\mathcal{O}_c(G, \mathfrak{h})_0$.*

We see that any module $M \in \mathcal{O}_c(G, \mathfrak{h})_0$ has a grading by generalized eigenvalues of $\mathbf{h}$: $M = \oplus_\beta M[\beta]$.

3.7. **Standard modules.** Let τ be a finite dimensional representation of G. The standard module over $H_{1,c}(G, \mathfrak{h})$ corresponding to τ (also called the Verma module) is

$$M_c(G, \mathfrak{h}, \tau) = M_c(\tau) = H_{1,c}(G, \mathfrak{h}) \otimes_{\mathbb{C}G \ltimes S\mathfrak{h}} \tau \ \in \mathcal{O}_c(G, \mathfrak{h})_0,$$

where $S\mathfrak{h}$ acts on τ by zero.

So from the PBW theorem, we have that as vector spaces, $M_c(\tau) \cong \tau \otimes S\mathfrak{h}^*$.

Remark 3.22. More generally, $\forall\lambda \in \mathfrak{h}^*$, let $G_\lambda = \mathrm{Stab}(\lambda)$, and τ be a finite dimensional representation of G_λ. Then we can define $M_{c,\lambda}(G, \mathfrak{h}, \tau) = H_{1,c}(G, \mathfrak{h}) \otimes_{\mathbb{C}G_\lambda \ltimes S\mathfrak{h}} \tau$, where $S\mathfrak{h}$ acts on τ by λ. These modules are called *the Whittaker modules.*

Let τ be irreducible, and let $h_c(\tau)$ be the number given by the formula

$$h_c(\tau) = \frac{\dim \mathfrak{h}}{2} - \sum_{s\in\mathcal{S}} \frac{2c_s}{1-\lambda_s} s|_\tau.$$

Then we see that $\mathbf{h}$ acts on $\tau \otimes S^m\mathfrak{h}^*$ by the scalar $h_c(\tau) + m$.

Definition 3.23. A vector v in an $H_{1,c}$-module M is singular if $y_i v = 0$ for all i.

Proposition 3.24. *Let U be an $H_{1,c}(G, \mathfrak{h})$-module. Let $\tau \subset U$ be a G-submodule consisting of singular vectors. Then there is a unique homomorphism $\phi : M_c(\tau) \to U$ of $\mathbb{C}[\mathfrak{h}]$-modules such that $\phi|_\tau$ is the identity, and it is an $H_{1,c}$-homomorphism.*

Proof. The first statement follows from the fact that $M_c(\tau)$ is a free module over $\mathbb{C}[\mathfrak{h}]$ generated by τ. Also, it follows from the Frobenius reciprocity that there must exist a map ϕ which is an $H_{1,c}$-homomorphism. This implies the proposition. □

3.8. **Finite length.**

Proposition 3.25. *$\exists K \in \mathbb{R}$ such that for any $M \subset N$ in $\mathcal{O}_c(G, \mathfrak{h})_0$, if $M[\beta] = N[\beta]$ for $\mathrm{Re}\,(\beta) \le K$, then $M = N$.*

Proof. Let $K = \max_\tau \mathrm{Re}\, h_c(\tau)$. Then if $M \neq N$, M/N begins in degree β_0 with $\mathrm{Re}\,\beta_0 > K$, which is impossible since by Proposition 3.24, β_0 must equal $h_c(\tau)$ for some τ. □

Corollary 3.26. *Any $M \in \mathcal{O}_c(G, \mathfrak{h})_0$ has finite length.*

Proof. Directly from the proposition. □

3.9. **Characters.** For $M \in \mathcal{O}_c(G, \mathfrak{h})_0$, define the character of M as the following formal series in t:

$$\mathrm{ch}_M(g, t) = \sum_\beta t^\beta \mathrm{Tr}_{M[\beta]}(g) = \mathrm{Tr}_M(g t^{\mathbf{h}}), \quad g \in G.$$

Proposition 3.27. *We have*

$$\mathrm{ch}_{M_c(\tau)}(g, t) = \frac{\chi_\tau(g) t^{h_c(\tau)}}{\det_{\mathfrak{h}^*}(1 - tg)}.$$

Proof. We begin with the following lemma.

Lemma 3.28 (MacMahon's Master theorem)**.** *Let V be a finite dimensional space, $A : V \to V$ a linear operator. Then*

$$\sum_{n\ge 0} t^n \mathrm{Tr}\,(S^n A) = \frac{1}{\det(1 - tA)}.$$

Proof of the lemma. If A is diagonalizable, this is obvious. The general statement follows by continuity. □

The lemma implies that $\mathrm{Tr}_{S\mathfrak{h}^*}(g t^D) = \dfrac{1}{\det(1 - gt)}$ where D is the degree operator. This implies the required statement. □

3.10. **Irreducible modules.** Let τ be an irreducible representation of G.

Proposition 3.29. *$M_c(\tau)$ has a maximal proper submodule $J_c(\tau)$.*

Proof. The proof is standard. $J_c(\tau)$ is the sum of all proper submodules of $M_c(\tau)$, and it is not equal to $M_c(\tau)$ because any proper submodule has a grading by generalized eigenspaces of $\mathbf{h}$, with eigenvalues β such that $\beta - h_c(\tau) > 0$. $\square$

We define $L_c(\tau) = M_c(\tau)/J_c(\tau)$, which is an irreducible module.

Proposition 3.30. *Any irreducible object of $\mathcal{O}_c(G, \mathfrak{h})_0$ has the form $L_c(\tau)$ for an unique τ.*

Proof. Let $L \in \mathcal{O}_c(G, \mathfrak{h})_0$ be irreducible, with lowest eigenspace of $\mathbf{h}$ containing an irreducible G-module τ. Then by Proposition 3.24, we have a nonzero homomorphism $M_c(\tau) \to L$, which is surjective, since L is irreducible. Then we must have $L = L_c(\tau)$. $\square$

Remark 3.31. Let χ be a character of G. Then we have an isomorphism $H_{1,c}(G, \mathfrak{h}) \to H_{1,c\chi}(G, \mathfrak{h})$, mapping $g \in G$ to $\chi^{-1}(g)g$. This automorphism maps $L_c(\tau)$ to $L_{c\chi}(\chi^{-1} \otimes \tau)$ isomorphically.

3.11. **The contragredient module.** Set $\bar{c}(s) = c(s^{-1})$. We have a natural isomorphism $\gamma : H_{1,\bar{c}}(G, \mathfrak{h}^*)^{\mathrm{op}} \to H_{1,c}(G, \mathfrak{h})$, acting trivially on $\mathfrak{h}$ and $\mathfrak{h}^*$, and sending $g \in G$ to g^{-1}.

Thus if M is an $H_{1,c}(G, \mathfrak{h})$-module, then the full dual space M^* is an $H_{1,\bar{c}}(M, \mathfrak{h}^*)$-module. If $M \in \mathcal{O}_c(G, \mathfrak{h})_0$, then we can define $M^\dagger$, which is the $\mathbf{h}$-finite part of M^*.

Proposition 3.32. *$M^\dagger$ belongs to $\mathcal{O}_{\bar{c}}(G, \mathfrak{h}^*)_0$.*

Proof. Clearly, if L is irreducible, then so is $L^\dagger$. Then $L^\dagger$ is generated by its lowest $\mathbf{h}$-eigenspace over $H_{1,\bar{c}}(G, \mathfrak{h}^*)$, hence over $S\mathfrak{h}^*$. Thus, $L^\dagger \in \mathcal{O}_{\bar{c}}(G, \mathfrak{h}^*)_0$. Now, let $M \in \mathcal{O}_c(G, \mathfrak{h})_0$ be any object. Since M has finite length, so does $M^\dagger$. Moreover, $M^\dagger$ has a finite filtration with successive quotients of the form $L^\dagger$, where $L \in \mathcal{O}_c(G, \mathfrak{h})_0$ is irreducible. This implies the required statement, since $\mathcal{O}_c(G, \mathfrak{h})_0$ is closed under taking extensions. $\square$

Clearly, $M^{\dagger\dagger} = M$. Thus, $M \mapsto M^\dagger$ is an equivalence of categories $\mathcal{O}_c(G, \mathfrak{h}) \to \mathcal{O}_{\bar{c}}(G, \mathfrak{h}^*)^{\mathrm{op}}$.

3.12. **The contravariant form.** Let τ be an irreducible representation of G. By Proposition 3.24, we have a unique homomorphism $\phi : M_c(G, \mathfrak{h}, \tau) \to M_{\bar{c}}(G, \mathfrak{h}^*, \tau^*)^\dagger$ which is the identity in the lowest $\mathbf{h}$-eigenspace. Thus, we have a pairing

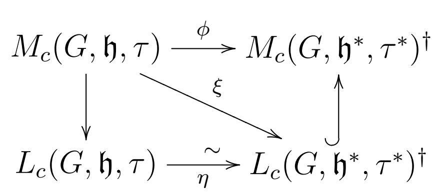
$$\beta_c : M_c(G, \mathfrak{h}, \tau) \times M_{\bar{c}}(G, \mathfrak{h}^*, \tau^*) \to \mathbb{C},$$

which is called *the contravariant form.*

Remark 3.33. If $G = W$ is a real reflection group, then $\mathfrak{h} \cong \mathfrak{h}^*$, $c = \bar{c}$, and $\tau \cong \tau^*$ via a symmetric form. So for real reflection groups, β_c is a symmetric form on $M_c(\tau)$.

Proposition 3.34. *The maximal proper submodule $J_c(\tau)$ is the kernel of ϕ (or, equivalently, of the contravariant form β_c).*

Proof. Let K be the kernel of the contravariant form. It suffices to show that $M_c(\tau)/K$ is irreducible. We have a diagram:



Indeed, a nonzero map ξ exists by Proposition 3.24, and it factors through $L_c(G,\mathfrak{h},\tau)$, with η being an isomorphism, since $L_c(G,\mathfrak{h}^*,\tau^*)^\dagger$ is irreducible. Now, by Proposition 3.24 (uniqueness of ϕ), the diagram must commute up to scaling, which implies the statement. $\square$

Proposition 3.35. *Assume that $h_c(\tau)-h_c(\tau')$ never equals a positive integer for any $\tau,\tau'\in$ Irrep$G$. Then $\mathcal{O}_c(G,\mathfrak{h})_0$ is semisimple, with simple objects $M_c(\tau)$.*

Proof. It is clear that in this situation, all $M_c(\tau)$ are simple. Also consider $\mathrm{Ext}^1(M_c(\tau),M_c(\tau'))$. If $h_c(\tau)-h_c(\tau')\notin\mathbb{Z}$, it is clearly 0. Otherwise, $h_c(\tau)=h_c(\tau')$, and again $\mathrm{Ext}^1(M_c(\tau),M_c(\tau'))=0$, since for any extension

$$0\to M_c(\tau')\to N\to M_c(\tau)\to 0,$$

by Proposition 3.24 we have a splitting $M_c(\tau)\to N$. $\square$

Remark 3.36. In fact, our argument shows that if $\mathrm{Ext}^1(M_c(\tau),M_c(\tau'))\neq 0$, then $h_c(\tau)-h_c(\tau')\in\mathbb{N}$.

3.13. **The matrix of multiplicities.** For $\tau,\sigma\in \mathrm{I}rrepG$, write $\tau<\sigma$ if

$$\mathrm{Re}\,h_c(\sigma)-\mathrm{Re}\,h_c(\tau)\in\mathbb{N}.$$

Proposition 3.37. *There exists a matrix of integers $N=(n_{\sigma,\tau})$, with $n_{\sigma,\tau}\geq 0$, such that $n_{\tau,\tau}=1$, $n_{\sigma,\tau}=0$ unless $\sigma<\tau$, and*

$$M_c(\sigma)=\sum n_{\sigma,\tau}L_c(\tau)\ \in \mathrm{K}_0(\mathcal{O}_c(G,\mathfrak{h})_0).$$

Proof. This follows from the Jordan-Hölder theorem and the fact that objects in $\mathcal{O}_c(G,\mathfrak{h})_0$ have finite length. $\square$

Corollary 3.38. *Let $N^{-1}=(\bar{n}_{\tau,\sigma})$. Then*

$$L_c(\tau)=\sum \bar{n}_{\tau,\sigma}M_c(\sigma).$$

Corollary 3.39. *We have*

$$\mathrm{ch}_{L_c(\tau)}(g,t)=\frac{\sum \bar{n}_{\tau,\sigma}\chi_\sigma(g)t^{h_c(\tau)}}{\det_{\mathfrak{h}^*}(1-tg)}.$$

Both of the corollaries can be obtained from the above proposition easily.

One of the main problems in the representation theory of rational Cherednik algebras is the following problem.

Problem: Compute the multiplicities $n_{\sigma,\tau}$ or, equivalently, $\mathrm{ch}_{L_c(\tau)}$ for all τ.

In general, this problem is open.

3.14. **Example: the rank 1 case.** Let $G=\mathbb{Z}/m\mathbb{Z}$ and λ be an m-th primitive root of 1. Then the algebra $H_{1,c}(G,\mathfrak{h})$ is generated by x,y,s with relations

$$[y,x]=1-2\sum_{j=1}^{m-1}c_js^j,\quad sxs^{-1}=\lambda x,\quad sys^{-1}=\lambda^{-1}y.$$

Consider the one-dimensional space $\mathbb{C}$ and let y act by 0 and $g\in G$ act by 1. We have $M_c(\mathbb{C})=\mathbb{C}[x]$. The contravariant form $\beta_{c,\mathbb{C}}$ on $M_c(\mathbb{C})$ is defined by

$$\beta_{c,\mathbb{C}}(x^n,x^n)=a_n;\quad \beta_{c,\mathbb{C}}(x^n,x^{n'})=0, n\neq n'.$$

Recall that $\beta_{c,\mathbb{C}}$ satisfies $\beta_{c,\mathbb{C}}(x^n, x^n) = \beta_{c,\mathbb{C}}(x^{n-1}, yx^n)$, which gives

$$a_n = a_{n-1}(n - b_n),$$

where b_n are new parameters:

$$b_n := 2\sum_{j=1}^{m-1} \frac{1-\lambda^{jn}}{1-\lambda^j} c_j \quad (b_0 = 0, b_{n+m} = b_n).$$

Thus we obtain the following proposition.

Proposition 3.40. (i) *$M_c(\mathbb{C})$ is irreducible if only if $n - b_n \neq 0$ for any $n \geq 1$.*

(ii) *Assume that r is the smallest positive integer such that $r = b_r$. Then $L_c(\mathbb{C})$ has dimension r (which can be any number not divisible by m) with basis $1, x, \dots, x^{r-1}$.*

Remark 3.41. According to Remark 3.31, this proposition in fact describes all the irreducible lowest weight modules.

Example 3.42. Consider the case $m = 2$. The $M_c(\mathbb{C})$ is irreducible unless $c \in 1/2 + \mathbb{Z}_{\geq 0}$. If $c = (2n+1)/2 \in 1/2 + \mathbb{Z}$, $n \geq 0$, then $L_c(\mathbb{C})$ has dimension $2n + 1$. A similar answer is obtained for lowest weight $\mathbb{C}_-$, replacing c by $-c$.

3.15. **The Frobenius property.** Let A be a $\mathbb{Z}_+$-graded commutative algebra. The algebra A is called Frobenius if the top degree $A[d]$ of A is 1-dimensional, and the multiplication map $A[m] \times A[d-m] \to A[d]$ is a nondegenerate pairing for any $0 \leq m \leq d$. In particular, the Hilbert polynomial of a Frobenius algebra A is palindromic.

Now, let us go back to considering modules over the rational Cherednik algebra $H_{1,c}$. Any submodule J of the polynomial representation $M_c(\mathbb{C}) = M_c = \mathbb{C}[\mathfrak{h}]$ is an ideal in $\mathbb{C}[\mathfrak{h}]$, so the quotient $A = M_c/J$ is a $\mathbb{Z}_+$-graded commutative algebra.

Now suppose that G preserves an inner product in $\mathfrak{h}$, i.e., $G \subseteq \mathrm{O}(\mathfrak{h})$.

Theorem 3.43. *If $A = M_c(\mathbb{C})/J$ is finite dimensional, then A is irreducible ($A = L_c(\mathbb{C})$) $\Longleftrightarrow$ A is a Frobenius algebra.*

Proof. 1) Suppose A is an irreducible $H_{1,c}$-module, i.e., $A = L_c(\mathbb{C})$. By Proposition 3.19, A is naturally a finite dimensional $\mathfrak{sl}_2$-module (in particular, it integrates to the group $\mathrm{SL}_2(\mathbb{C})$). Hence, by the representation theory of $\mathfrak{sl}_2$, the top degree of A is 1-dimensional. Let $\phi \in A^*$ denote a nonzero linear function on the top component. Let β_c be the contravariant form on $M_c(\mathbb{C})$. Consider the form

$$(v_1, v_2) \mapsto E(v_1, v_2) := \beta_c(v_1, gv_2), \text{ where } g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

Then $E(xv_1, v_2) = E(v_1, xv_2)$. So for any $p, q \in M_c(\mathbb{C}) = \mathbb{C}[\mathfrak{h}]$, $E(p, q) = \phi(p(x)q(x))$ (for a suitable normalization of ϕ).

Since E is a nondegenerate form, A is a Frobenius algebra.

2) Suppose A is Frobenius. Then the highest component is 1-dimensional, and $E : A \otimes A \to \mathbb{C}$, $E(a, b) = \phi(ab)$ is nondegenerate. We have $E(xa, b) = E(a, xb)$. So set $\beta(a, b) = E(a, g^{-1}b)$. Then β satisfies $\beta(a, x_i b) = \beta(y_i a, b)$. Thus, for all $p, q \in \mathbb{C}[\mathfrak{h}]$, $\beta(p(x), q(x)) = \beta(q(y)p(x), 1)$. So $\beta = \beta_c$ up to scaling. Thus, β_c is nondegenerate and A is irreducible. □

Remark 3.44. If $G \nsubseteq \mathrm{O}(\mathfrak{h})$, this theorem is false, in general.

Now consider the Frobenius property of $L_c(\mathbb{C})$ for any $G \subset \mathrm{GL}(\mathfrak{h})$.

Theorem 3.45. *Let $U \subset M_c(\mathbb{C}) = \mathbb{C}[\mathfrak{h}]$ be a G-subrepresentation of dimension $l = \dim \mathfrak{h}$, sitting in degree r, which consists of singular vectors. Let $J = \langle U \rangle$. Assume that $A = M_c/J$ is finite dimensional. Then*

(i) *A is Frobenius.*

(ii) *A admits a BGG type resolution:*

$$A \leftarrow M_c(\mathbb{C}) \leftarrow M_c(U) \leftarrow M_c(\wedge^2 U) \leftarrow \cdots \leftarrow M_c(\wedge^l U) \leftarrow 0.$$

(iii) *The character of A is given by the formula*

$$\chi_A(g,t) = t^{\frac{l}{2} - \sum_{s\in S} \frac{2c_s}{1-\lambda_s}} \frac{\det_U(1-gt^r)}{\det_{\mathfrak{h}^*}(1-gt)}.$$

In particular, $\dim A = r^l$.

(iv) *If G preserves an inner product, then A is irreducible.*

Proof. (i) Since $\operatorname{Spec} A$ is a complete intersection, A is Frobenius.

(ii) We will use the following theorem:

Theorem 3.46 (Serre)**.** *Let $f_1, \dots, f_n \in \mathbb{C}[t_1, \dots, t_n]$ be homogeneous polynomials, and assume that $\mathbb{C}[t_1, \dots, t_n]$ is a finitely generated module over $\mathbb{C}[f_1, \dots, f_n]$. Then this is a free module.*

Consider $SU \subset S\mathfrak{h}^*$. Then $S\mathfrak{h}^*$ is a finitely generated SU-module (as $S\mathfrak{h}^*/\langle U \rangle$ is finite dimensional). By Serre's theorem, we know that $S\mathfrak{h}^*$ is a free SU-module. The rank of this module is r^l. Consider the Koszul complex attached to this module. Since the module is free, the Koszul complex is exact (i.e., it is a resolution of the zero fiber). At the level of SU-modules, it looks exactly like we want in (3.45).

So we only need to show that the maps of the resolution are morphisms over $H_{1,c}$. This is shown by induction. Namely, let $\delta_j : M_c(\wedge^j U) \to M_c(\wedge^{j-1} U)$ be the corresponding differentials (so that $\delta_0 : M_c(\mathbb{C}) \to A$ is the projection). Then δ_0 is an $H_{1,c}$-morphism, which is the base of induction. If δ_j is an $H_{1,c}$-morphism, then the kernel of δ_j is a submodule $K_j \subset M_c(\wedge^j U)$. Its lowest degree part is $\wedge^{j+1} U$ sitting in degree $(j+1)r$ and consisting of singular vectors. Now, δ_{j+1} is a morphism over $S\mathfrak{h}^*$ which maps $\wedge^{j+1} U$ identically to itself. By Proposition 3.24, there is only one such morphism, and it must be an $H_{1,c}$-morphism. This completes the induction step.

(iii) follows from (ii) by the Euler-Poincaré formula.

(iv) follows from Theorem 3.43.

□

3.16. **Representations of $H_{1,c}$ of type A.** Let us now apply the above results to the case of type A. We will follow the paper [CE].

Let $G = \mathfrak{S}_n$, and $\mathfrak{h}$ be its reflection representation. In this case the function c reduces to one number. We will denote the rational Cherednik algebra $H_{1,c}(\mathfrak{S}_n)$ by $H_c(n)$. It is generated by $x_1, \dots, x_n$, $y_1, \dots, y_n$ and $\mathbb{C}\mathfrak{S}_n$ with the following relations:

$$\sum y_i = 0, \quad \sum x_i = 0, \quad [y_i, x_j] = -\frac{1}{n} + c s_{ij}, i \neq j,$$

$$[y_i, x_i] = \frac{n-1}{n} - c\sum_{j\neq i} s_{ij}.$$

The polynomial representation $M_c(\mathbb{C})$ of this algebra is the space of $\mathbb{C}[x_1, \ldots, x_n]^T$ of polynomials of $x_1, \ldots, x_n$, which are invariant under simultaneous translation $T : x_i \mapsto x_i + a$. In other words, it is the space of regular functions on $\mathfrak{h} = \mathbb{C}^n/\Delta$, where Δ is the diagonal.

Proposition 3.47 (C. Dunkl)**.** *Let r be a positive integer not divisible by n, and $c = r/n$. Then $M_c(\mathbb{C})$ contains a copy of the reflection representation $\mathfrak{h}$ of $\mathfrak{S}_n$, which consists of singular vectors (i.e. those killed by $y \in \mathfrak{h}$). This copy sits in degree r and is spanned by the functions*

$$f_i(x_1, \ldots, x_n) = \mathrm{Res}_\infty[(z-x_1)\cdots(z-x_n)]^{\frac{r}{n}} \frac{\mathrm{d}z}{z-x_i}.$$

(the symbol Res_∞ *denotes the residue at infinity).*

Remark 3.48. The space spanned by f_i is $(n-1)$-dimensional, since $\sum_i f_i = 0$ (this sum is the residue of an exact differential).

Proof. This proposition can be proved by a straightforward computation. The functions f_i are a special case of Jack polynomials. □

Let I_c be the submodule of $M_c(\mathbb{C})$ generated by f_i. Consider the $H_c(n)$-module $V_c = M_c(\mathbb{C})/I_c$, and regard it as a $\mathbb{C}[\mathfrak{h}]$-module. We have the following results.

Theorem 3.49. *Let $d = (r, n)$ denote the greatest common divisor of r and n. Then the (set-theoretical) support of V_c is the union of $\mathfrak{S}_n$-translates of the subspaces of $\mathbb{C}^n/\Delta$, defined by the equations*

$$x_1 = x_2 = \cdots = x_{\frac{n}{d}}; \quad x_{\frac{n}{d}+1} = \cdots = x_{2\frac{n}{d}}; \quad \ldots \quad x_{(d-1)\frac{n}{d}+1} = \cdots = x_n.$$

In particular, the Gelfand-Kirillov dimension of V_c is $d-1$.

Corollary 3.50 ([BEG])**.** *If $d = 1$ then the module V_c is finite dimensional, irreducible, admits a BGG type resolution, and its character is*

$$\chi_{V_c}(g,t) = t^{(1-r)(n-1)/2} \frac{\det|_{\mathfrak{h}}(1-gt^r)}{\det|_{\mathfrak{h}}(1-gt)}.$$

Proof. For $d = 1$ Theorem 3.49 says that the support of $M_c(\mathbb{C})/I_c$ is $\{0\}$. This implies that $M_c(\mathbb{C})/I_c$ is finite dimensional. The rest follows from Theorem 3.45. □

Proof of Theorem 3.49. The support of V_c is the zero set of I_c, i.e. the common zero set of f_i. Fix $x_1, \ldots, x_n \in \mathbb{C}$. Then $f_i(x_1, \ldots, x_n) = 0$ for all i iff $\sum_{i=1}^n \lambda_i f_i = 0$ for all λ_i, i.e.

$$\mathrm{Res}_\infty \left(\prod_{j=1}^n (z-x_j)^{\frac{r}{n}} \sum_{i=1}^n \frac{\lambda_i}{z-x_i} \right) \mathrm{d}z = 0.$$

Assume that $x_1, \ldots x_n$ take distinct values $y_1, \ldots, y_p$ with positive multiplicities $m_1, \ldots, m_p$. The previous equation implies that the point $(x_1, \ldots, x_n)$ is in the zero set iff

$$\mathrm{R}es_\infty \prod_{j=1}^p (z-y_j)^{m_j\frac{r}{n}-1} \left(\sum_{i=1}^p \nu_i (z-y_1)\cdots\widehat{(z-y_i)}\cdots(z-y_p) \right) \mathrm{d}z = 0 \quad \forall \nu_i.$$

Since ν_i are arbitrary, this is equivalent to the condition

$$\mathrm{R}es_\infty \prod_{j=1}^{p} (z-y_j)^{m_j \frac{r}{n}-1} z^i \mathrm{d}z = 0, \quad i = 0, \ldots, p-1.$$

We will now need the following lemma.

Lemma 3.51. *Let* $a(z) = \prod_{j=1}^{p} (z-y_j)^{\mu_j}$*, where* $\mu_j \in \mathbb{C}$*,* $\sum_j \mu_j \in \mathbb{Z}$ *and* $\sum_j \mu_j > -p$*. Suppose*

$$\mathrm{R}es_\infty a(z) z^i \mathrm{d}z = 0, \quad i = 0, 1, \ldots, p-2.$$

Then $a(z)$ *is a polynomial.*

Proof. Let $g(z)$ be a polynomial. Then

$$0 = \mathrm{R}es_\infty \mathrm{d}(g(z) \cdot a(z)) = \mathrm{Res}_\infty (g'(z)a(z) + a'(z)g(z))\mathrm{d}z,$$

and hence

$$\mathrm{R}es_\infty \left(g'(z) + \sum_i \frac{\mu_j}{z-y_j} g(z) \right) a(z) \mathrm{d}z = 0.$$

Let $g(z) = z^l \prod_j (z-y_j)$. Then $g'(z) + \sum_j \frac{\mu_j}{z-y_j} g(z)$ is a polynomial of degree $l+p-1$ with highest coefficient $l + p + \sum \mu_j \neq 0$ (as $\sum \mu_j > -p$). This means that for every $l \geq 0$, $\mathrm{R}es_\infty z^{l+p-1} a(z)\mathrm{d}z$ is a linear combination of residues of $z^q a(z)\mathrm{d}z$ with $q < l+p-1$. By the assumption of the lemma, this implies by induction in l that all such residues are 0 and hence a is a polynomial. ☐

In our case $\sum (m_j r/n - 1) = r - p$ (since $\sum m_j = n$) and the conditions of the lemma are satisfied. Hence $(x_1, \ldots, x_n)$ is in the zero set of I_c iff $\prod_{j=1}^{p} (z-y_j)^{m_j \frac{r}{n}-1}$ is a polynomial. This is equivalent to saying that all m_j are divisible by n/d.

We have proved that $(x_1, \ldots, x_n)$ is in the zero set of I_c if and only if $(z-x_1)\cdots(z-x_n)$ is the (n/d)-th power of a polynomial of degree d. This implies the theorem. ☐

Remark 3.52. For $c > 0$, the above representations are the only irreducible finite dimensional representations of $H_{1,c}(\mathfrak{S}_n)$. Namely, it is proved in [BEG] that the only finite dimensional representations of $H_{1,c}(\mathfrak{S}_n)$ are multiples of $L_c(\mathbb{C})$ for $c = r/n$, and of $L_c(\mathbb{C}_-)$ (where $\mathbb{C}_-$ is the sign representation) for $c = -r/n$, where r is a positive integer relatively prime to n.

3.17. **Notes.** The discussion of the definition of rational Cherednik algebras and their basic properties follows Section 7 of [E4]. The discussion of the category $\mathcal{O}$ for rational Cherednik algebras follows Section 11 of [E4]. The material in Sections 3.14-3.16 is borrowed from [CE].

4. The Macdonald-Mehta integral

4.1. **Finite Coxeter groups and the Macdonald-Mehta integral.** Let W be a finite Coxeter group of rank r with real reflection representation $\mathfrak{h}_{\mathbb{R}}$ equipped with a Euclidean W-invariant inner product $(\cdot,\cdot)$. Denote by $\mathfrak{h}$ the complexification of $\mathfrak{h}_{\mathbb{R}}$. The reflection hyperplanes subdivide $\mathfrak{h}_{\mathbb{R}}$ into $|W|$ chambers; let us pick one of them to be the dominant chamber and call its interior D. For each reflection hyperplane, pick the perpendicular vector $\alpha \in \mathfrak{h}_{\mathbb{R}}$ with $(\alpha,\alpha) = 2$ which has positive inner products with elements of D, and call it the positive root corresponding to this hyperplane. The walls of D are then defined by the equations $(\alpha_i, v) = 0$, where α_i are simple roots. Denote by $\mathcal{S}$ the set of reflections in W, and for a reflection $s \in \mathcal{S}$ denote by α_s the corresponding positive root. Let

$$\delta(\mathbf{x}) = \prod_{s\in\mathcal{S}} (\alpha_s, \mathbf{x})$$

be the corresponding discriminant polynomial. Let $d_i, i = 1, \ldots, r$, be the degrees of the generators of the algebra $\mathbb{C}[\mathfrak{h}]^W$. Note that $|W| = \prod_i d_i$.

Let $H_{1,c}(W,\mathfrak{h})$ be the rational Cherednik algebra of W. Here we choose $c = -k$ as a constant function. Let $M_c = M_c(\mathbb{C})$ be the polynomial representation of $H_{1,c}(W,\mathfrak{h})$, and β_c be the contravariant form on M_c defined in Section 3.12. We normalize it by the condition $\beta_c(1,1) = 1$.

Theorem 4.1. (i) *(The Macdonald-Mehta integral) For* $\mathrm{Re}\,(k) \geq 0$*, one has*

$$(2\pi)^{-r/2} \int_{\mathfrak{h}_{\mathbb{R}}} \mathbf{e}^{-(\mathbf{x},\mathbf{x})/2} |\delta(\mathbf{x})|^{2k} \mathrm{d}\mathbf{x} = \prod_{i=1}^{r} \frac{\Gamma(1+kd_i)}{\Gamma(1+k)}. \tag{4.1}$$

(ii) *Let* $b(k) := \beta_c(\delta,\delta)$*. Then*

$$b(k) = |W| \prod_{i=1}^{r} \prod_{m=1}^{d_i-1} (kd_i + m).$$

For Weyl groups, this theorem was proved by E. Opdam [Op1]. The non-crystallographic cases were done by Opdam in [Op2] using a direct computation in the rank 2 case (reducing (4.1) to the beta integral by passing to polar coordinates), and a computer calculation by F. Garvan for H_3 and H_4.

Example 4.2. In the case $W = \mathfrak{S}_n$, we have the following integral (the Mehta integral):

$$(2\pi)^{-(n-1)/2} \int_{\{\mathbf{x}\in\mathbb{R}^n \mid \sum_i x_i = 0\}} \mathbf{e}^{-(\mathbf{x},\mathbf{x})/2} \prod_{i\neq j} |x_i - x_j|^{2k} \mathrm{d}\mathbf{x} = \prod_{d=2}^{n} \frac{\Gamma(1+kd)}{\Gamma(1+k)}.$$

In the next subsection, we give a uniform proof of Theorem 4.1 which is given in [E2]. We emphasize that many parts of this proof are borrowed from Opdam's previous proof of this theorem.

4.2. Proof of Theorem 4.1.

Proposition 4.3. *The function b is a polynomial of degree at most $|\mathcal{S}|$, and the roots of b are negative rational numbers.*

Proof. Since δ has degree $|\mathcal{S}|$, it follows from the definition of b that it is a polynomial of degree $\le |\mathcal{S}|$.

Suppose that $b(k) = 0$ for some $k \in \mathbb{C}$. Then $\beta_c(\delta, P) = 0$ for any polynomial P. Indeed, if there exists a P such that $\beta_c(\delta, P) \neq 0$, then there exists such a P which is antisymmetric of degree $|\mathcal{S}|$. Then P must be a multiple of δ which contradicts the equality $\beta_c(\delta, \delta) = 0$.

Thus, M_c is reducible and hence has a singular vector, i.e. a nonzero homogeneous polynomial f of positive degree d living in an irreducible representation τ of W killed by y_a. Applying the element $\mathbf{h} = \sum_i x_{a_i} y_{a_i} + r/2 + k \sum_{s\in\mathcal{S}} s$ to f, we get

$$k = -\frac{d}{m_\tau},$$

where m_τ is the eigenvalue of the operator $T := \sum_{s\in\mathcal{S}}(1 - s)$ on τ. But it is clear (by computing the trace of T) that $m_\tau \ge 0$ and $m_\tau \in \mathbb{Q}$. This implies that any root of b is negative rational. ☐

Denote the Macdonald-Mehta integral by $F(k)$.

Proposition 4.4. *One has*

$$F(k + 1) = b(k)F(k).$$

Proof. Let $\mathbf{F} = \sum_i y_{a_i}^2/2$. Introduce *the Gaussian inner product* on M_c as follows:

Definition 4.5. *The Gaussian inner product* γ_c on M_c is given by the formula

$$\gamma_c(v, v') = \beta_c(\exp(\mathbf{F})v, \exp(\mathbf{F})v').$$

This makes sense because the operator $\mathbf{F}$ is locally nilpotent on M_c. Note that δ is a nonzero W-antisymmetric polynomial of the smallest possible degree, so $(\sum y_{a_i}^2)\delta = 0$ and hence

$$\gamma_c(\delta, \delta) = \beta_c(\delta, \delta) = b(k). \tag{4.2}$$

For $a \in \mathfrak{h}$, let $x_a \in \mathfrak{h}^* \subset H_{1,c}(W, \mathfrak{h})$, $y_a \in \mathfrak{h} \subset H_{1,c}(W, \mathfrak{h})$ be the corresponding generators of the rational Cherednik algebra.

Proposition 4.6. *Up to scaling, γ_c is the unique W-invariant symmetric bilinear form on M_c satisfying the condition*

$$\gamma_c((x_a - y_a)v, v') = \gamma_c(v, y_a v'),\ a \in \mathfrak{h}.$$

Proof. We have

$$\begin{aligned} &\gamma_c((x_a - y_a)v, v') = \beta_c(\exp(\mathbf{F})(x_a - y_a)v, \exp(\mathbf{F})v') = \beta_c(x_a \exp(\mathbf{F})v, \exp(\mathbf{F})v') \\ = \ &\beta_c(\exp(\mathbf{F})v, y_a \exp(\mathbf{F})v') = \beta_c(\exp(\mathbf{F})v, \exp(\mathbf{F})y_a v') = \gamma_c(v, y_a v'). \end{aligned}$$

Let us now show uniqueness. If γ is any W-invariant symmetric bilinear form satisfying the condition of the Proposition, then let $\beta(v, v') = \gamma(\exp(-\mathbf{F})v, \exp(-\mathbf{F})v')$. Then β is contravariant, so it's a multiple of β_c, hence γ is a multiple of γ_c. ☐

Now we will need the following known result (see [Du2], Theorem 3.10).

Proposition 4.7. *For* $\mathrm{Re}\,(k) \geq 0$ *we have*

$$\gamma_c(f,g) = F(k)^{-1}\int_{\mathfrak{h}_{\mathbb{R}}} f(\mathbf{x})g(\mathbf{x})\mathrm{d}\mu_c(\mathbf{x}) \tag{4.3}$$

where

$$\mathrm{d}\mu_c(\mathbf{x}) := \mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}|\delta(\mathbf{x})|^{2k}\mathrm{d}\mathbf{x}.$$

Proof. It follows from Proposition 4.6 that γ_c is uniquely, up to scaling, determined by the condition that it is W-invariant, and $y_a^\dagger = x_a - y_a$. These properties are easy to check for the right hand side of (4.3), using the fact that the action of y_a is given by Dunkl operators. □

Now we can complete the proof of Proposition 4.4. By Proposition 4.7, we have

$$F(k+1) = F(k)\gamma_c(\delta,\delta),$$

so by (4.2) we have

$$F(k+1) = F(k)b(k).$$

□

Let

$$b(k) = b_0\prod(k+k_i)^{n_i}.$$

We know that $k_i > 0$, and also $b_0 > 0$ (because the inner product β_0 on real polynomials is positive definite).

Corollary 4.8. *We have*

$$F(k) = b_0^k\prod_i\left(\frac{\Gamma(k+k_i)}{\Gamma(k_i)}\right)^{n_i}.$$

Proof. Denote the right hand side by $F_*(k)$ and let $\phi(k) = F(k)/F_*(k)$. Clearly, $\phi(0) = 1$. Proposition 4.4 implies that $\phi(k)$ is a 1-periodic positive function on $[0,\infty)$. Also by the Cauchy-Schwarz inequality,

$$F(k)F(k') \geq F((k+k')/2)^2,$$

so $\log F(k)$ is convex for $k \geq 0$. This implies that $\phi = 1$, since $(\log F_*(k))'' \to 0$ as $k \to +\infty$. □

Remark 4.9. The proof of this corollary is motivated by the standard proof of the following well known characterization of the Γ function.

Proposition 4.10. *The* Γ *function is determined by three properties:*

(i) $\Gamma(x)$ *is positive on* $[1,+\infty)$ *and* $\Gamma(1) = 1$*;*
(ii) $\Gamma(x+1) = x\Gamma(x)$*;*
(iii) $\log\Gamma(x)$ *is a convex function on* $[1,+\infty)$.

Proof. It is easy to see from the definition $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ that the Γ function has properties (i) and (ii); property (iii) follows from this definition and the Cauchy-Schwarz inequality.

Conversely, suppose we have a function $F(x)$ satisfying the above properties, then we have $F(x) = \phi(x)\Gamma(x)$ for some 1-periodic function $\phi(x)$ with $\phi(x) > 0$. Thus, we have

$$(\log F)'' = (\log\phi)'' + (\log\Gamma)''.$$

Since $\lim_{x\to+\infty}(\log\Gamma)''=0$, $(\log F)''\geq 0$, and ϕ is periodic, we have $(\log\phi)''\geq 0$. Since $\int_n^{n+1}(\log\phi)''\mathrm{d}x=0$, we see that $(\log\phi)''\equiv 0$. So we have $\phi(x)\equiv 1$. □

In particular, we see from Corollary 4.8 and the multiplication formulas for the Γ function that part (ii) of Theorem 4.1 implies part (i).

It remains to establish (ii).

Proposition 4.11. *The polynomial b has degree exactly $|\mathcal{S}|$.*

Proof. By Proposition 4.3, b is a polynomial of degree at most $|\mathcal{S}|$. To see that the degree is precisely $|\mathcal{S}|$, let us make the change of variable $\mathbf{x}=k^{1/2}\mathbf{y}$ in the Macdonald-Mehta integral and use the steepest descent method. We find that the leading term of the asymptotics of $\log F(k)$ as $k\to+\infty$ is $|\mathcal{S}|k\log k$. This together with the Stirling formula and Corollary 4.8 implies the statement. □

Proposition 4.12. *The function*

$$G(k):=F(k)\prod_{j=1}^{r}\frac{1-\mathbf{e}^{2\pi\mathbf{i}kd_j}}{1-\mathbf{e}^{2\pi\mathbf{i}k}}$$

analytically continues to an entire function of k.

Proof. Let $\xi\in D$ be an element. Consider the real hyperplane $C_t=\mathbf{i}t\xi+\mathfrak{h}_{\mathbb{R}}$, $t>0$. Then C_t does not intersect reflection hyperplanes, so we have a continuous branch of $\delta(\mathbf{x})^{2k}$ on C_t which tends to the positive branch in D as $t\to 0$. Then, it is easy to see that for any $w\in W$, the limit of this branch in the chamber $w(D)$ will be $e^{2\pi\mathbf{i}k\ell(w)}|\delta(\mathbf{x})|^{2k}$, where $\ell(w)$ is the length of w. Therefore, by letting $t=0$, we get

$$(2\pi)^{-r/2}\int_{C_t}\mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}\delta(\mathbf{x})^{2k}\mathrm{d}\mathbf{x}=\frac{1}{|W|}F(k)(\sum_{w\in W}e^{2\pi\mathbf{i}k\ell(w)})$$

(as this integral does not depend on t by Cauchy's theorem). But it is well known that

$$\sum_{w\in W}\mathbf{e}^{2\pi\mathbf{i}k\ell(w)}=\prod_{j=1}^{r}\frac{1-\mathbf{e}^{2\pi\mathbf{i}kd_j}}{1-\mathbf{e}^{2\pi\mathbf{i}k}},$$

([Hu], p.73), so

$$(2\pi)^{-r/2}|W|\int_{C_t}\mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}\delta(\mathbf{x})^{2k}\mathrm{d}\mathbf{x}=G(k).$$

Since $\int_{C_t}\mathbf{e}^{-(\mathbf{x},\mathbf{x})/2}\delta(\mathbf{x})^{2k}\mathrm{d}\mathbf{x}$ is clearly an entire function, the statement is proved.

□

Corollary 4.13. *For every $k_0\in[-1,0]$ the total multiplicity of all the roots of b of the form k_0-p, $p\in\mathbb{Z}_+$, equals the number of ways to represent k_0 in the form $-m/d_i$, $m=1,\dots,d_i-1$. In other words, the roots of b are $k_{i,m}=-m/d_i-p_{i,m}$, $1\leq m\leq d_i-1$, where $p_{i,m}\in\mathbb{Z}_+$.*

Proof. We have

$$G(k-p)=\frac{F(k)}{b(k-1)\cdots b(k-p)}\prod_{j=1}^{r}\frac{1-\mathbf{e}^{2\pi\mathbf{i}kd_j}}{1-\mathbf{e}^{2\pi\mathbf{i}k}},$$

Now plug in $k = 1 + k_0$ and a large positive integer p. Since by Proposition 4.12 the left hand side is regular, so must be the right hand side, which implies the claimed upper bound for the total multiplicity, as $F(1 + k_0) > 0$. The fact that the bound is actually attained follows from the fact that the polynomial b has degree exactly $|\mathcal{S}|$ (Proposition 4.11), and the fact that all roots of b are negative rational (Proposition 4.3). □

It remains to show that in fact in Corollary 4.13, $p_{i,m} = 0$ for all i, m; this would imply (ii) and hence (i).

Proposition 4.14. *Identity* (4.1) *of Theorem 4.1 is satisfied in* $\mathbb{C}[k]/k^2$.

Proof. Indeed, we clearly have $F(0) = 1$. Next, a rank 1 computation gives $F'(0) = -\gamma|\mathcal{S}|$, where γ is the Euler constant (i.e. $\gamma = \lim_{n\to+\infty}(1 + \cdots + 1/n - \log n)$), while the derivative of the right hand side of (4.1) at zero equals to

$$-\gamma \sum_{i=1}^{r} (d_i - 1).$$

But it is well known that

$$\sum_{i=1}^{r} (d_i - 1) = |\mathcal{S}|,$$

([Hu], p.62), which implies the result. □

Proposition 4.15. *Identity* (4.1) *of Theorem 4.1 is satisfied in* $\mathbb{C}[k]/k^3$.

Note that Proposition 4.15 immediately implies (ii), and hence the whole theorem. Indeed, it yields that

$$(\log F)''(0) = \sum_{i=1}^{r} \sum_{m=1}^{d_i-1} (\log\Gamma)''(m/d_i),$$

so by Corollary 4.13

$$\sum_{i=1}^{r} \sum_{m=1}^{d_i-1} (\log\Gamma)''(m/d_i + p_{i,m}) = \sum_{i=1}^{r} \sum_{m=1}^{d_i-1} (\log\Gamma)''(m/d_i),$$

which implies that $p_{i,m} = 0$ since $(\log\Gamma)''$ is strictly decreasing on $[0, \infty)$.

To prove Proposition 4.15, we will need the following result about finite Coxeter groups.

Let $\psi(W) = 3|\mathcal{S}|^2 - \sum_{i=1}^{r}(d_i^2 - 1)$.

Lemma 4.16. *One has*

$$\psi(W) = \sum_{G \in \mathrm{Par}_2(W)} \psi(G), \tag{4.4}$$

where $\mathrm{Par}_2(W)$ *is the set of parabolic subgroups of* W *of rank 2.*

Proof. Let

$$Q(q) = |W| \prod_{i=1}^{r} \frac{1-q}{1-q^{d_i}}.$$

It follows from Chevalley's theorem that

$$Q(q) = (1-q)^r \sum_{w \in W} \det(1 - qw|_{\mathfrak{h}})^{-1}.$$

Let us subtract the terms for $w = 1$ and $w \in \mathcal{S}$ from both sides of this equation, divide both sides by $(q-1)^2$, and set $q = 1$ (cf. [Hu], p.62, formula (21)). Let W_2 be the set of elements of W that can be written as a product of two different reflections. Then by a straightforward computation we get

$$\frac{1}{24}\psi(W) = \sum_{w \in W_2} \frac{1}{r - \mathrm{Tr}_{\mathfrak{h}}(w)}.$$

In particular, this is true for rank 2 groups. The result follows, as any element $w \in W_2$ belongs to a unique parabolic subgroup G_w of rank 2 (namely, the stabilizer of a generic point $\mathfrak{h}^w$, [Hu], p.22). □

Proof of Proposition 4.15. Now we are ready to prove the proposition. By Proposition 4.14, it suffices to show the coincidence of the second derivatives of (4.1) at $k = 0$. The second derivative of the right hand side of (4.1) at zero is equal to

$$\frac{\pi^2}{6}\sum_{i=1}^{r}(d_i^2 - 1) + \gamma^2|\mathcal{S}|^2.$$

On the other hand, we have

$$F''(0) = (2\pi)^{-r/2} \sum_{\alpha,\beta \in \mathcal{S}} \int_{\mathfrak{h}_{\mathbb{R}}} \mathbf{e}^{-(\mathbf{x},\mathbf{x})/2} \log \alpha^2(\mathbf{x}) \log \beta^2(\mathbf{x}) \mathrm{d}\mathbf{x}.$$

Thus, from a rank 1 computation we see that our job is to establish the equality

$$(2\pi)^{-r/2} \sum_{\alpha \neq \beta \in \mathcal{S}} \int_{\mathfrak{h}_{\mathbb{R}}} \mathbf{e}^{-(\mathbf{x},\mathbf{x})/2} \log \alpha^2(\mathbf{x}) \log \frac{\beta^2(\mathbf{x})}{\alpha^2(\mathbf{x})} \mathrm{d}\mathbf{x} = \frac{\pi^2}{6}(\sum_{i=1}^{r}(d_i^2 - 1) - 3|\mathcal{S}|^2) = -\frac{\pi^2}{6}\psi(W).$$

Since this equality holds in rank 2 (as in this case (4.1) reduces to the beta integral), in general it reduces to equation (4.4) (as for any $\alpha \neq \beta \in S$, s_α and s_β are contained in a unique parabolic subgroup of W of rank 2). The proposition is proved. □

4.3. **Application: the supports of $L_c(\mathbb{C})$.** In this subsection we will use the Macdonald-Mehta integral to computation of the support of the irreducible quotient of the polynoamial representation of a rational Cherednik algebra (with equal parameters). We will follow the paper [E3].

First note that the vector space $\mathfrak{h}$ has a stratification labeled by parabolic subgroups of W. Indeed, for a parabolic subgroup $W' \subset W$, let $\mathfrak{h}_{\mathrm{reg}}^{W'}$ be the set of points in $\mathfrak{h}$ whose stabilizer is W'. Then

$$\mathfrak{h} = \coprod_{W' \in \mathrm{Par}(W)} \mathfrak{h}_{\mathrm{reg}}^{W'},$$

where $\mathrm{Par}(W)$ is the set of parabolic subgroups in W.

For a finitely generated module M over $\mathbb{C}[\mathfrak{h}]$, denote the support of M by $\mathrm{supp}\,(M)$.

The following theorem is proved in [Gi1], Section 6 and in [BE] with different method. We will recall the proof from [BE] later.

Theorem 4.17. *Consider the stratification of* $\mathfrak{h}$ *with respect to stabilizers of points in* W*. Then the support* $\operatorname{supp}(M)$ *of any object* M *of* $\mathcal{O}_c(W,\mathfrak{h})$ *in* $\mathfrak{h}$ *is a union of strata of this stratification.*

This makes one wonder which strata occur in $\operatorname{supp}(L_c(\tau))$, for given c and τ. In [VV], Varagnolo and Vasserot gave a partial answer for $\tau = \mathbb{C}$. Namely, they determined (for W being a Weyl group) when $L_c(\mathbb{C})$ is finite dimensional, which is equivalent to $\operatorname{supp}(L_c(\mathbb{C})) = 0$. For the proof (which is quite complicated), they used the geometry affine Springer fibers. Here we will give a different (and simpler) proof. In fact, we will prove a more general result.

Recall that for any Coxeter group W, we have the Poincaré polynomial:

$$P_W(q) = \sum_{w\in W} q^{\ell(w)} = \prod_{i=1}^{r} \frac{1-q^{d_i(W)}}{1-q}, \text{ where } d_i(W) \text{ are the degrees of } W.$$

Lemma 4.18. *If* $W' \subset W$ *is a parabolic subgroup of* W*, then* P_W *is divisible by* $P_{W'}$*.*

Proof. By Chevalley's theorem, $\mathbb{C}[\mathfrak{h}]$ is a free module over $\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[\mathfrak{h}]^{W'}$ is a direct summand in this module. So $\mathbb{C}[\mathfrak{h}]^{W'}$ is a projective module, thus free (since it is graded).

Hence, there exists a polynomial $Q(q)$ such that we have

$$Q(q)h_{\mathbb{C}[\mathfrak{h}]^W}(q) = h_{\mathbb{C}[\mathfrak{h}]^{W'}}(q),$$

where $h_V(q)$ denotes the Hilbert series of a graded vector space V. Notice that we have $h_{\mathbb{C}[\mathfrak{h}]^W}(q) = \dfrac{1}{P_W(q)(1-q)^r}$, so we have

$$\frac{Q(q)}{P_W(q)} = \frac{1}{P_{W'}(q)}, \text{ i.e. } Q(q) = P_W(q)/P_{W'}(q).$$

□

Corollary 4.19. *If* $m \geq 2$ *then we have the following inequality:*

$$\#\{i|m \text{ divides } d_i(W)\} \geq \#\{i|m \text{ divides } d_i(W')\}.$$

Proof. This follows from Lemma 4.18 by looking at the roots of the polynomials P_W and $P_{W'}$. □

Our main result is the following theorem.

Theorem 4.20. [E3] *Let* $c \geq 0$*. Then* $a \in \operatorname{supp}(L_c(\mathbb{C}))$ *if and only if*

$$\frac{P_W}{P_{W_a}}(\mathbf{e}^{2\pi\mathbf{i}c}) \neq 0.$$

We can obtain the following corollary easily.

Corollary 4.21.
(i) $L_c(\mathbb{C}) \neq M_c(\mathbb{C})$ *if and only if* $c \in \mathbb{Q}_{>0}$ *and the denominator* m *of* c *divides* d_i *for some* i*;*
(ii) $L_c(\mathbb{C})$ *is finite dimensional if and only if* $\dfrac{P_W}{P_{W'}}(\mathbf{e}^{2\pi\mathbf{i}c}) = 0$*, i.e., iff*

$$\#\{i|m \text{ divides } d_i(W)\} > \#\{i|m \text{ divides } d_i(W')\}.$$

for any maximal parabolic subgroup $W' \subset W$*.*

Remark 4.22. Varagnolo and Vasserot prove that $L_c(\mathbb{C})$ is finite dimensional if and only if there exists a regular elliptic element in W of order m. Case-by-case inspection shows that this condition is equivalent to the combinatorial condition of (2). Also, a uniform proof of this equivalence is given in the appendix to [E3], written by S. Griffeth.

Example 4.23. For type A_{n-1}, i.e., $W = \mathfrak{S}_n$, we get that $L_c(\mathbb{C})$ is finite dimensional if and only if the denominator of c is n. This agrees with our previous results in type A_{n-1}.

Example 4.24. Suppose W is the Coxeter group of type E_7. Then we have the following list of maximal parabolic subgroups and the degrees (note that E_7 itself is not a maximal parabolic).

Subgroups	E_7	D_6	$A_3 \times A_2 \times A_1$	A_6
Degrees	2,6,8,10,12,14,18	2,4,6,6,8,10	2,3,4,2,3,2	2,3,4,5,6,7
Subgroups	$A_4 \times A_2$	E_6	$D_5 \times A_1$	$A_5 \times A_1$
Degrees	2,3,4,5,2,3	2,5,6,8,9,12	2,4,5,6,8,2	2,3,4,5,6,2

So $L_c(\mathbb{C})$ is finite dimensional if and only if the denominator of c is $2, 6, 14, 18$.

The rest of the subsection is dedicated to the proof of Theorem 4.20. First we recall some basic facts about the Schwartz space and tempered distributions.

Let $\mathcal{S}(\mathbb{R}^n)$ be the set of Schwartz functions on $\mathbb{R}^n$, i.e.

$$\mathcal{S}(\mathbb{R}^n) = \{f \in C^\infty(\mathbb{R}^n) | \forall \alpha, \beta, \sup |\mathbf{x}^\alpha \partial^\beta f(\mathbf{x})| < \infty\}.$$

This space has a natural topology.

A tempered distribution on $\mathbb{R}^n$ is a continuous linear functional on $\mathcal{S}(\mathbb{R}^n)$. Let $\mathcal{S}'(\mathbb{R}^n)$ denote the space of tempered distributions.

We will use the following well known lemma.

Lemma 4.25. (i) $\mathbb{C}[\mathbf{x}]\mathbf{e}^{-\mathbf{x}^2/2} \subset \mathcal{S}(\mathbb{R}^n)$ *is a dense subspace.*

(ii) *Any tempered distribution ξ has finite order, i.e., $\exists N = N(\xi)$ such that if $f \in \mathcal{S}(\mathbb{R}^n)$ satisfying $f = \mathrm{d}f = \cdots = \mathrm{d}^{N-1}f = 0$ on $\operatorname{supp}\xi$, then $\langle \xi, f\rangle = 0$.*

Proof of Theorem 4.20. Recall that on $M_c(\mathbb{C})$, we have the Gaussian form γ_c from Section 4.2. We have for $\operatorname{Re}(c) \le 0$,

$$\gamma_c(P, Q) = \frac{(2\pi)^{-r/2}}{F_W(-c)} \int_{\mathfrak{h}_\mathbb{R}} \mathbf{e}^{-\mathbf{x}^2/2} |\delta(\mathbf{x})|^{-2c} P(\mathbf{x})Q(\mathbf{x})\mathrm{d}\mathbf{x},$$

where P, Q are polynomials and

$$F_W(k) = (2\pi)^{-r/2} \int_{\mathfrak{h}_\mathbb{R}} \mathbf{e}^{-\mathbf{x}^2/2} |\delta(\mathbf{x})|^{2k} \mathrm{d}\mathbf{x}$$

is the Macdonald-Mehta integral.

Consider the distribution:

$$\xi_c^W = \frac{(2\pi)^{-r/2}}{F_W(-c)} |\delta(\mathbf{x})|^{-2c}.$$

It is well-known that this distribution is meromorphic in c (Bernstein's theorem). Moreover, since $\gamma_c(P, Q)$ is a polynomial in c for any P and Q, this distribution is in fact holomorphic in $c \in \mathbb{C}$.

Proposition 4.26.

$$\begin{aligned}\operatorname{supp}(\xi_c^W) &= \{a\in\mathfrak{h}_{\mathbb{R}}|\frac{F_{W_a}}{F_W}(-c)\neq 0\} = \{a\in\mathfrak{h}_{\mathbb{R}}|\frac{P_W}{P_{W_a}}(e^{2\pi \mathrm{i}c})\neq 0\}\\ &= \{a\in\mathfrak{h}_{\mathbb{R}}|\#\{i|\textit{denominator of } c \textit{ divides } d_i(W)\}\\ &\qquad\qquad = \#\{i|\textit{denominator of } c \textit{ divides } d_i(W_a)\}\}.\end{aligned}$$

Proof. First note that the last equality follows from the product formula for the Poincaré polynomial, and the second equality from the Macdonald-Mehta identity. Now let us prove the first equality.

Look at ξ_c^W near $a\in\mathfrak{h}$. Equivalently, we can consider

$$\xi_c^W(\mathbf{x}+a) = \frac{(2\pi)^{-r/2}}{F_W(-c)}|\delta(\mathbf{x}+a)|^{-2c}$$

with $\mathbf{x}$ near 0. We have

$$\begin{aligned}\delta_W(\mathbf{x}+a) &= \prod_{s\in\mathcal{S}}\alpha_s(\mathbf{x}+a) = \prod_{s\in\mathcal{S}}(\alpha_s(\mathbf{x})+\alpha_s(a))\\ &= \prod_{s\in\mathcal{S}\cap W_a}\alpha_s(\mathbf{x})\cdot\prod_{s\in\mathcal{S}\setminus\mathcal{S}\cap W_a}(\alpha_s(\mathbf{x})+\alpha_s(a))\\ &= \delta_{W_a}(\mathbf{x})\cdot\Psi(\mathbf{x}),\end{aligned}$$

where Ψ is a nonvanishing function near a (since $\alpha_s(a)\neq 0$ if $s\notin\mathcal{S}\cap W_a$).

So near a, we have

$$\xi_c^W(\mathbf{x}+a) = \frac{F_{W_a}}{F_W}(-c)\cdot\xi_c^{W_a}(\mathbf{x})\cdot|\Psi|^{-2c},$$

and the last factor is well defined since Ψ is nonvanishing. Thus $\xi_c^W(\mathbf{x})$ is nonzero near a if and only if $\frac{F_{W_a}}{F_W}(-c)\neq 0$ which finishes the proof. □

Proposition 4.27. *For $c\geq 0$,*

$$\operatorname{supp}(\xi_c^W) = \operatorname{supp} L_c(\mathbb{C})_{\mathbb{R}},$$

where the right hand side stands for the real points of the support.

Proof. Let $a\notin\operatorname{supp} L_c(\mathbb{C})$ and assume $a\in\operatorname{supp}\xi_c^W$. Then we can find a $P\in J_c(\mathbb{C})=\ker\gamma_c$ such that $P(a)\neq 0$. Pick a compactly supported test function $\phi\in C_c^\infty(\mathfrak{h}_{\mathbb{R}})$ such that P does not vanish anywhere on $\operatorname{supp}\phi$, and $\langle\xi_c^W,\phi\rangle\neq 0$ (this can be done since $P(a)\neq 0$ and ξ_c^W is nonzero near a). Then we have $\phi/P\in\mathcal{S}(\mathfrak{h}_{\mathbb{R}})$. Thus from Lemma 4.25 (i) it follows that there exists a sequence of polynomials P_n such that

$$P_n(\mathbf{x})\mathbf{e}^{-\mathbf{x}^2/2}\to\frac{\phi}{P}\text{ in }\mathcal{S}(\mathfrak{h}_{\mathbb{R}}),\text{ when } n\to\infty.$$

So $PP_n\mathbf{e}^{-\mathbf{x}^2/2}\to\phi$ in $\mathcal{S}(\mathfrak{h}_{\mathbb{R}})$, when $n\to\infty$.

But we have $\langle\xi_c^W, PP_n\mathbf{e}^{-\mathbf{x}^2/2}\rangle=\gamma_c(P,P_n)=0$ which is a contradiction. This implies that $\operatorname{supp}\xi_c^W\subset(\operatorname{supp}L_c(\mathbb{C}))_{\mathbb{R}}$.

To show the opposite inclusion, let P be a polynomial on $\mathfrak{h}$ which vanishes identically on $\operatorname{supp}\xi_c^W$. By Lemma 4.25 (ii), there exists N such that $\langle\xi_c^W,P^N(\mathbf{x})Q(\mathbf{x})\mathbf{e}^{-\mathbf{x}^2/2}\rangle=0$. Thus, for any polynomial Q, $\gamma_c(P^N, Q) = 0$, i.e. $P^N \in \operatorname{Ker}\gamma_c$. Thus, $P|_{\operatorname{supp} L_c(\mathbb{C})} = 0$. This implies the required inclusion, since $\operatorname{supp}\xi_c^W$ is a union of strata. □

Theorem 4.20 follows from Proposition 4.26 and Proposition 4.27. □

4.4. **Notes.** Our exposition in Sections 4.1 and 4.2 follows the paper [E2]; Section 4.3 follows the paper [E3].

5. Parabolic induction and restriction functors for rational Cherednik algebras

5.1. **A geometric approach to rational Cherednik algebras.** An important property of the rational Cherednik algebra $H_{1,c}(G,\mathfrak{h})$ is that it can be sheafified, as an algebra, over $\mathfrak{h}/G$ (see [E1]). More specifically, the usual sheafification of $H_{1,c}(G,\mathfrak{h})$ as a $\mathcal{O}_{\mathfrak{h}/G}$-module is in fact a quasicoherent sheaf of algebras, $H_{1,c,G,\mathfrak{h}}$. Namely, for every affine open subset $U\subset\mathfrak{h}/G$, the algebra of sections $H_{1,c,G,\mathfrak{h}}(U)$ is, by definition, $\mathbb{C}[U]\otimes_{\mathbb{C}[\mathfrak{h}]^G}H_{1,c}(G,\mathfrak{h})$.

The same sheaf can be defined more geometrically as follows (see [E1], Section 2.9). Let $\widetilde{U}$ be the preimage of U in $\mathfrak{h}$. Then the algebra $H_{1,c,G,\mathfrak{h}}(U)$ is the algebra of linear operators on $\mathcal{O}(\widetilde{U})$ generated by $\mathcal{O}(\widetilde{U})$, the group G, and Dunkl operators

$$\partial_a-\sum_{s\in\mathcal{S}}\frac{2c_s}{1-\lambda_s}\frac{\alpha_s(a)}{\alpha_s}(1-s),\ \text{where } a\in\mathfrak{h}.$$

5.2. **Completion of rational Cherednik algebras.** For any $b\in\mathfrak{h}$ we can define the completion $\widehat{H_{1,c}}(G,\mathfrak{h})_b$ to be the algebra of sections of the sheaf $H_{1,c,G,\mathfrak{h}}$ on the formal neighborhood of the image of b in $\mathfrak{h}/G$. Namely, $\widehat{H_{1,c}}(G,\mathfrak{h})_b$ is generated by regular functions on the formal neighborhood of the G-orbit of b, the group G, and Dunkl operators.

The algebra $\widehat{H_{1,c}}(G,\mathfrak{h})_b$ inherits from $H_{1,c}(G,\mathfrak{h})$ the natural filtration $F^\bullet$ by order of differential operators, and each of the spaces $F^n\widehat{H_{1,c}}(G,\mathfrak{h})_b$ has a projective limit topology; the whole algebra is then equipped with the topology of the nested union (or inductive limit).

Consider the completion of the rational Cherednik algebra at zero, $\widehat{H_{1,c}}(G,\mathfrak{h})_0$. It naturally contains the algebra $\mathbb{C}[[\mathfrak{h}]]$. Define the category $\widehat{\mathcal{O}}_c(G,\mathfrak{h})$ of representations of $\widehat{H_{1,c}}(G,\mathfrak{h})_0$ which are finitely generated over $\mathbb{C}[[\mathfrak{h}]]_0=\mathbb{C}[[\mathfrak{h}]]$.

We have a completion functor $\widehat{\ }\colon\mathcal{O}_c(G,\mathfrak{h})\to\widehat{\mathcal{O}}_c(G,\mathfrak{h})$, defined by

$$\widehat{M}=\widehat{H_{1,c}}(G,\mathfrak{h})_0\otimes_{H_{1,c}(G,\mathfrak{h})}M=\mathbb{C}[[\mathfrak{h}]]\otimes_{\mathbb{C}[\mathfrak{h}]}M.$$

Also, for $N\in\widehat{\mathcal{O}}_c(G,\mathfrak{h})$, let $E(N)$ be the subspace spanned by generalized eigenvectors of $\mathbf{h}$ in N where $\mathbf{h}$ is defined by (3.2). Then it is easy to see that $E(N)\in\mathcal{O}_c(G,\mathfrak{h})_0$.

Theorem 5.1. *The restriction of the completion functor* $\widehat{\ }$ *to* $\mathcal{O}_c(G,\mathfrak{h})_0$ *is an equivalence of categories* $\mathcal{O}_c(G,\mathfrak{h})_0\to\widehat{\mathcal{O}}_c(G,\mathfrak{h})$. *The inverse equivalence is given by the functor* E.

Proof. It is clear that $M\subset\widehat{M}$, so $M\subset E(\widehat{M})$ (as M is spanned by generalized eigenvectors of $\mathbf{h}$). Let us demonstrate the opposite inclusion. Pick generators $m_1,\dots,m_r$ of M which are generalized eigenvectors of $\mathbf{h}$ with eigenvalues $\mu_1,\dots,\mu_r$. Let $0\neq v\in E(\widehat{M})$. Then $v=\sum_i f_im_i$, where $f_i\in\mathbb{C}[[\mathfrak{h}]]$. Assume that $(\mathbf{h}-\mu)^Nv=0$ for some N. Then $v=\sum_i f_i^{(\mu-\mu_i)}m_i$, where for $f\in\mathbb{C}[[\mathfrak{h}]]$ we denote by $f^{(d)}$ the degree d part of f. Thus $v\in M$, so $M=E(\widehat{M})$.

It remains to show that $\widehat{E(N)}=N$, i.e. that N is the closure of $E(N)$. In other words, letting $\mathfrak{m}$ denote the maximal ideal in $\mathbb{C}[[\mathfrak{h}]]$, we need to show that the natural map $E(N)\to N/\mathfrak{m}^jN$ is surjective for every j.

To do so, note that $\mathbf{h}$ preserves the descending filtration of N by subspaces $\mathfrak{m}^jN$. On the other hand, the successive quotients of these subspaces, $\mathfrak{m}^jN/\mathfrak{m}^{j+1}N$, are finite dimensional, which implies that $\mathbf{h}$ acts locally finitely on their direct sum $\mathrm{gr}N$, and moreover each generalized eigenspace is finite dimensional. Now for each $\beta \in \mathbb{C}$ denote by $N_{j,\beta}$ the generalized β-eigenspace of $\mathbf{h}$ in $N/\mathfrak{m}^j N$. We have surjective homomorphisms $N_{j+1,\beta} \to N_{j,\beta}$, and for large enough j they are isomorphisms. This implies that the map $E(N) \to N/\mathfrak{m}^j N$ is surjective for every j, as desired. □

Example. Suppose that $c = 0$. Then Theorem 5.1 specializes to the well known fact that the category of G-equivariant local systems on $\mathfrak{h}$ with a locally nilpotent action of partial differentiations is equivalent to the category of all G-equivariant local systems on the formal neighborhood of zero in $\mathfrak{h}$. In fact, both categories in this case are equivalent to the category of finite dimensional representations of G.

We can now define the composition functor $\mathcal{J} : \mathcal{O}_c(G, \mathfrak{h}) \to \mathcal{O}_c(G, \mathfrak{h})_0$, by the formula $\mathcal{J}(M) = E(\widehat{M})$. The functor $\mathcal{J}$ is called the Jacquet functor ([Gi2]).

5.3. **The duality functor.** Recall that in Section 3.11, for any $H_{1,c}(G, \mathfrak{h})$-module M, the full dual space M^* is naturally an $H_{1,\bar{c}}(G, \mathfrak{h}^*)$-module, via $\pi_{M^*}(a) = \pi_M(\gamma(a))^*$.

It is clear that the duality functor $*$ defines an equivalence between the category $\mathcal{O}_c(G, \mathfrak{h})_0$ and $\widehat{\mathcal{O}}_{\bar{c}}(G, \mathfrak{h}^*)^{\mathrm{op}}$, and that $M^\dagger = E(M^*)$ (where $M^\dagger$ is the contragredient, or restricted dual module to M defined in Section 3.11).

5.4. **Generalized Jacquet functors.**

Proposition 5.2. *For any $M \in \widehat{\mathcal{O}}_c(G, \mathfrak{h})$, a vector $v \in M$ is $\mathbf{h}$-finite if and only if it is $\mathfrak{h}$-nilpotent.*

Proof. The "if" part follows from Theorem 3.20. To prove the "only if" part, assume that $(\mathbf{h} - \mu)^N v = 0$. Then for any $u \in S^r\mathfrak{h} \cdot v$, we have $(\mathbf{h} - \mu + r)^N u = 0$. But by Theorem 5.1, the real parts of generalized eigenvalues of $\mathbf{h}$ in M are bounded below. Hence $S^r\mathfrak{h} \cdot v = 0$ for large enough r, as desired. □

According to Proposition 5.2, the functor E can be alternatively defined by setting $E(M)$ to be the subspace of M which is locally nilpotent under the action of $\mathfrak{h}$.

This gives rise to the following generalization of E: for any $\lambda \in \mathfrak{h}^*$ we define the functor $E_\lambda : \widehat{\mathcal{O}}_c(G, \mathfrak{h}) \to \mathcal{O}_c(G, \mathfrak{h})_\lambda$ by setting $E_\lambda(M)$ to be the space of generalized eigenvectors of $\mathbb{C}[\mathfrak{h}^*]^G$ in M with eigenvalue λ. This way, we have $E_0 = E$.

We can also define the generalized Jacquet functor $\mathcal{J}_\lambda : \mathcal{O}_c(G, \mathfrak{h}) \to \mathcal{O}_c(G, \mathfrak{h})_\lambda$ by the formula $\mathcal{J}_\lambda(M) = E_\lambda(\widehat{M})$. Then we have $\mathcal{J}_0 = \mathcal{J}$, and one can show that the restriction of $\mathcal{J}_\lambda$ to $\mathcal{O}_c(G, \mathfrak{h})_\lambda$ is the identity functor.

5.5. **The centralizer construction.** For a finite group H, let $\mathsf{e}_H = |H|^{-1} \sum_{g \in H} g$ be the symmetrizer of H.

If $G \supset H$ are finite groups, and A is an algebra containing $\mathbb{C}[H]$, then define the algebra $Z(G, H, A)$ to be the centralizer $\mathrm{End}_A(P)$ of A in the right A-module $P = \mathrm{F}un_H(G, A)$ of H-invariant A-valued functions on G, i.e. such functions $f : G \to A$ that $f(hg) = hf(g)$. Clearly, P is a free A-module of rank $|G/H|$, so the algebra $Z(G, H, A)$ is isomorphic to $\mathrm{M}at_{|G/H|}(A)$, but this isomorphism is not canonical.

The following lemma is trivial.

Lemma 5.3. *The functor $N \mapsto I(N) := P \otimes_A N = \mathrm{F}un_H(G, N)$ defines an equivalence of categories $A - \mathrm{mod} \to Z(G, H, A) - \mathrm{mod}$.*

5.6. **Completion of rational Cherednik algebras at arbitrary points of $\mathfrak{h}/G$.** The following result is, in essence, a consequence of the geometric approach to rational Cherednik algebras, described in Subsection 5.1. It should be regarded as a direct generalization to the case of Cherednik algebras of Theorem 8.6 of [L] for affine Hecke algebras.

Let $b \in \mathfrak{h}$. Abusing notation, denote the restriction of c to the set $\mathcal{S}_b$ of reflections in G_b also by c.

Theorem 5.4. *One has a natural isomorphism*

$$\theta : \widehat{H_{1,c}}(G,\mathfrak{h})_b \to Z(G, G_b, \widehat{H_{1,c}}(G_b,\mathfrak{h})_0),$$

defined by the following formulas. Suppose that $f \in P = \mathrm{Fun}_{G_b}(G, \widehat{H_{1,c}}(G_b,\mathfrak{h})_0)$. Then

$$(\theta(u)f)(w) = f(wu), u \in G;$$

for any $\alpha \in \mathfrak{h}^*$,

$$(\theta(x_\alpha)f)(w) = (x^{(b)}_{w\alpha} + (w\alpha, b))f(w),$$

where $x_\alpha \in \mathfrak{h}^* \subset H_{1,c}(G,\mathfrak{h})$, $x^{(b)}_\alpha \in \mathfrak{h}^* \subset H_{1,c}(G_b,\mathfrak{h})$ are the elements corresponding to α; and for any $a \in \mathfrak{h}$,

$$(5.1) \qquad (\theta(y_a)f)(w) = y^{(b)}_{wa}f(w) - \sum_{s\in\mathcal{S}:s\notin G_b} \frac{2c_s}{1-\lambda_s}\frac{\alpha_s(wa)}{x^{(b)}_{\alpha_s} + \alpha_s(b)}(f(w) - f(sw)).$$

where $y_a \in \mathfrak{h} \subset H_{1,c}(G,\mathfrak{h})$, $y^{(b)}_a \in \mathfrak{h} \subset H_{1,c}(G_b,\mathfrak{h})$.

Proof. The proof is by a direct computation. We note that in the last formula, the fraction $\alpha_s(wa)/(x^{(b)}_{\alpha_s} + \alpha_s(b))$ is viewed as a power series (i.e., an element of $\mathbb{C}[[\mathfrak{h}]]$), and that only the entire sum, and not each summand separately, is in the centralizer algebra. $\square$

Remark. Let us explain how to see the existence of θ without writing explicit formulas, and how to guess the formula (5.1) for θ. It is explained in [E1] (see e.g. [E1], Section 2.9) that the sheaf of algebras obtained by sheafification of $H_{1,c}(G,\mathfrak{h})$ over $\mathfrak{h}/G$ is generated (on every affine open set in $\mathfrak{h}/G$) by regular functions on $\mathfrak{h}$, elements of G, and Dunkl operators. Therefore, this statement holds for formal neighborhoods, i.e., it is true on the formal neighborhood of the image in $\mathfrak{h}/G$ of any point $b \in \mathfrak{h}$. However, looking at the formula for Dunkl operators near b, we see that the summands corresponding to $s \in \mathcal{S}, s \notin G_b$ are actually regular at b, so they can be safely deleted without changing the generated algebra (as all regular functions on the formal neighborhood of b are included into the system of generators). But after these terms are deleted, what remains is nothing but the Dunkl operators for $(G_b,\mathfrak{h})$, which, together with functions on the formal neighborhood of b and the group G_b, generate the completion of $H_{1,c}(G_b,\mathfrak{h})$. This gives a construction of θ without using explicit formulas.

Also, this argument explains why θ should be defined by formula (5.1) of Theorem 5.4. Indeed, what this formula does is just restores the terms with $s \notin G_b$ that have been previously deleted.

The map θ defines an equivalence of categories

$$\theta_* : \widehat{H_{1,c}}(G,\mathfrak{h})_b - \mathrm{mod} \to Z(G, G_b, \widehat{H_{1,c}}(G_b,\mathfrak{h})_0) - \mathrm{mod}.$$

Corollary 5.5. *We have a natural equivalence of categories*

$$\psi_\lambda : \mathcal{O}_c(G, \mathfrak{h})_\lambda \to \mathcal{O}_c(G_\lambda, \mathfrak{h}/\mathfrak{h}^{G_\lambda})_0.$$

Proof. The category $\mathcal{O}_c(G, \mathfrak{h})_\lambda$ is the category of modules over $H_{1,c}(G, \mathfrak{h})$ which are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and extend by continuity to the completion of the algebra $H_{1,c}(G, \mathfrak{h})$ at λ. So it follows from Theorem 5.4 that we have an equivalence $\mathcal{O}_c(G, \mathfrak{h})_\lambda \to \mathcal{O}_c(G_\lambda, \mathfrak{h})_0$. Composing this equivalence with the equivalence $\zeta : \mathcal{O}_c(G_\lambda, \mathfrak{h})_0 \to \mathcal{O}_c(G_\lambda, \mathfrak{h}/\mathfrak{h}^{G_\lambda})_0$, we obtain the desired equivalence ψ_λ. □

Remark 5.6. Note that in this proof, we take the completion of $H_{1,c}(G, \mathfrak{h})$ at a point of $\lambda \in \mathfrak{h}^*$ rather than $b \in \mathfrak{h}$.

5.7. **The completion functor.** Let $\widehat{\mathcal{O}}_c(G, \mathfrak{h})^b$ be the category of modules over $\widehat{H_{1,c}}(G, \mathfrak{h})_b$ which are finitely generated over $\widehat{\mathbb{C}[\mathfrak{h}]}_b$.

Proposition 5.7. *The duality functor $*$ defines an anti-equivalence of categories $\mathcal{O}_c(G, \mathfrak{h})_\lambda \to \widehat{\mathcal{O}}_{\bar{c}}(G, \mathfrak{h}^*)^\lambda$.*

Proof. This follows from the fact (already mentioned above) that $\mathcal{O}_c(G, \mathfrak{h})_\lambda$ is the category of modules over $H_{1,c}(G, \mathfrak{h})$ which are finitely generated over $\mathbb{C}[\mathfrak{h}]$ and extend by continuity to the completion of the algebra $H_{1,c}(G, \mathfrak{h})$ at λ. □

Let us denote the functor inverse to $*$ also by $*$; it is the functor of continuous dual (in the formal series topology).

We have an exact functor of completion at b, $\mathcal{O}_c(G, \mathfrak{h})_0 \to \widehat{\mathcal{O}}_c(G, \mathfrak{h})^b$, $M \mapsto \widehat{M}_b$. We also have a functor $E^b : \widehat{\mathcal{O}}_c(G, \mathfrak{h})^b \to \mathcal{O}_c(G, \mathfrak{h})_0$ in the opposite direction, sending a module N to the space $E^b(N)$ of $\mathfrak{h}$-nilpotent vectors in N.

Proposition 5.8. *The functor E^b is right adjoint to the completion functor $\widehat{\ }_b$.*

Proof. We have

$$\mathrm{Hom}_{\widehat{H_{1,c}}(G,\mathfrak{h})_b}(\widehat{M}_b, N) = \mathrm{Hom}_{\widehat{H_{1,c}}(G,\mathfrak{h})_b}(\widehat{H_{1,c}}(G, \mathfrak{h})_b \otimes_{H_{1,c}(G,\mathfrak{h})} M, N)$$

$$= \mathrm{Hom}_{H_{1,c}(G,\mathfrak{h})}(M, N|_{H_{1,c}(G,\mathfrak{h})}) = \mathrm{Hom}_{H_{1,c}(G,\mathfrak{h})}(M, E^b(N)).$$

□

Remark 5.9. Recall that by Theorem 5.1, if $b = 0$ then these functors are not only adjoint but also inverse to each other.

Proposition 5.10. (i) *For $M \in \mathcal{O}_{\bar{c}}(G, \mathfrak{h}^*)_b$, one has $E^b(M^*) = (\widehat{M})^*$ in $\mathcal{O}_c(G, \mathfrak{h})_0$.*
(ii) *For $M \in \mathcal{O}_c(G, \mathfrak{h})_0$, $(\widehat{M}_b)^* = E_b(M^*)$ in $\mathcal{O}_{\bar{c}}(G, \mathfrak{h}^*)_b$.*
(iii) *The functors E_b, E^b are exact.*

Proof. (i),(ii) are straightforward from the definitions. (iii) follows from (i),(ii), since the completion functors are exact. □

5.8. **Parabolic induction and restriction functors for rational Cherednik algebras.** Theorem 5.4 allows us to define analogs of parabolic restriction functors for rational Cherednik algebras.

Namely, let $b \in \mathfrak{h}$, and $G_b = G'$. Define a functor $\mathrm{Res}_b : \mathcal{O}_c(G, \mathfrak{h})_0 \to \mathcal{O}_c(G', \mathfrak{h}/\mathfrak{h}^{G'})_0$ by the formula

$$\mathrm{Res}_b(M) = (\zeta \circ E \circ I^{-1} \circ \theta_*)(\widehat{M_b}).$$

We can also define the parabolic induction functors in the opposite direction. Namely, let $N \in \mathcal{O}_c(G', \mathfrak{h}/\mathfrak{h}^{G'})_0$. Then we can define the object $\mathrm{Ind}_b(N) \in \mathcal{O}_c(G, \mathfrak{h})_0$ by the formula

$$\mathrm{Ind}_b(N) = (E^b \circ \theta_*^{-1} \circ I)(\widehat{\zeta^{-1}(N)_0}).$$

Proposition 5.11. (i) *The functors* Ind_b, Res_b *are exact.*

(ii) *One has* $\mathrm{Ind}_b(\mathrm{Res}_b(M)) = E^b(\widehat{M_b})$.

Proof. Part (i) follows from the fact that the functor E^b and the completion functor $\widehat{\ }_b$ are exact (see Proposition 5.10). Part (ii) is straightforward from the definition. □

Theorem 5.12. *The functor* Ind_b *is right adjoint to* Res_b.

Proof. We have

$$\begin{aligned} & \mathrm{Hom}(\mathrm{Res}_b(M), N) = \mathrm{Hom}((\zeta \circ E \circ I^{-1} \circ \theta_*)(\widehat{M_b}), N) = \mathrm{Hom}((E \circ I^{-1} \circ \theta_*)(\widehat{M_b}), \zeta^{-1}(N)) \\ = \; & \mathrm{Hom}((I^{-1} \circ \theta_*)(\widehat{M_b}), \widehat{\zeta^{-1}(N)_0}) = \mathrm{Hom}(\widehat{M_b}, (\theta_*^{-1} \circ I)(\widehat{\zeta^{-1}(N)_0})) \\ = \; & \mathrm{Hom}(M, (E^b \circ \theta_*^{-1} \circ I)(\widehat{\zeta^{-1}(N)_0})) = \mathrm{Hom}(M, \mathrm{Ind}_b(N)). \end{aligned}$$

At the end we used Proposition 5.8. □

Then we can obtain the following corollary easily.

Corollary 5.13. *The functor* Res_b *maps projective objects to projective ones, and the functor* Ind_b *maps injective objects to injective ones.*

We can also define functors $\mathrm{res}_\lambda : \mathcal{O}_c(G, \mathfrak{h})_0 \to \mathcal{O}_c(G', \mathfrak{h}/\mathfrak{h}^{G'})_0$ and $\mathrm{ind}_\lambda : \mathcal{O}_c(G', \mathfrak{h}/\mathfrak{h}^{G'})_0 \to \mathcal{O}_c(G, \mathfrak{h})_0$, attached to $\lambda \in \mathfrak{h}^{*G'}_{\mathrm{reg}}$, by

$$\mathrm{res}_\lambda := \dagger \circ \mathrm{Res}_\lambda \circ \dagger, \mathrm{ind}_\lambda := \dagger \circ \mathrm{Ind}_\lambda \circ \dagger,$$

where † is as in Subsection 5.3.

Corollary 5.14. *The functors* res_λ, ind_λ *are exact. The functor* ind_λ *is left adjoint to* res_λ. *The functor* ind_λ *maps projective objects to projective ones, and the functor* res_λ *injective objects to injective ones.*

Proof. Easy to see from the definition of the functors and the Theorem 5.12. □

We also have the following proposition, whose proof is straightforward.

Proposition 5.15. *We have*

$$\mathrm{ind}_\lambda(N) = (\mathcal{J} \circ \psi_\lambda^{-1})(N), \quad \textit{and} \quad \mathrm{res}_\lambda(M) = (\psi_\lambda \circ E_\lambda)(\widehat{M}),$$

where ψ_λ *is defined in Corollary 5.5.*

5.9. **Some evaluations of the parabolic induction and restriction functors.** For generic c, the category $\mathcal{O}_c(G,\mathfrak{h})$ is semisimple, and naturally equivalent to the category Rep$G$ of finite dimensional representations of G, via the functor $\tau \mapsto M_c(G,\mathfrak{h},\tau)$. (If G is a Coxeter group, the exact set of such c (which are called regular) is known from [GGOR] and [Gy]).

Proposition 5.16. (i) *Suppose that c is generic. Upon the above identification, the functors* Ind_b, ind_λ *and* Res_b, res_λ *go to the usual induction and restriction functors between categories* Rep$G$ *and* RepG'. *In other words, we have*

$$\mathrm{Res}_b(M_c(G,\mathfrak{h},\tau)) = \oplus_{\xi\in\widehat{G'}} n_{\tau\xi} M_c(G',\mathfrak{h}/\mathfrak{h}^{G'},\xi),$$

and

$$\mathrm{Ind}_b(M_c(G',\mathfrak{h}/\mathfrak{h}^{G'},\xi)) = \oplus_{\tau\in\widehat{G}} n_{\tau\xi} M_c(G,\mathfrak{h},\tau),$$

where $n_{\tau\xi}$ is the multiplicity of occurrence of ξ in $\tau|_{G'}$, and similarly for res_λ, ind_λ.

(ii) *The equations of (i) hold at the level of Grothendieck groups for all c.*

Proof. Part (i) is easy for $c = 0$, and is obtained for generic c by a deformation argument. Part (ii) is also obtained by deformation argument, taking into account that the functors Res_b and Ind_b are exact and flat with respect to c. □

Example 5.17. Suppose that $G' = 1$. Then $\mathrm{Res}_b(M)$ is the fiber of M at b, while $\mathrm{Ind}_b(\mathbb{C}) = P_{KZ}$, the object defined in [GGOR], which is projective and injective (see Remark 5.22). This shows that Proposition 5.16 (i) does not hold for special c, as P_{KZ} is not, in general, a direct sum of standard modules.

5.10. **Dependence of the functor Res_b on b.** Let $G' \subset G$ be a parabolic subgroup. In the construction of the functor Res_b, the point b can be made a variable which belongs to the open set $\mathfrak{h}^{G'}_{\mathrm{reg}}$.

Namely, let $\widehat{\mathfrak{h}^{G'}_{\mathrm{reg}}}$ be the formal neighborhood of the locally closed set $\mathfrak{h}^{G'}_{\mathrm{reg}}$ in $\mathfrak{h}$, and let $\pi : \widehat{\mathfrak{h}^{G'}_{\mathrm{reg}}} \to \mathfrak{h}/G$ be the natural map (note that this map is an étale covering of the image with the Galois group $N_G(G')/G'$, where $N_G(G')$ is the normalizer of G' in G). Let $\widehat{H_{1,c}}(G,\mathfrak{h})_{\mathfrak{h}^{G'}_{\mathrm{reg}}}$ be the pullback of the sheaf $H_{1,c,G,\mathfrak{h}}$ under π. We can regard it as a sheaf of algebras over $\mathfrak{h}^{G'}_{\mathrm{reg}}$. Similarly to Theorem 5.4 we have an isomorphism

$$\theta : \widehat{H_{1,c}}(G,\mathfrak{h})_{\mathfrak{h}^{G'}_{\mathrm{reg}}} \to Z(G,G',\widehat{H_{1,c}}(G',\mathfrak{h}/\mathfrak{h}^{G'})_0)\hat{\otimes}\mathcal{D}(\mathfrak{h}^{G'}_{\mathrm{reg}}),$$

where $\mathcal{D}(\mathfrak{h}^{G'}_{\mathrm{reg}})$ is the sheaf of differential operators on $\mathfrak{h}^{G'}_{\mathrm{reg}}$, and $\hat{\otimes}$ is an appropriate completion of the tensor product.

Thus, repeating the construction of Res_b, we can define the functor

$$\mathrm{Res} : \mathcal{O}_c(G,\mathfrak{h})_0 \to \mathcal{O}_c(G',\mathfrak{h}/\mathfrak{h}^{G'})_0 \boxtimes \mathrm{Loc}(\mathfrak{h}^{G'}_{\mathrm{reg}}),$$

where $\mathrm{Loc}(\mathfrak{h}^{G'}_{\mathrm{reg}})$ stands for the category of local systems (i.e. $\mathcal{O}$-coherent $\mathcal{D}$-modules) on $\mathfrak{h}^{G'}_{\mathrm{reg}}$. This functor has the property that Res_b is the fiber of Res at b. Namely, the functor Res is defined by the formula

$$\mathrm{Res}(M) = (E \circ I^{-1} \circ \theta_*)(\widehat{M}_{\mathfrak{h}^{G'}_{\mathrm{reg}}}),$$

where $\widehat{M}_{\mathfrak{h}^{G'}_{\mathrm{reg}}}$ is the restriction of the sheaf M on $\mathfrak{h}$ to the formal neighborhood of $\mathfrak{h}^{G'}_{\mathrm{reg}}$.

Remark 5.18. If G' is the trivial group, the functor Res is just the KZ functor from [GGOR], which we will discuss later. Thus, Res is a relative version of the KZ functor.

Remark 5.19. Note that the object $\mathrm{R}es(M)$ is naturally equivariant under the group $N_G(G')/G'$.

Thus, we see that the functor Res_b does not depend on b, up to an isomorphism. A similar statement is true for the functors Ind_b, res_λ, ind_λ.

Conjecture 5.20. For any $b \in \mathfrak{h}, \lambda \in \mathfrak{h}^*$ such that $G_b = G_\lambda$, we have isomorphisms of functors $\mathrm{Res}_b \cong \mathrm{res}_\lambda$, $\mathrm{Ind}_b \cong \mathrm{ind}_\lambda$.

Remark 5.21. Conjecture 5.20 would imply that Ind_b is left adjoint to Res_b, and that Res_b maps injective objects to injective ones, while Ind_b maps projective objects to projective ones.

Remark 5.22. If b and λ are generic (i.e., $G_b = G_\lambda = 1$) then the conjecture holds. Indeed, in this case the conjecture reduces to showing that we have an isomorphism of functors $\mathrm{F}iber_b(M) \cong \mathrm{F}iber_\lambda(M^\dagger)^*$ ($M \in \mathcal{O}_c(G, \mathfrak{h})$). Since both functors are exact functors to the category of vector spaces, it suffices to check that $\dim \mathrm{F}iber_b(M) = \dim \mathrm{F}iber_\lambda(M^\dagger)$. But this is true because both dimensions are given by the leading coefficient of the Hilbert polynomial of M (characterizing the growth of M).

It is important to mention, however, that although Res_b is isomorphic to $\mathrm{Res}_{b'}$ if $G_b = G_{b'}$, this isomorphism is not canonical. So let us examine the dependence of Res_b on b a little more carefully.

Theorem 5.16 implies that if c is generic, then

$$\mathrm{Res}(M_c(G, \mathfrak{h}, \tau)) = \oplus_\xi M_c(G', \mathfrak{h}/\mathfrak{h}^{G'}, \xi) \otimes \mathcal{L}_{\tau\xi},$$

where $\mathcal{L}_{\tau\xi}$ is a local system on $\mathfrak{h}^{G'}_{\mathrm{reg}}$ of rank $n_{\tau\xi}$. Let us characterize the local system $\mathcal{L}_{\tau\xi}$ explicitly.

Proposition 5.23. *The local system $\mathcal{L}_{\tau\xi}$ is given by the connection on the trivial bundle given by the formula*

$$\nabla = \mathrm{d} - \sum_{s \in \mathcal{S}: s \notin G'} \frac{2c_s}{1 - \lambda_s} \frac{\mathrm{d}\alpha_s}{\alpha_s}(1 - s).$$

with values in $\mathrm{Hom}_{G'}(\xi, \tau|_{G'})$.

Proof. This follows immediately from formula (5.1). □

Definition 5.24. We will call the connection of Proposition 5.23 the parabolic KZ (Knizhnik-Zamolodchikov) connection.

Example 5.25. Let $G = \mathfrak{S}_n$ and $G' = \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k}$ with $n_1 + \cdots + n_k = n$. In this case, there is only one parameter c.

Let $\mathfrak{h} = \mathbb{C}^n$ be the permutation representation of G. Then

$$\mathfrak{h}^{G'} = (\mathbb{C}^n)^{G'} = \{v \in \mathfrak{h} | v = (\underbrace{z_1, \ldots, z_1}_{n_1}, \underbrace{z_2, \ldots, z_2}_{n_2}, \ldots, \underbrace{z_k, \ldots, z_k}_{n_k})\}.$$

Thus, the parabolic KZ connection on the trivial bundle with fiber being a representation τ of $\mathfrak{S}_n$ has the form

$$\mathrm{d} - c \sum_{1\le p<q\le k} \sum_{i=n_1+\cdots+n_{p-1}+1}^{n_1+\cdots+n_p} \sum_{j=n_1+\cdots+n_{q-1}+1}^{n_1+\cdots+n_q} \frac{\mathrm{d}z_p - \mathrm{d}z_q}{z_p - z_q}(1 - s_{ij}).$$

So the differential equations for a flat section $F(z)$ of this bundle have the form

$$\frac{\partial F}{\partial z_p} = c\sum_{q\neq p} \sum_{i=n_1+\cdots+n_{p-1}+1}^{n_1+\cdots+n_p} \sum_{j=n_1+\cdots+n_{q-1}+1}^{n_1+\cdots+n_q} \frac{(1-s_{ij})F}{z_p - z_q}.$$

So $F(z) = G(z)\prod_{p<q}(z_p - z_q)^{cn_pn_q}$, where the function G satisfies the differential equation

$$\frac{\partial G}{\partial z_p} = -c\sum_{q\neq p} \sum_{i=n_1+\cdots+n_{p-1}+1}^{n_1+\cdots+n_p} \sum_{j=n_1+\cdots+n_{q-1}+1}^{n_1+\cdots+n_q} \frac{s_{ij}G}{z_p - z_q}.$$

Let $\tau = V^{\otimes n}$ where V is a finite dimensional space with $\dim V = N$ (this class of representations contains as summands all irreducible representations of $\mathfrak{S}_n$). Let $V_p = V^{\otimes n_p}$, so that $\tau = V_1 \otimes \cdots \otimes V_k$. Then the equation for G can be written as

$$\frac{\partial G}{\partial z_p} = -c\sum_{q\neq p}\frac{\Omega_{pq}G}{z_p - z_q}, \quad p = 1, \ldots, k,$$

where $\Omega = \sum_{s,t=1}^{N} E_{s,t} \otimes E_{t,s}$ is the Casimir element for $\mathfrak{gl}_N$ ($E_{i,j}$ is the N by N matrix with the only 1 at the (i,j)-th place, and the rest of the entries being 0).

This is nothing but the well known Knizhnik-Zamolodchikov system of equations of the WZW conformal field theory, for the Lie algebra $\mathfrak{gl}_N$, see [EFK]. (Note that the representations V_i are "the most general" in the sense that any irreducible finite dimensional representation of $\mathfrak{gl}_N$ occurs in $V^{\otimes r}$ for some r, up to tensoring with a character.)

This motivates the term "parabolic KZ connection".

5.11. **Supports of modules.** The following two basic propositions are proved in [Gi1], Section 6. We will give different proofs of them, based on the restriction functors.

Proposition 5.26. *Consider the stratification of $\mathfrak{h}$ with respect to stabilizers of points in G. Then the (set-theoretical) support* $\mathrm{Supp}M$ *of any object M of $\mathcal{O}_c(G,\mathfrak{h})$ in $\mathfrak{h}$ is a union of strata of this stratification.*

Proof. This follows immediately from the existence of the flat connection along the set of points b with a fixed stabilizer G' on the bundle $\mathrm{Res}_b(M)$. □

Proposition 5.27. *For any irreducible object M in $\mathcal{O}_c(G,\mathfrak{h})$, $\mathrm{Supp}M/G$ is an irreducible algebraic variety.*

Proof. Let X be a component of $\mathrm{Supp}M/G$. Let M' be the subspace of elements of M whose restriction to a neighborhood of a generic point of X is zero. It is obvious that M' is an $H_{1,c}(G,\mathfrak{h})$-submodule in M. By definition, it is a proper submodule. Therefore, by the irreducibility of M, we have $M' = 0$. Now let $f \in \mathbb{C}[\mathfrak{h}]^G$ be a function that vanishes on X. Then there exists a positive integer N such that f^N maps M to M', hence acts by zero on M. This implies that $\mathrm{Supp}M/G = X$, as desired. □

Propositions 5.26 and 5.27 allow us to attach to every irreducible module $M \in \mathcal{O}_c(G, \mathfrak{h})$, a conjugacy class of parabolic subgroups, $C_M \in \mathrm{Par}(G)$, namely, the conjugacy class of the stabilizer of a generic point of the support of M. Also, for a parabolic subgroup $G' \subset G$, denote by $\mathcal{X}(G')$ the set of points $b \in \mathfrak{h}$ whose stabilizer contains a subgroup conjugate to G'.

The following proposition is immediate.

Proposition 5.28. (i) *Let $M \in \mathcal{O}_c(G, \mathfrak{h})_0$ be irreducible. If b is such that $G_b \in C_M$, then* $\mathrm{Res}_b(M)$ *is a nonzero finite dimensional module over $H_{1,c}(G_b, \mathfrak{h}/\mathfrak{h}^{G_b})$.*

(ii) *Conversely, let $b \in \mathfrak{h}$, and L be a finite dimensional module $H_{1,c}(G_b, \mathfrak{h}/\mathfrak{h}^{G_b})$. Then the support of* $\mathrm{Ind}_b(L)$ *in $\mathfrak{h}$ is $\mathcal{X}(G_b)$.*

Let $\mathrm{FD}(G, \mathfrak{h})$ be the set of c for which $H_{1,c}(G, \mathfrak{h})$ admits a finite dimensional representation.

Corollary 5.29. *Let G' be a parabolic subgroup of G. Then $\mathcal{X}(G')$ is the support of some irreducible representation from $\mathcal{O}_c(G, \mathfrak{h})_0$ if and only if $c \in \mathrm{FD}(G', \mathfrak{h}/\mathfrak{h}^{G'})$.*

Proof. Immediate from Proposition 5.28. □

Example 5.30. Let $G = \mathfrak{S}_n$, $\mathfrak{h} = \mathbb{C}^{n-1}$. In this case, the set $\mathrm{Par}(G)$ is the set of partitions of n. Assume that $c = r/m$, $(r, m) = 1$, $2 \le m \le n$. By a result of [BEG], finite dimensional representations of $H_c(G, \mathfrak{h})$ exist if and only if $m = n$. Thus the only possible classes C_M for irreducible modules M have stabilizers $\mathfrak{S}_m \times \cdots \times \mathfrak{S}_m$, i.e., correspond to partitions into parts, where each part is equal to m or 1. So there are $[n/m] + 1$ possible supports for modules, where $[a]$ denotes the integer part of a.

5.12. **Notes.** Our discussion of the geometric approach to rational Cherednik algebras in Section 5.1 follows [E1] and Section 2.2 of [BE]. Our exposition in the other sections follows the corresponding parts of the paper [BE].

6. The Knizhnik-Zamolodchikov functor

6.1. **Braid groups and Hecke algebras.** Let G be a complex reflection group and let $\mathfrak{h}$ be its reflection representation. For any reflection hyperplane $H \subset \mathfrak{h}$, its pointwise stabilizer is a cyclic group of order m_H. Fix a collection of nonzero constants $q_{1,H}, \ldots, q_{m_H-1,H}$ which are G-invariant, namely, if H and H' are conjugate to each other under some element in G, then $q_{i,H} = q_{i,H'}$ for $i = 1, \ldots, m_H - 1$.

Let $B_G = \pi_1(\mathfrak{h}_{\text{reg}}/G, \mathbf{x}_0)$ be the braid group of G, and $T_H \in B_G$ be a representative of the conjugacy class defined by a small circle around the image of H in $\mathfrak{h}/G$ oriented in the counterclockwise direction.

The following theorem follows from elementary algebraic topology.

Proposition 6.1. *The group G is the quotient of the braid group B_G by the relations*

$$T_H^{m_H} = 1$$

for all reflection hyperplanes H.

Proof. See, e.g., [BMR] Proposition 2.17. □

Definition 6.2. The Hecke algebra of G is defined to be

$$\mathcal{H}_q(G) = \mathbb{C}[B_G]/\langle (T_H - 1) \prod_{j=1}^{m_H-1} (T_H - \exp(2\pi \mathbf{i} j/m_H) q_{j,H}), \text{ for all } H \rangle.$$

Thus, by Proposition 6.1 we have an isomorphism

$$\mathcal{H}_1(G) \cong \mathbb{C}G.$$

So $\mathcal{H}_q(G)$ is a deformation of $\mathbb{C}G$.

Example 6.3 (Coxeter group case)**.** Now let W be a Coxeter group. Let $\mathcal{S}$ be the set of reflections and let $\alpha_s = 0$ be the reflection hyperplane corresponding to $s \in \mathcal{S}$. The Hecke algebra $\mathcal{H}_q(W)$ is the quotient of $\mathbb{C}[B_W]$ by the relations

$$(T_s - 1)(T_s + q_s) = 0,$$

for all reflections s where T_s is a small counterclockwise circle around the image of the hyperplane $\alpha_s = 0$ in $\mathfrak{h}/W$.

6.2. **KZ functors.** For a complex reflection group G, let $\text{Loc}(\mathfrak{h}_{\text{reg}})$ be the category of local systems (i.e., $\mathcal{O}$-coherent $\mathcal{D}$-modules) on $\mathfrak{h}_{\text{reg}}$, and let $\text{Loc}(\mathfrak{h}_{\text{reg}})^G$ be the category of G-equivariant local systems on $\mathfrak{h}_{\text{reg}}$, i.e. of local systems on $\mathfrak{h}_{\text{reg}}/G$.

Suppose that $G' = 1$ is the trivial subgroup in G. Then the restriction functor defined in Section 5.10 defines a functor $\text{Res} : \mathcal{O}_c(G, \mathfrak{h})_0 \to \text{Loc}(\mathfrak{h}_{\text{reg}}/G)$. Also, we have the monodromy functor $\text{M}on : \text{Loc}(\mathfrak{h}_{\text{reg}}/G) \cong \text{Rep}(B_G)$. The composition of these two functors is a functor from $\mathcal{O}_c(G, \mathfrak{h})_0$ to $\text{Rep}(B_G)$, which is exactly the KZ functor defined in [GGOR]. We will denote this functor by KZ.

Theorem 6.4 (Ginzburg, Guay, Opdam, Rouquier, [GGOR])**.** *The KZ functor factors through*
$\text{Rep}\mathcal{H}_q(G)$*, where*

$$q_{j,H} = \exp(2\pi \mathbf{i} b_{j,H}/m_H), \quad \textit{and} \quad b_{j,H} = 2 \sum_{\ell=1}^{m_H-1} \frac{c_{s_H^\ell}(1 - \mathbf{e}^{2\pi \mathbf{i} j\ell/m_H})}{1 - \mathbf{e}^{-2\pi \mathbf{i}\ell/m_H}}.$$

Proof. Assume first that c is generic. Then the category $\mathcal{O}_c(G,\mathfrak{h})_0$ is semisimple, with simple objects $M_c(\tau)$, so it is enough to check the statement on $M_c(\tau)$. Consider the trivial bundle over $\mathfrak{h}_{\text{reg}}$ with fiber τ. The KZ connection on it has the form

$$\mathrm{d} - \sum_{s\in\mathcal{S}} \frac{2c_s}{1-\lambda_s}\frac{\mathrm{d}\alpha_s}{\alpha_s}(1-s).$$

The residue of the connection form of this connection on the hyperplane H on the j-th irreducible representation of $\mathbb{Z}/m_H\mathbb{Z}$ is

$$2\sum_{\ell=1}^{m_H-1} \frac{c_{s_H^\ell}}{1-\mathbf{e}^{-2\pi\mathbf{i}\ell/m_H}}(1-\mathbf{e}^{2\pi\mathbf{i}j\ell/m_H}).$$

Therefore, the monodromy operator around this hyperplane is diagonalizable, and the eigenvalues of this operator are 1 and $\exp(2\pi\mathbf{i}j/m_H)q_{j,H}$, as desired.

For special c, introduce the generalized Verma module

$$M_{c,n}(\tau) = H_c(G,\mathfrak{h}) \otimes_{\mathbb{C}G\ltimes S\mathfrak{h}} (\tau\otimes S\mathfrak{h}/\mathfrak{m}^{n+1}),$$

where $\mathfrak{m}\subset S\mathfrak{h}$ is the maximal ideal of 0, $n\geq 0$. Clearly, $M_{c,0}=M_c(\tau)$. Moreover, $M_{c,n}\in\mathcal{O}_c(G,\mathfrak{h})_0$ for any n, since it has a finite filtration whose successive quotients are Verma modules.

Theorem 6.5. *For large enough n, $M_{c,n}(\mathbb{C}G)$ contains a direct summand which is a projective generator of $\mathcal{O}_c(G,\mathfrak{h})_0$.*

Proof. From the definition, $M_{c,n} = S\mathfrak{h}^*\otimes\mathbb{C}G\otimes S\mathfrak{h}/\mathfrak{m}^{n+1}$. Let ∂ be the degree operator on $M_{c,n}(\mathbb{C}G)$ with $\deg\mathfrak{h}^*=1, \deg\mathfrak{h}=-1$, and $\deg G = 0$, i.e., we have

$$[\partial, x] = x, [\partial, y] = -y, \text{ where } x\in\mathfrak{h}^*, y\in\mathfrak{h}.$$

So $\mathbf{h}-\partial$ is a module endomorphism of $M_{c,n}(\mathbb{C}G)$ where $\mathbf{h}$ is the operator defined in (3.2). Moreover, $\mathbf{h}-\partial$ acts locally finitely. In particular, we have a decomposition of $M_{c,n}(\mathbb{C}G)$ into generalized eigenspaces of $\mathbf{h}-\partial$:

$$M_{c,n}(\mathbb{C}G) = \bigoplus_{\beta\in\mathbb{C}} M_{c,n}^\beta(\mathbb{C}G).$$

We have

$$\operatorname{Hom}(M_{c,n}(\mathbb{C}G), N) = \{\text{vectors in } N \text{ which are killed by } \mathfrak{m}^{n+1}\},$$

and

$$\operatorname{Hom}(M_{c,n}^\beta(\mathbb{C}G), N) = \{\text{vectors in } N \text{ which are killed by } \mathfrak{m}^{n+1} \text{ and are generalized eigenvectors of } \mathbf{h} \text{ with generalized eigenvalue } \beta\}.$$

Let $\Sigma = \{h_c(\tau) | \tau \text{ is a irreducible representation of } G\}$ (recall that $h_c(\tau) = \frac{\dim\mathfrak{h}}{2} - \sum_{s\in\mathcal{S}}\frac{2c_s}{1-\lambda_s}s|_\tau$), and let

$$M_{c,n}^\Sigma(\mathbb{C}G) = \bigoplus_{\beta\in\Sigma} M_{c,n}^\beta(\mathbb{C}G).$$

Claim: for large n, $M_{c,n}^\Sigma(\mathbb{C}G)$ is a projective generator of $\mathcal{O}_c(G,\mathfrak{h})_0$.

Proof of the claim. First, for any β, there exists n such that $M^{\Sigma}_{c,n}(\mathbb{C}G)$ is projective (since the condition of being killed by $\mathfrak{m}^{n+1}$ is vacuous for large n).

Secondly, consider the functor $\mathrm{Hom}(M^{\Sigma}_{c,n}(\mathbb{C}G), \bullet)$. For any module $N \in \mathcal{O}_c(G,\mathfrak{h})_0$, if $\mathrm{Hom}(M^{\Sigma}_{c,n}(\mathbb{C}G), N) = 0$, then $\oplus_{\beta\in\Sigma} N[\beta] = 0$. So $N = 0$. Thus this functor does not kill nonzero objects, and so $M^{\Sigma}_{c,n}(\mathbb{C}G)$ is a generator. □

Theorem 6.5 is proved. □

Corollary 6.6. (i) *$\mathcal{O}_c(G,\mathfrak{h})_0$ has enough projectives, so it is equivalent to the category of modules over a finite dimensional algebra.*

(ii) *Any object of $\mathcal{O}_c(G,\mathfrak{h})_0$ is a quotient of a multiple of $M_{c,n}(\mathbb{C}G)$ for large enough n.*

Proof. Directly from the definition and the above theorem. □

Now we can finish the proof of Theorem 6.4. We have shown that for generic c, $\mathrm{KZ}(M_{c,n}(\mathbb{C}G)) \in \mathrm{Rep}\mathcal{H}_q(G)$. Hence this is true for any c, since $M_{c,n}(\mathbb{C}G)$ is a flat family of modules over $H_c(G,\mathfrak{h})$. Then, $\mathrm{KZ}(M)$ is a $\mathcal{H}_q(G)$-module for all M, since any M is a quotient of $M_{c,n}(\mathbb{C}G)$ and the functor KZ is exact. □

Corollary 6.7 (Broué, Malle, Rouquier, [BMR])**.** *Let $q_{j,H} = \exp(t_{j,H})$ where $t_{j,H}$'s are formal parameters. Then $\mathcal{H}_q(G)$ is a free module over $\mathbb{C}[[t_{j,H}]]$ of rank $|G|$.*

Proof. We have
$$\mathcal{H}_q(G)/(t) = \mathcal{H}_1(G) = \mathbb{C}G.$$
So it remains to show that $\mathcal{H}_q(G)$ is free. To show this, it is sufficient to show that any $\tau \in \mathrm{Irrep}G$ admits a flat deformation τ_q to a representation of $\mathcal{H}_q(G)$. We can define this deformation by letting $\tau_q = \mathrm{KZ}(M_c(\tau))$. □

Remark 6.8. 1. The validity of this Corollary in characteristic zero implies that it is also valid over a field positive characteristic.

2. It is not known in general if the Corollary holds for numerical q (even generically). This is a conjecture of Broué, Malle, and Rouquier. But it is known for many cases (including all Coxeter groups).

3. The proof of the Corollary is analytic (it is based on the notion of monodromy). There is no known algebraic proof, except in special cases, and in the case of Coxeter groups, which we'll discuss later.

6.3. **The image of the KZ functor.** First, let us recall the definition of a quotient category. Let $\mathcal{A}$ be an abelian category and $\mathcal{B} \subset \mathcal{A}$ a full abelian subcategory.

Definition 6.9. $\mathcal{B}$ is a Serre subcategory if it is closed under subquotients and extensions (i.e., if two terms in a short exact sequence are in $\mathcal{B}$, so is the third one).

If $\mathcal{B} \subset \mathcal{A}$ is a Serre subcategory, one can define a category $\mathcal{A}/\mathcal{B}$ as follows:
$$\text{objects in } \mathcal{A}/\mathcal{B} = \text{ objects in } \mathcal{A},$$
$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(X,Y) = \varinjlim_{Y', X/X' \in \mathcal{B}} \mathrm{Hom}_{\mathcal{A}}(X', Y/Y').$$

The category $\mathcal{A}/\mathcal{B}$ is an abelian category with the following universal property: any exact functor $F : \mathcal{A} \to \mathcal{C}$ that kills $\mathcal{B}$ must factor through $\mathcal{A}/\mathcal{B}$.

Now let $\mathcal{O}_c(G,\mathfrak{h})_0^{\rm tor}$ be the full subcategory of $\mathcal{O}_c(G,\mathfrak{h})_0$ consisting of modules supported on the reflection hyperplanes. It is a Serre subcategory, and $\ker(\mathrm{KZ}) = \mathcal{O}_c(G,\mathfrak{h})_0^{\rm tor}$. Thus we have a functor:

$$\overline{\mathrm{KZ}} : \mathcal{O}_c(G,\mathfrak{h})_0/\mathcal{O}_c(G,\mathfrak{h})_0^{\rm tor} \to \mathrm{Rep}\mathcal{H}_q(G).$$

Theorem 6.10 (Ginzburg, Guay, Opdam, Rouquier, [GGOR])**.** *If* $\dim \mathcal{H}_q(G) = |G|$*, the functor* $\overline{\mathrm{KZ}}$ *is an equivalence of categories.*

Proof. See [GGOR], Theorem 5.14. □

6.4. **Example: the symmetric group $\mathfrak{S}_n$.** Let $\mathfrak{h} = \mathbb{C}^n$, $G = \mathfrak{S}_n$. Then we have (for $q \in \mathbb{C}^*$):

$$\mathcal{H}_q(\mathfrak{S}_n) = \langle T_1, \dots, T_{n-1}\rangle / \langle \text{the braid relations and } (T_i - 1)(T_i + q) = 0\rangle.$$

The following facts are known:

(1) $\dim \mathcal{H}_q(\mathfrak{S}_n) = n!$;
(2) $\mathcal{H}_q(\mathfrak{S}_n)$ is semisimple if and only if $\mathrm{ord}(q) \neq 2, 3, \dots, n$.

Now suppose q is generic. Let λ be a partition of n. Then we can define an $\mathcal{H}_q(\mathfrak{S}_n)$-module S_λ, the Specht module for the Hecke algebra in the sense of [DJ]. This is a certain deformation of the classical irreducible Specht module for the symmetric group. The Specht module carries an inner product $\langle\cdot,\cdot\rangle$. Denote $D_\lambda = S_\lambda/\mathrm{Rad}\langle\cdot,\cdot\rangle$.

Theorem 6.11 (Dipper, James, [DJ])**.** *D_λ is either an irreducible $\mathcal{H}_q(\mathfrak{S}_n)$-module, or* 0*. $D_\lambda \neq 0$ if and only if λ is e-regular where* $e = \mathrm{ord}(q)$ *(i.e., every part of λ occurs less than e times).*

Proof. See [DJ], Theorem 6.3, 6.8. □

Now let $M_c(\lambda)$ be the Verma module associated to the Specht module for $\mathfrak{S}_n$ and $L_c(\lambda)$ be its irreducible quotient. Then we have the following theorem.

Theorem 6.12. *If $c \le 0$, then* $\mathrm{KZ}(M_c(\lambda)) = S_\lambda$ *and* $\mathrm{KZ}(L_c(\lambda)) = D_\lambda$.

Proof. See Section 6.2 of [GGOR]. □

Corollary 6.13. *If $c \le 0$, then* $\mathrm{Supp}L_c(\lambda) = \mathbb{C}^n$ *if and only if λ is e-regular. If $c > 0$, then* $\mathrm{Supp}L_c(\lambda) = \mathbb{C}^n$ *if and only if $\lambda^\vee$ is e-regular, or equivalently, λ is e-restricted (i.e., $\lambda_i - \lambda_{i+1} < e$ for $i = 1, \dots, n-1$).*

Proof. Directly from the definition and the above theorem. □

6.5. **Notes.** The references for this section are [GGOR], [BMR].

7. Rational Cherednik algebras and Hecke algebras for varieties with group actions

7.1. **Twisted differential operators.** Let us recall the theory of twisted differential operators (see [BB], section 2).

Let X be a smooth affine algebraic variety over $\mathbb{C}$. Given a closed 2-form ω on X, the algebra $\mathcal{D}_\omega(X)$ of differential operators on X twisted by ω can be defined as the algebra generated by $\mathcal{O}_X$ and "Lie derivatives" $\mathbf{L}_v$, $v \in \mathrm{Vect}(X)$, with defining relations

$$f\mathbf{L}_v = \mathbf{L}_{fv},\ [\mathbf{L}_v, f] = L_v f,\ [\mathbf{L}_v, \mathbf{L}_w] = \mathbf{L}_{[v,w]} + \omega(v, w).$$

This algebra depends only on the cohomology class $[\omega]$ of ω, and equals the algebra $\mathcal{D}(X)$ of usual differential operators on X if $[\omega] = 0$.

An important special case of twisted differential operators is the algebra of differential operators on a line bundle. Namely, let L be a line bundle on X. Since X is affine, L admits an algebraic connection ∇ with curvature ω, which is a closed 2-form on X. Then it is easy to show that the algebra $\mathcal{D}(X, L)$ of differential operators on L is isomorphic to $\mathcal{D}_\omega(X)$.

If the variety X is smooth but not necessarily affine, then (sheaves of) algebras of twisted differential operators are classified by the space $\mathrm{H}^2(X, \Omega_X^{\geq 1})$, where $\Omega_X^{\geq 1}$ is the two-step complex of sheaves $\Omega_X^1 \to \Omega_X^{2,\mathrm{cl}}$, given by the De Rham differential acting from 1-forms to closed 2-forms (sitting in degrees 1 and 2, respectively). If X is projective then this space is isomorphic to $\mathrm{H}^{2,0}(X, \mathbb{C}) \oplus \mathrm{H}^{1,1}(X, \mathbb{C})$. We refer the reader to [BB], Section 2, for details.

Remark 7.1. One can show that $\mathcal{D}_\omega(X)$ is the universal deformation of $\mathcal{D}(X)$ (see [E1]).

7.2. **Some algebraic geometry preliminaries.** Let Z be a smooth hypersurface in a smooth affine variety X. Let $i : Z \to X$ be the corresponding closed embedding. Let N denote the normal bundle of Z in X (a line bundle). Let $\mathcal{O}_X(Z)$ denote the module of regular functions on $X \setminus Z$ which have a pole of at most first order at Z. Then we have a natural map of $\mathcal{O}_X$-modules $\phi : \mathcal{O}_X(Z) \to i_*N$. Indeed, we have a natural residue map $\eta : \mathcal{O}_X(Z) \otimes_{\mathcal{O}_X} \Omega_X^1 \to i_*\mathcal{O}_Z$ (where Ω_X^1 is the module of 1-forms), hence a map $\eta' : \mathcal{O}_X(Z) \to i_*\mathcal{O}_Z \otimes_{\mathcal{O}_X} TX = i_*(TX|_Z)$ (where TX is the tangent bundle). The map ϕ is obtained by composing η' with the natural projection $TX|_Z \to N$.

We have an exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(Z) \xrightarrow{\phi} i_*N \to 0$$

Thus we have a natural surjective map of $\mathcal{O}_X$-modules $\xi_Z : TX \to \mathcal{O}_X(Z)/\mathcal{O}_X$.

7.3. **The Cherednik algebra of a variety with a finite group action.** We will now generalize the definition of $H_{t,c}(G, \mathfrak{h})$ to the global case. Let X be an affine algebraic variety over $\mathbb{C}$, and G be a finite group of automorphisms of X. Let E be a G-invariant subspace of the space of closed 2-forms on X, which projects isomorphically to $\mathrm{H}^2(X, \mathbb{C})$. Consider the algebra $G \ltimes \mathcal{O}_{T^*X}$, where T^*X is the cotangent bundle of X. We are going to define a deformation $H_{t,c,\omega}(G, X)$ of this algebra parametrized by

1. complex numbers t,
2. G-invariant functions c on the (finite) set $\mathcal{S}$ of pairs $s = (Y, g)$, where $g \in G$, and Y is a connected component of the set of fixed points X^g such that $codim Y = 1$, and
3. elements $\omega \in E^G = \mathrm{H}^2(X, \mathbb{C})^G$.

If all the parameters are zero, this algebra will conicide with $G \ltimes \mathcal{O}_{T^*X}$.

Let $t, c = \{c(Y,g)\}, \omega \in E^G$ be variables. Let $\mathcal{D}_{\omega/t}(X)_r$ be the algebra (over $\mathbb{C}[t, t^{-1}, \omega]$) of twisted (by ω/t) differential operators on X with rational coefficients.

Definition 7.2. A Dunkl-Opdam operator for (X, G) is an element of $\mathcal{D}_{\omega/t}(X)_r[c]$ given by the formula

$$D := t\mathbf{L}_v - \sum_{(Y,g)\in\mathcal{S}} \frac{2c(Y,g)}{1-\lambda_{Y,g}} \cdot f_Y(x) \cdot (1-g), \tag{7.1}$$

where $\lambda_{Y,g}$ is the eigenvalue of g on the conormal bundle to Y, $v \in \Gamma(X, TX)$ is a vector field on X, and $f_Y \in \mathcal{O}_X(Z)$ is an element of the coset $\xi_Y(v) \in \mathcal{O}_X(Z)/\mathcal{O}_X$ (recall that ξ_Y is defined in Subsection 7.2).

Definition 7.3. The algebra $H_{t,c,\omega}(X, G)$ is the subalgebra of $G \ltimes \mathcal{D}_{\omega/t}(X)_r[c]$ generated (over $\mathbb{C}[t, c, \omega]$) by the function algebra $\mathcal{O}_X$, the group G, and the Dunkl-Opdam operators.

By specializing t, c, ω to numerical values, we can define a family of algebras over $\mathbb{C}$, which we will also denote $H_{t,c,\omega}(G, X)$. Note that when we set $t = 0$, the term $t\mathbf{L}_v$ does not become 0 but turns into the classical momentum.

Definition 7.4. $H_{t,c,\omega}(G, X)$ is called *the Cherednik algebra of the orbifold X/G.*

Remark 7.5. One has $H_{1,0,\omega}(G, X) = G \ltimes \mathcal{D}_\omega(X)$. Also, if $\lambda \neq 0$ then $H_{\lambda t,\lambda c,\lambda\omega}(G, X) = H_{t,c,\omega}(G, X)$.

Example 7.6. $X = \mathfrak{h}$ is a vector space and G is a subgroup in $\mathrm{GL}(\mathfrak{h})$. Let v be a constant vector field, and let $f_Y(x) = (\alpha_Y, v)/\alpha_Y(x)$, where $\alpha_Y \in \mathfrak{h}^*$ is a nonzero functional vanishing on Y. Then the operator D is just the usual Dunkl-Opdam operator D_v in the complex reflection case (see Section 2.5). This implies that all the Dunkl-Opdam operators in the sense of Definition 7.2 have the form $\sum f_i D_{y_i} + a$, where $f_i \in \mathbb{C}[\mathfrak{h}]$, $a \in G \ltimes \mathbb{C}[\mathfrak{h}]$, and D_{y_i} are the usual Dunkl-Opdam operators (for some basis y_i of $\mathfrak{h}$). So the algebra $H_{t,c}(G, \mathfrak{h}) = H_{t,c,0}(G, X)$ is the rational Cherednik algebra for $(G, \mathfrak{h})$, see Section 3.1.

The algebra $H_{t,c,\omega}(G, X)$ has a filtration $F^\bullet$ which is defined on generators by $\deg(\mathcal{O}_X) = \deg(G) = 0$, $\deg(D) = 1$ for Dunkl-Opdam operators D.

Theorem 7.7 (the PBW theorem)**.** *We have*

$$\mathrm{gr}_F(H_{t,c,\omega}(G, X)) = G \ltimes \mathcal{O}(T^*X)[t, c, \omega].$$

Proof. Suppose first that $X = \mathfrak{h}$ is a vector space and G is a subgroup in $\mathrm{GL}(\mathfrak{h})$. Then, as we mentioned, $H_{t,c,\omega}(G, \mathfrak{h}) = H_{t,c}(G, \mathfrak{h})$ is the rational Cherednik algebra for $G, \mathfrak{h}$. So in this case the theorem is true.

Now consider arbitrary X. We have a homomorphism of graded algebras

$$\psi : \mathrm{gr}_F(H_{t,c,\omega}(G, X)) \to G \ltimes \mathcal{O}(T^*X)[t, c, \omega] \quad \text{(the principal symbol homomorphism).}$$

The homomorphism ψ is clearly surjective, and our job is to show that it is injective (this is the nontrivial part of the proof). In each degree, ψ is a morphism of finitely generated $\mathcal{O}_X^G$-modules. Therefore, to check its injectivity, it suffices to check the injectivity on the formal neighborhood of each point $z \in X/G$.

Let x be a preimage of z in X, and G_x be the stabilizer of x in G. Then G_x acts on the formal neighborhood U_x of x in X.

Lemma 7.8. *Any action of a finite group on a formal polydisk over $\mathbb{C}$ is linearizable.*

Proof. Let $\mathcal{D}$ be a formal polydisk over $\mathbb{C}$. Suppose we have an action of a finite group G on $\mathcal{D}$. Then we have a group homomorphism:

$$\rho : G \to \mathrm{Aut}(\mathcal{D}) = \mathrm{GL}_n(\mathbb{C}) \ltimes \mathrm{Aut}_U(\mathcal{D}),$$

where $\mathrm{Aut}_U(\mathcal{D})$ is the group of unipotent automorphisms of $\mathcal{D}$ (i.e. those whose derivative at the origin is 1), which is a prounipotent algebraic group.

Our job is to show that the image of G under ρ can be conjugated into $\mathrm{GL}_n(\mathbb{C})$. The obstruction to this is in the cohomology group $\mathrm{H}^1(G, \mathrm{Aut}_U(\mathcal{D}))$, which is trivial since G is finite and $\mathrm{Aut}_U(\mathcal{D})$ is prounipotent over $\mathbb{C}$. □

It follows from Lemma 7.8 that it suffices to prove the theorem in the linear case, which has been accomplished already. We are done. □

Remark 7.9. The following remark is meant to clarify the proof of Theorem 7.7. In the case $X = \mathfrak{h}$, the proof of Theorem 7.7 is based, essentially, on the (fairly nontrivial) fact that the usual Dunkl-Opdam operators D_v commute with each other. It is therefore very important to note that in contrast with the linear case, for a general X we do **not** have any natural commuting family of Dunkl-Opdam operators. Instead, the operators (7.1) satisfy a weaker property, which is still sufficient to validate the PBW theorem. This property says that if D_1, D_2, D_3 are Dunkl-Opdam operators corresponding to vector fields $v_1, v_2, v_3 := [v_1, v_2]$ and some choices of the functions f_Y, then $[D_1, D_2] - D_3 \in G \ltimes \mathcal{O}(X)$ (i.e., it has no poles). To prove this property, it is sufficient to consider the case when X is a formal polydisk, with a linear action of G. But in this case everything follows from the commutativity of the usual Dunkl operators D_v.

Example 7.10. (1) Suppose $G = 1$. Then for $t \neq 0$, $H_{t,0,\omega}(G, X) = \mathcal{D}_{\omega/t}(X)$.

(2) Suppose G is a Weyl group and $X = H$ the corresponding torus. Then $H_{1,c,0}(G, H)$ is called the *trigonometric Cherednik algebra.*

7.4. **Globalization.** Let X be any smooth algebraic variety, and $G \subset \mathrm{Aut}(X)$. Assume that X admits a cover by affine G-invariant open sets. Then the quotient variety X/G exists.

For any affine open set U in X/G, let U' be the preimage of U in X. Then we can define the algebra $H_{t,c,0}(G, U')$ as above. If $U \subset V$, we have an obvious restriction map $H_{t,c,0}(G, V') \to H_{t,c,0}(G, U')$. The gluing axiom is clearly satisfied. Thus the collection of algebras $H_{t,c,0}(G, U')$ can be extended (by sheafification) to a sheaf of algebras on X/G. We are going to denote this sheaf by $H_{t,c,0,G,X}$ and call it the sheaf of Cherednik algebras on X/G. Thus, $H_{t,c,0,G,X}(U) = H_{t,c,0}(G, U')$.

Similarly, if $\psi \in \mathrm{H}^2(X, \Omega_X^{\geq 1})^G$, we can define the sheaf of twisted Cherednik algebras $H_{t,c,\psi,G,X}$. This is done similarly to the case of twisted differential operators (which is the case $G = 1$).

Remark 7.11. (1) The construction of $H_{t,c,\omega}(G, X)$ and the PBW theorem extend in a straightforward manner to the case when the ground field is not $\mathbb{C}$ but an algebraically closed field k of positive characteristic, provided that the order of the group G is relatively prime to the characteristic.

(2) The construction and main properties of the (sheaves of) Cherednik algebras of algebraic varieties can be extended without significant changes to the case when X is a complex analytic manifold, and G is not necessarily finite but acts properly discontinuously. In the following lectures, we will often work in this generalized setting.

7.5. **Modified Cherednik algebra.** It will be convenient for us to use a slight modification of the sheaf $H_{t,c,\psi,G,X}$. Namely, let η be a function on the set of conjugacy classes of Y such that $(Y,g) \in \mathcal{S}$. We define $H_{t,c,\eta,\psi,G,X}$ in the same way as $H_{t,c,\psi,G,X}$ except that the Dunkl-Opdam operators are defined by the formula

$$D := t\mathbf{L}_v + \sum_{(Y,g)\in\mathcal{S}} f_Y(x)\frac{2c(Y,g)}{1-\lambda_{Y,g}}(g-1) + \sum_Y f_Y(x)\eta(Y). \tag{7.2}$$

The following result shows that this modification is in fact tautological. Let ψ_Y be the class in $\mathrm{H}^2(X, \Omega_X^{\geq 1})$ defined by the line bundle $\mathcal{O}_X(Y)^{-1}$, whose sections are functions vanishing on Y.

Proposition 7.12. *One has an isomorphism*

$$H_{t,c,\eta,\psi,G,X} \to H_{t,c,\psi+\sum_Y \eta(Y)\psi_Y,G,X}.$$

Proof. Let $y \in Y$ and z be a function on the formal neighborhood of y such that $z|_Y = 0$ and $\mathrm{d}z_y \neq 0$. Extend it to a system of local formal coordinates $z_1 = z, z_2, \ldots, z_d$ near y. A Dunkl-Opdam operator near y for the vector field $\frac{\partial}{\partial z}$ can be written in the form

$$D = \frac{\partial}{\partial z} + \frac{1}{z}\left(\sum_{m=1}^{n-1} \frac{2c(Y,g^m)}{1-\lambda_{Y,g}^m}(g^m-1) + \eta(Y)\right).$$

Conjugating this operator by the formal expression $z^{\eta(Y)} := (z^m)^{\eta(Y)/m}$, we get

$$z^{\eta(Y)} \circ D \circ z^{-\eta(Y)} = \frac{\partial}{\partial z} + \frac{1}{z}\sum_{m=1}^{n-1} \frac{2c(Y,g^m)}{1-\lambda_{Y,g}^m}(g^m-1)$$

This implies the required statement. □

We note that the sheaf $H_{1,c,\eta,0,G,X}$ localizes to $G \ltimes \mathcal{D}_X$ on the complement of all the hypersurfaces Y. This follows from the fact that the line bundle $\mathcal{O}_X(Y)$ is trivial on the complement of Y.

7.6. **Orbifold Hecke algebras.** Let X be a connected and simply connected complex manifold, and G is a discrete group of automorphisms of X which acts properly discontinuously. Then X/G is a complex orbifold. Let $X' \subset X$ be the set of points with trivial stabilizer. Fix a base point $x_0 \in X'$. Then the braid group of X/G is defined to be $B_G = \pi_1(X'/G, x_0)$. We have an exact sequence $1 \to K \to B_G \to G \to 1$.

Now let $\mathcal{S}$ be the set of pairs (Y,g) such that Y is a component of X^g of codimension 1 in X (such Y will be called a reflection hypersurface). For $(Y,g) \in \mathcal{S}$, let G_Y be the subgroup of G whose elements act trivially on Y. This group is obviously cyclic; let $n_Y = |G_Y|$. Let C_Y be the conjugacy class in B_G corresponding to a small circle going counterclockwise around the image of Y in X/G, and T_Y be a representative in C_Y.

The following theorem follows from elementary topology:

Theorem 7.13. *K is defined by relations $T_Y^{n_Y} = 1$, for all reflection hypersurfaces Y (i.e., K is the intersection of all normal subgroups of B_G containing $T_Y^{n_Y}$).*

Proof. See, e.g., [BMR] Proposition 2.17. □

For any conjugacy class of hypersurfaces Y such that $(Y, g) \in \mathcal{S}$ we introduce formal parameters $\tau_{1Y}, \ldots, \tau_{n_Y Y}$. The entire collection of these parameters will be denoted by τ. Let $A_0 = \mathbb{C}[G]$.

Definition 7.14. We define the Hecke algebra of (G, X), denoted $A = \mathcal{H}_\tau(G, X, x_0)$, to be the quotient of the group algebra of the braid group, $\mathbb{C}[B_G][[\tau]]$, by the relations

$$\prod_{j=1}^{n_Y}(T - \mathbf{e}^{2\pi j\mathbf{i}/n_Y}\mathbf{e}^{\tau_{jY}}) = 0,\ T \in C_Y \tag{7.3}$$

(i.e., by the closed ideal in the formal series topology generated by these relations).

Thus, A is a deformation of A_0.

It is clear that up to an isomorphism this algebra is independent on the choice of x_0, so we will sometimes drop x_0 form the notation.

The main result of this section is the following theorem.

Theorem 7.15. *Assume that $\mathrm{H}^2(X, \mathbb{C}) = 0$. Then $A = \mathcal{H}_\tau(G, X)$ is a* **flat** *formal deformation of A_0, which means $A = A_0[[\tau]]$ as a module over $\mathbb{C}[[\tau]]$.*

Example 7.16. Let $\mathfrak{h}$ be a finite dimensional vector space, and G be a complex reflection group in $\mathrm{GL}(\mathfrak{h})$. Then $\mathcal{H}_\tau(G, \mathfrak{h})$ is the Hecke algebra of G studied in [BMR]. It follows from Theorem 7.15 that this Hecke algebra is flat. This proof of flatness is in fact the same as the original proof of this result given in [BMR] (based on the Dunkl-Opdam-Cherednik operators, and explained above).

Example 7.17. Let $\mathfrak{h}$ be a universal covering of a maximal torus of a simply connected simple Lie group G, $Q^\vee$ be the dual root lattice, and $\widehat{G} = G \ltimes Q^\vee$ be its affine Weyl group. Then $\mathcal{H}_\tau(\mathfrak{h}, \widehat{G})$ is the affine Hecke algebra. This algebra is also flat by Theorem 7.15. In fact, its flatness is a well known result from representation theory; our proof of flatness is essentially due to Cherednik [Ch].

Example 7.18. Let $G, \mathfrak{h}, Q^\vee$ be as in the previous example, $\eta \in \mathbb{C}_+$ be a complex number with a positive imaginary part, and $\widehat{\widehat{G}} = G \ltimes (Q^\vee \oplus \eta Q^\vee)$ be the double affine Weyl group. Then $\mathcal{H}_\tau(\mathfrak{h}, \widehat{\widehat{G}})$ is (one of the versions of) the double affine Hecke algebra of Cherednik ([Ch]), and it is flat by Theorem 7.15. The fact that this algebra is flat was proved by Cherednik, Sahi, Noumi, Stokman (see [Ch],[Sa],[NoSt],[St]) using a different approach (q-deformed Dunkl operators).

7.7. **Hecke algebras attached to Fuchsian groups.** Let H be a simply connected complex Riemann surface (i.e., Riemann sphere, Euclidean plane, or Lobachevsky plane), and Γ be a cocompact lattice in $\mathrm{A}ut(H)$ (i.e., a Fuchsian group). Let $\Sigma = H/\Gamma$. Then Σ is a compact complex Riemann surface. When Γ contains elliptic elements (i.e., nontrivial elements of finite order), we are going to regard Σ as an orbifold: it has special points P_i, $i = 1, \ldots, m$ with stabilizers $\mathbb{Z}_{n_i}$. Then Γ is the orbifold fundamental group of Σ.[1]

Let g be the genus of Σ, and $a_l, b_l, l = 1, \ldots, g$, be the a-cycles and b-cycles of Σ. Let c_j be the counterclockwise loops around P_j. Then Γ is generated by a_l, b_l, c_j with relations

$$c_j^{n_j} = 1,\ c_1c_2\cdots c_m = \prod_l a_lb_la_l^{-1}b_l^{-1}.$$

For each j, introduce formal parameters τ_{kj}, $k = 1, \ldots, n_j$. Define the Hecke algebra $\mathcal{H}_\tau(\Sigma)$ of Σ to be generated over $\mathbb{C}[[\tau]]$ by the same generators a_l, b_l, c_j with defining relations

$$\prod_{k=1}^{n_j}(c_j - \mathbf{e}^{2\pi j\mathbf{i}/n_j}\mathbf{e}^{\tau_{kj}}) = 0,\ c_1c_2\cdots c_m = \prod_l a_lb_la_l^{-1}b_l^{-1}.$$

Thus $\mathcal{H}_\tau(\Sigma)$ is a deformation of $\mathbb{C}[\Gamma]$.

This deformation is flat if H is a Euclidean plane or a Lobachevsky plane. Indeed, $\mathcal{H}_\tau(\Sigma) = \mathcal{H}_\tau(\Gamma, H)$, so the result follows from Theorem 7.15 and the fact that $\mathrm{H}^2(H, \mathbb{C}) = 0$.

On the other hand, if H is the Riemann sphere (so that the condition $\mathrm{H}^2(H, \mathbb{C}) = 0$ is violated) and $\Gamma \neq 1$ then this deformation is not flat. Indeed, let $\tau = \tau(\hbar)$ be a 1-parameter subdeformation of $\mathcal{H}_\tau(\Sigma)$ which is flat. Let us compute the determinant of the product $c_1 \cdots c_m$ in the regular representation of this algebra (which is finite dimensional if H is the sphere). On the one hand, it is 1, as $c_1 \cdots c_m$ is a product of commutators. On the other hand, the eigenvalues of c_j in this representation are $\mathbf{e}^{2\pi j\mathbf{i}/n_j}\mathbf{e}^{\tau_{kj}}$ with multiplicity $|\Gamma|/n_j$. Computing determinants as products of eigenvalues, we get a nontrivial equation on $\tau_{kj}(\hbar)$, which means that the deformation $\mathcal{H}_\tau$ is not flat.

Thus, we see that $\mathcal{H}_\tau(\Sigma)$ fails to be flat in the following "forbidden" cases:

$$g = 0,\ m = 2, (n_1, n_2) = (n, n);$$

$$m = 3, (n_1, n_2, n_3) = (2, 2, n), (2, 3, 3), (2, 3, 4), (2, 3, 5).$$

Indeed, the orbifold Euler characteristic of a closed surface Σ of genus g with m special points $x_1, \ldots, x_m$ whose orders are $n_1, \ldots, n_m$ is

$$\chi^{\mathrm{orb}}(\Sigma, x_1, \ldots, x_m) = 2 - 2g - m + \sum_{i=1}^{m} \frac{1}{n_i},$$

and above solutions are the solutions of the inequality

$$\chi^{\mathrm{orb}}(\mathbb{C}P^1, x_1, \ldots, x_m) > 0.$$

(note that the solutions for $m = 1$ and solutions (n_1, n_2) with $n_1 \neq n_2$ don't arise, since they don't correspond to any orbifolds).

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[1]Let X be a connected topological space on with a properly discontinuous action of a discrete group G. Then the orbifold fundamental group of the orbifold X/G with base point $x \in X$, denoted $\pi_1^{orb}(X/G, x)$, is the set of pairs (g, γ), where $g \in G$ and γ is a homotopy class of paths leading from x to gx, with multiplication law $(g_1, \gamma_1)(g_2, \gamma_2) = (g_1g_2, \gamma)$, where γ is γ_1 followed by $g_1(\gamma_2)$. Obviously, in this situation we have an exact sequence

$$1 \to \pi_1(X, x) \to \pi_1^{orb}(X/G, x) \to G \to 1.$$

7.8. **Hecke algebras of wallpaper groups and del Pezzo surfaces.** The case when H is the Euclidean plane (i.e., Γ is a wallpaper group) deserves special attention. If there are elliptic elements, this reduces to the following configurations: $g = 0$ and

$$m = 3, (n_1, n_2, n_3) = (3, 3, 3), (2, 4, 4), (2, 3, 6) \text{ (cases } E_6,\ E_7,\ E_8),$$

or

$$m = 4, (n_1, n_2, n_3, n_4) = (2, 2, 2, 2) \text{ (case } D_4).$$

In these cases, the algebra $\mathcal{H}_\tau(\Gamma, H)$ (for numerical τ) has Gelfand-Kirillov dimension 2, so it can be interpreted in terms of the theory of noncommutative surfaces.

Recall that a del Pezzo surface (or a Fano surface) is a smooth projective surface, whose anticanonical line bundle is ample. It is known that such surfaces are $\mathbb{CP}^1 \times \mathbb{CP}^1$, or a blow-up of $\mathbb{CP}^2$ at up to 8 generic points. The degree of a del Pezzo surface X is by definition the self intersection number $K \cdot K$ of its canonical class K. For example, a del Pezzo surface of degree 3 is a cubic surface in $\mathbb{CP}^3$, and the degree of $\mathbb{CP}^2$ with n generic points blown up is $d = 9 - n$.

Now suppose τ is numerical. Let $\hbar = \sum_{j,k} n_j^{-1} \tau_{kj}$. Also let n be the largest of n_j, and c be the corresponding c_j. Let $\mathsf{e} \in \mathbb{C}[c] \subset \mathcal{H}_\tau(\Gamma, H)$ be the projector to an eigenspace of c. Consider the "spherical" subalgebra $B_\tau(\Gamma, H) := \mathsf{e}\mathcal{H}_\tau(\Gamma, H)\mathsf{e}$.

Theorem 7.19 (Etingof, Oblomkov, Rains, [EOR])**.** (i) *If* $\hbar = 0$ *then the algebra* $B_\tau(\Gamma, H)$ *is commutative, and its spectrum is an affine del Pezzo surface. More precisely, in the case* $(2, 2, 2, 2)$, *it is a del Pezzo surface of degree 3 (a cubic surface) with a triangle of lines removed; in the cases* $(3, 3, 3), (2, 4, 4), (2, 3, 6)$ *it is a del Pezzo surface of degrees 3,2,1 respectively with a nodal rational curve removed.*

(ii) *The algebra* $B_\tau(\Gamma, H)$ *for* $\hbar \neq 0$ *is a quantization of the unique algebraic symplectic structure on the surface from (i) with Planck's constant* $\hbar$.

Proof. See [EOR]. □

Remark 7.20. In the case $(2, 2, 2, 2)$, $\mathcal{H}_\tau(\Gamma, \Gamma)$ is the Cherednik-Sahi algebra of rank 1; it controls the theory of Askey-Wilson polynomials.

Example 7.21. This is a "multivariate" version of the Hecke algebras attached to Fuchsian groups, defined in the previous subsection. Namely, letting Γ, H be as in the previous subsection, and $N \geq 1$, we consider the manifold $X = H^N$ with the action of $\Gamma_N = \mathfrak{S}_N \ltimes \Gamma^N$. If H is a Euclidean or Lobachevsky plane, then by Theorem 7.15 $\mathcal{H}_\tau(\Gamma_N, X^N)$ is a flat deformation of the group algebra $\mathbb{C}[\Gamma_N]$. If $N > 1$, this algebra has one more essential parameter than for $N = 1$ (corresponding to reflections in $\mathfrak{S}_N$). In the Euclidean case, one expects that an appropriate "spherical" subalgebra of this algebra is a quantization of the Hilbert scheme of a del Pezzo surface.

7.9. **The Knizhnik-Zamolodchikov functor.** In this subsection we will define a global analog of the KZ functor defined in [GGOR]. This functor will be used as a tool of proof of Theorem 7.15.

Let X be a simply connected complex manifold, and G a discrete group of holomorphic transformations of X acting on X properly discontinuously. Let $X' \subset X$ be the set of points with trivial stabilizer. Then we can define the sheaf of Cherednik algebras $H_{1,c,\eta,0,G,X}$ on X/G. Note that the restriction of this sheaf to X'/G is the same as the restriction of the sheaf $G \ltimes \mathcal{D}_X$ to X'/G (i.e. on X'/G, the dependence of the sheaf on the parameters c and η disappears). This follows from the fact that the line bundles $\mathcal{O}_X(Y)$ become trivial when restricted to X'.

Now let M be a module over $H_{1,c,\eta,0,G,X}$ which is a locally free coherent sheaf when restricted to X'/G. Then the restriction of M to X'/G is a G-equivariant $\mathcal{D}$-module on X' which is coherent and locally free as an $\mathcal{O}$-module. Thus, M corresponds to a locally constant sheaf (local system) on X'/G, which gives rise to a monodromy representation of the braid group $\pi_1(X'/G, x_0)$ (where x_0 is a base point). This representation will be denoted by $\mathrm{KZ}(M)$. This defines a functor KZ, which is analogous to the one in [GGOR].

It follows from the theory of $\mathcal{D}$-modules that any $\mathcal{O}_{X/G}$-coherent $H_{1,c,\eta,0,G,X}$-module is locally free when restricted to X'/G. Thus the KZ functor acts from the abelian category $\mathcal{C}_{c,\eta}$ of $\mathcal{O}_{X/G}$-coherent $H_{1,c,\eta,0,G,X}$-modules to the category of representations of $\pi_1(X'/G, x_0)$. It is easy to see that this functor is exact.

For any Y, let g_Y be the generator of G_Y which has eigenvalue $\mathbf{e}^{2\pi \mathbf{i}/n_Y}$ in the conormal bundle to Y. Let $(c, \eta) \to \tau(c, \eta)$ be the invertible linear transformation defined by the formula

$$\tau_{jY} = 2\pi\mathbf{i}(2\sum_{m=1}^{n_Y-1} c(Y, g_Y^m)\frac{1-\mathbf{e}^{2\pi jm\mathbf{i}/n_Y}}{1-\mathbf{e}^{-2\pi m\mathbf{i}/n_Y}} - \eta(Y))/n_Y.$$

Proposition 7.22. *The functor* KZ *maps the category* $\mathcal{C}_{c,\eta}$ *to the category of representations of the algebra* $\mathcal{H}_{\tau(c,\eta)}(G, X)$.

Proof. The result follows from the corresponding result in the linear case (which we have already proved) by restricting M to the union of G-translates of a neighborhood of a generic point $y \in Y$, and then linearizing the action of G_Y on this neighborhood. □

7.10. **Proof of Theorem 7.15.** Consider the module $M = \mathrm{Ind}_{\mathcal{D}_X}^{G\ltimes\mathcal{D}_X}\mathcal{O}_X$. Then $\mathrm{KZ}(M)$ is the regular representation of G which is denoted by $\mathrm{reg}G$. We want to show that M deforms uniquely (up to an isomorphism) to a module over $H_{1,c,0,\eta,G,X}$ for formal c, η. The obstruction to this deformation is in $\mathrm{Ext}^2_{G\ltimes\mathcal{D}_X}(M, M)$ and the freedom of this deformation is in $\mathrm{Ext}^1_{G\ltimes\mathcal{D}_X}(M, M)$. Since

$$\mathrm{Ext}^i_{G\ltimes\mathcal{D}_X}(M, M) = \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{O}_X, \mathrm{Res}M) = \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X \otimes \mathbb{C}G)$$

$$= \mathrm{Ext}^i_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) \otimes \mathbb{C}G = \mathrm{H}^i(X, \mathbb{C}) \otimes \mathbb{C}G,$$

and X is simply connected, we have

$$\mathrm{Ext}^1_{G\ltimes\mathcal{D}_X}(M, M) = 0, \text{ and } \mathrm{Ext}^2_{G\ltimes\mathcal{D}_X}(M, M) = 0 \text{ if } \mathrm{H}^2(X, \mathbb{C}) = 0.$$

Thus such deformation exists and is unique if $\mathrm{H}^2(X, \mathbb{C}) = 0$.

Now let $M_{c,\eta}$ be the deformation. Then $\mathrm{KZ}(M_{c,\eta})$ is a $\mathcal{H}_{\tau(c,\eta)}(G, X)$-module from Proposition 7.22 and it deforms flatly the module $\mathrm{reg}G$. This implies $\mathcal{H}_{\tau(c,\eta)}(G, X)$ is flat over $\mathbb{C}[[\tau]]$.

Remark 7.23. When X is not simply connected, the theorem is still true under the assumption $\pi_2(X) \otimes \mathbb{C} = 0$ (i.e. $\mathrm{H}^2(\widetilde{X}, \mathbb{C}) = 0$, where $\widetilde{X}$ is the universal cover of X), and the proof is contained in [E1].

7.11. **Example: the simplest case of double affine Hecke algebras.** Now let $G = \mathbb{Z}_2 \ltimes \mathbb{Z}^2$ acting on $\mathbb{C}$. Then the conjugacy classes of reflection hyperplanes are four points: $0, 1/2, 1/2+\eta/2, \eta/2$, where we suppose the lattice in $\mathbb{C}$ is $\mathbb{Z} \oplus \mathbb{Z}\eta$. Correspondingly, the presentation of G is as follows:

$$\textbf{generators: } T_1, T_2, T_3, T_4; \qquad \textbf{relations: } T_1T_2T_3T_4 = 1, T_i^2 = 1.$$

Thus, the corresponding orbifold Hecke algebra is the following deformation of $\mathbb{C}G$:

$$\textbf{generators: } T_1, T_2, T_3, T_4; \qquad \textbf{relations: } T_1T_2T_3T_4 = 1, (T_i - p_i)(T_i - q_i) = 0,$$

where p_i, q_i ($i = 1, \dots, 4$), are parameters.

If we renormalize the T_i, these relations turn into

$$(T_i - t_i)(T_i + t_i^{-1}) = 0, \quad T_1T_2T_3T_4 = q,$$

and we get the type $C^\vee C_1$ double affine Hecke algebra. If we set three of the four T_i's satisfying the undeformed relation $T_i^2 = 1$, we get the double affine Hecke algebra of type A_1. More precisely, this algebra is generated by $T_1, \dots, T_4$ with relations

$$T_2^2 = T_3^2 = T_4^2 = 1, \quad (T_1 - t)(T_1 + t^{-1}) = 0, \quad T_1T_2T_3T_4 = q.$$

Another presentation of this algebra (which is more widely used) is as follows. Let $E = \mathbb{C}/\mathbb{Z}^2$, an elliptic curve with an $\mathbb{Z}_2$ action defined by $z \mapsto -z$. Define *the partial braid group*

$$B = \pi_1^{\mathrm{orb}}(E \backslash \{0\}/\mathbb{Z}_2, x),$$

where x is a generic point. Notice that comparing to the usual braid group, we do not delete three of the four reflection points. The generators of the group $\pi_1(E \setminus \{0\}, x)$ (the fundamental group of a punctured 2-torus) are X (corresponding to the a-cycle on the torus), Y (corresponding to the b-cycle on the torus) and C (corresponding to the loop around 0). In order to construct B, which is an extension of $\mathbb{Z}_2$ by $\pi_1(E \setminus \{0\}, x)$, we introduce an element T s.t. $T^2 = C$ (the half-loop around the puncture). Then X, Y, T satisfy the following relations:

$$TXT = X^{-1}, \quad T^{-1}YT^{-1} = Y^{-1}, \quad Y^{-1}X^{-1}YXT^2 = 1.$$

The Hecke algebra of the partial braid group is then defined to be the group algebra of B plus an extra relation: $(T - q_1)(T + q_2) = 0$.

A common way to present this Hecke algebra is to renormalize the generators so that one has the following relations:

$$TXT = X^{-1}, T^{-1}YT^{-1} = Y^{-1}, Y^{-1}X^{-1}YXT^2 = q, (T - t)(T + t^{-1}) = 0.$$

This is Cherednik's definition for $\mathcal{H}(q, t)$, the double affine Hecke algebra of type A_1 which depends on two parameters q, t.

There are two degenerations of the algebra $\mathcal{H}(q, t)$.

1. **The trigonometric degeneration.**

Set $Y = \mathbf{e}^{\hbar y}$, $q = \mathbf{e}^{\hbar}$, $t = \mathbf{e}^{\hbar c}$ and $T = s\mathbf{e}^{\hbar cs}$, where $s \in \mathbb{Z}_2$ is the reflection. Then s, X, y satisfy the following relations modulo $\hbar$:

$$s^2 = 1, \quad sXs^{-1} = X^{-1}, \quad sy + ys = 2c, \quad X^{-1}yX - y = 1 - 2cs.$$

The algebra generated by s, X, y with these relations is called the type A_1 trigonometric Cherednik algebra. It is easy to show that it is isomorphic to the Cherednik algebra $H_{1,c}(\mathbb{Z}_2, \mathbb{C}^*)$, where $\mathbb{Z}_2$ acts on $\mathbb{C}^*$ by $z \to z^{-1}$.

2. **The rational degeneration.**

In the trigonometric Cherednik algebra, set $X = \mathbf{e}^{\hbar x}$ and $y = \hat{y}/\hbar$. Then $s, x, \hat{y}$ satisfy the following relations modulo $\hbar$:

$$s^2 = 1, sx = -xs, s\hat{y} = -\hat{y}s, \hat{y}x - x\hat{y} = 1 - 2cs.$$

The algebra generated by $s, x, \hat{y}$ with these relations is the rational Cherednik algebra $H_{1,c}(\mathbb{Z}_2, \mathbb{C})$ with the action of $\mathbb{Z}_2$ on $\mathbb{C}$ is given by $z \to -z$.

7.12. **Affine and extended affine Weyl groups.** Let $R = \{\alpha\} \subset \mathbb{R}^n$ be a root system with respect to a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ on $\mathbb{R}^n$. We will assume that R is reduced. Let $\{\alpha_i\}_{i=1}^n \subset R$ be the set of simple roots and R_+ (respectively R_-) be the set of positive (respectively negative) roots. The coroots are denoted by $\alpha^\vee = 2\alpha/(\alpha,\alpha)$. Let $Q^\vee = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i^\vee$ be the coroot lattice and $P^\vee = \bigoplus_{i=1}^n \mathbb{Z}\omega_i^\vee$ the coweight lattice, where $\omega_i^\vee$'s are the fundamental coweights, i.e., $(\omega_i^\vee, \alpha_j) = \delta_{ij}$. Let θ be the maximal positive root, and assume that the bilinear form is normalized by the condition $(\theta,\theta) = 2$. Let $\overline{W}$ be the Weyl group which is generated by the reflections s_α $(\alpha \in R)$.

By definition, the affine root system is

$$R^a = \{\tilde{\alpha} = [\alpha, j] \in \mathbb{R}^n \times \mathbb{R} | \text{ where } \alpha \in R, j \in \mathbb{Z}\}.$$

The set of positive affine roots is $R_+^a = \{[\alpha, j] \,|\, j \in \mathbb{Z}_{>0}\} \cup \{[\alpha, 0] \,|\, \alpha \in R_+\}$. Define $\alpha_0 = [-\theta, 1]$. We will identify $\alpha \in R$ with $\tilde{\alpha} = [\alpha, 0] \in R^a$.

For an arbitrary affine root $\tilde{\alpha} = [\alpha, j]$ and a vector $\tilde{z} = [z, \zeta] \in \mathbb{R}^n \times \mathbb{R}$, the corresponding affine reflection is defined as follows:

$$s_{\tilde{\alpha}}(\tilde{z}) = \tilde{z} - 2\frac{(z,\alpha)}{(\alpha,\alpha)}\,\tilde{\alpha} = \tilde{z} - (z, \alpha^\vee)\,\tilde{\alpha}.$$

The affine Weyl group $\overline{W}_a$ is generated by the affine reflections $\{s_{\tilde{\alpha}} \,|\, \tilde{\alpha} \in \widetilde{R}_+\}$, and we have an isomorphism:

$$\overline{W}_a \cong \overline{W} \ltimes Q^\vee,$$

where the translation $\alpha^\vee \in Q^\vee$ is naturally identified with the composition $s_{[-\alpha,1]}s_\alpha \in \overline{W}_a$.

Define the extended affine Weyl group to be $\overline{W}_a^{\text{ext}} = \overline{W} \ltimes P^\vee$ acting on $\mathbb{R}^{n+1}$ via $b(\tilde{z}) = [z, \zeta - (b, z)]$ for $\tilde{z} = [z, \zeta]$, $b \in P^\vee$. Then $\overline{W}_a \subset \overline{W}_a^{\text{ext}}$. Moreover, $\overline{W}_a$ is a normal subgroup of $\overline{W}_a^{\text{ext}}$ and $\overline{W}_a^{\text{ext}}/\overline{W}_a = P^\vee/Q^\vee$. The latter group can be identified with the group $\Pi = \{\pi_r\}$ of the elements of $\overline{W}_a^{\text{ext}}$ permuting simple affine roots under their action in $\mathbb{R}^{n+1}$. It is a normal commutative subgroup of $\text{Aut} = \text{Aut}(\text{Dyn}^a)$ (Dyn^a denotes the affine Dynkin diagram). The quotient Aut/Π is isomorphic to the group of the automorphisms preserving α_0, i.e. the group AutDyn of automorphisms of the finite Dynkin diagram.

7.13. **Cherednik's double affine Hecke algebra of a root system.** In this subsection, we will give an explicit presentation of Cherednik's DAHA for a root system, defined in Example 7.18. This is done by giving an explicit presentation of the corresponding braid group (which is called *the elliptic braid group*), and then imposing quadratic relations on the generators corresponding to reflections.

For a root system R, let $m = 2$ if R is of type D_{2k}, $m = 1$ if R is of type B_{2k}, C_k, and otherwise $m = |\Pi|$. Let m_{ij} be the number of edges between vertex i and vertex j in the affine Dynkin diagram of R^a. Let X_i $(i = 1, \dots, n)$ be a family of pairwise commutative and algebraically independent elements. Set

$$X_{[b,j]} = \prod_{i=1}^{n} X_i^{\ell_i} q^j, \text{ where } b = \sum_{i=1}^{n} \ell_i \omega_i \in P, j \in \mathbb{Z}/m\mathbb{Z}.$$

For an element $\hat{w} \in \overline{W}_a^{\text{ext}}$, we can define an action on these $X_{[b,j]}$ by $\hat{w} X_{[b,j]} = X_{\hat{w}[b,j]}$.

Definition 7.24 (Cherednik)**.** *The double affine Hecke algebra* (DAHA) of the root system R, denoted by $\mathcal{H}$, is an algebra defined over the field $\mathbb{C}_{q,t} = \mathbb{C}(q^{1/m}, t^{1/2})$, generated by $T_i, i = 0, \dots, n, \Pi, X_b, b \in P$, subject to the following relations:

(1) $T_i T_j T_i \cdots = T_j T_i T_j \cdots$, m_{ij} factors each side;
(2) $(T_i - t_i)(T_i + t_i^{-1}) = 0$ for $i = 0, \dots, n$;
(3) $\pi T_i \pi^{-1} = T_{\pi(i)}$, for $\pi \in \Pi$ and $i = 0, \dots, n$;
(4) $\pi X_b \pi^{-1} = X_{\pi(b)}$, for $\pi \in \Pi, b \in P$;
(5) $T_i X_b T_i = X_b X_{\alpha_i}^{-1}$, if $i > 0$ and $(b, \alpha_i^\vee) = 1$; $T_i X_b = X_b T_i$, if $i > 0$ and $(b, \alpha_i^\vee) = 0$;
(6) $T_0 X_b T_0 = X_{b-\alpha_0}$ if $(b, \theta) = -1$; $T_0 X_b = X_b T_0$ if $(b, \theta) = 0$.

Here t_i are parameters attached to simple affine roots (so that roots of the same length give rise to the same parameters).

The degenerate double affine Hecke algebra (trigonometric Cherednik algebra) $\mathcal{H}_{trig}$ is generated by the group algebra of $\overline{W}_a^{\text{ext}}$, Π and pairwise commutative $y_{\tilde{b}} = \sum_{i=1}^{n} (b, \alpha^\vee) y_i + u$ for $\tilde{b} = [b, u] \in P \times \mathbb{Z}$, with the following relations:

$$s_i y_b - y_{s_i(b)} s_i = -k_i (b, \alpha_i^\vee), \text{ for } i = 1, \dots, n,$$
$$s_0 y_b - y_{s_0(b)} s_0 = k_0 (b, \theta), \qquad \pi_r y_b \pi_r^{-1} = y_{\pi_r(b)} \text{ for } \pi_r \in \Pi.$$

Remark 7.25. This degeneration can be obtained from the DAHA similarly to the case of A_1, which is described above.

7.14. **Algebraic flatness of Hecke algebras of polygonal Fuchsian groups.** Let W be the Coxeter group of rank r corresponding to a Coxeter datum:

$$m_{ij}\, (i, j = 1, \dots, r, i \neq j), \text{ such that } 2 \leq m_{ij} \leq \infty \text{ and } m_{ij} = m_{ji}.$$

So the group W has generators s_i $i = 1, \dots, r$, and defining relations

$$s_i^2 = 1, \ (s_i s_j)^{m_{ij}} = 1 \text{ if } m_{ij} \neq \infty.$$

It has a sign character $\xi : W \to \{\pm 1\}$ given by $\xi(s_i) = -1$. Denote by W_+ the kernel of ξ (the even subgroup of W). It is generated by $a_{ij} = s_i s_j$ with relations:

$$a_{ij} = a_{ji}^{-1}, \quad a_{ij} a_{jk} a_{ki} = 1, \quad a_{ij}^{m_{ij}} = 1.$$

We can deform the group algebra $\mathbb{C}[W]$ as follows. Define the algebra $A(W)$ with invertible generators s_i, and $t_{ij,k}$, $i, j = 1, \dots, r$, $k \in \mathbb{Z}_{m_{ij}}$ for (i, j) such that $m_{ij} < \infty$ and defining relations

$$t_{ij,k} = t_{ji,-k}^{-1}, \quad s_i^2 = 1, \quad [t_{ij,k}, t_{i'j',k'}] = 0, \quad s_p t_{ij,k} = t_{ji,k} s_p,$$
$$\prod_{k=1}^{m_{ij}} (s_i s_j - t_{ij,k}) = 0 \text{ if } m_{ij} < \infty.$$

Notice that if we set $t_{ij,k} = \exp(2\pi k \mathbf{i}/m_{ij})$, we get $\mathbb{C}[W]$.

Define also the algebra $A_+(W)$ over $\mathcal{R} := \mathbb{C}[t_{ij,k}]$ ($t_{ij,k} = t^{-1}_{ji,-k}$) by generators a_{ij}, $i \neq j$ ($a_{ij} = a^{-1}_{ji}$), and relations

$$\prod_{k=1}^{m_{ij}}(a_{ij} - t_{ij,k}) = 0 \text{ if } m_{ij} < \infty, \quad a_{ij}a_{jp}a_{pi} = 1.$$

If w is a word in letters s_i, let T_w be the corresponding element of $A(W)$. Choose a function $w(x)$ which attaches to every element $x \in W$, a reduced word $w(x)$ representing x in W.

Theorem 7.26 (Etingof, Rains, [ER])**.** (i) *The elements $T_{w(x)}$, $x \in W$, form a spanning set in $A(W)$ as a left $\mathcal{R}$-module.*

(ii) *The elements $T_{w(x)}$, $x \in W_+$, form a spanning set in $A_+(W)$ as a left $\mathcal{R}$-module.*

(iii) *The elements $T_{w(x)}$, $x \in W$, are linearly independent if W has no finite parabolic subgroups of rank* 3*.*

Proof. We only give the proof of (i). Statement (ii) follows from (i). Proof of (iii), which is quite nontrivial, can be found in [ER] (it uses the geometry of constructible sheaves on the Coxeter complex of W).

Let us write the relation

$$\prod_{k=1}^{m_{ij}}(s_is_j - t_{ij,k}) = 0$$

as a deformed braid relation:

$$s_js_is_j \ldots + S.L.T. = t_{ij}s_is_js_i \ldots + S.L.T.,$$

where $t_{ij} = (-1)^{m_{ij}+1}t_{ij,1}\cdots t_{ij,m_{ij}}$, S.L.T. mean "smaller length terms", and the products on both sides have length m_{ij}. This can be done by multiplying the relation by $s_is_j\cdots$ (m_{ij} factors).

Now let us show that $T_{w(x)}$ span $A(W)$ over $\mathcal{R}$. Clearly, T_w for all words w span $A(W)$. So we just need to take any word w and express T_w via $T_{w(x)}$.

It is well known from the theory of Coxeter groups (see e.g. [B]) that using the braid relations, one can turn any non-reduced word into a word that is not square free, and any reduced expression of a given element of W into any other reduced expression of the same element. Thus, if w is non-reduced, then by using the deformed braid relations we can reduce T_w to a linear combination of T_u with words u of smaller length than w. On the other hand, if w is a reduced expression for some element $x \in W$, then using the deformed braid relations we can reduce T_w to a linear combination of T_u with u shorter than w, and $T_{w(x)}$. Thus $T_{w(x)}$ are a spanning set. This proves (i). $\square$

Thus, $A_+(W)$ is a "deformation" of $\mathbb{C}[W_+]$ over $\mathcal{R}$, and similarly $A(W)$ is a "twisted deformation" of $\mathbb{C}[W]$.

Now let $\Gamma = \Gamma(m_1, \ldots, m_r)$, $r \geq 3$, be the Fuchsian group defined by generators c_j, $j = 1, \ldots, r$, with defining relations

$$c_j^{m_j} = 1, \ \prod_{j=1}^{r} c_j = 1.$$

Here $2 \leq m_j < \infty$.

Suppose Γ acts on H where H is a simply connected complex Riemann surface as in Section 7.7. We have the Hecke algebra of Γ, $\mathcal{H}_\tau(H,\Gamma)$, defined by the same (invertible) generators c_j and relations

$$\prod_k (c_j - \exp(2\pi \mathbf{i} k/n_j) q_{jk}) = 0, \; \prod_{j=1}^r c_j = 1,$$

where $q_{jk} = \exp(\tau_{jk})$.

We saw above (Theorem 7.15) that if τ_{jk}'s are formal, the algebra $\mathcal{H}_\tau(\Gamma, H)$ is flat in τ if $|\Gamma|$ is infinite (i.e., H is Euclidean or hyperbolic). Here is a much stronger non-formal version of this theorem.

Theorem 7.27. *The algebra $\mathcal{H}_\tau(\Gamma, H)$ is free as a left module over $R := \mathbb{C}[q_{jk}^{\pm 1}]$ if and only if $\sum_j (1 - 1/m_j) \geq 2$ (i.e., H is Euclidean or hyperbolic).*

Proof. Let us consider the Coxeter datum: m_{ij}, $i, j = 1, \ldots, r$, such that $m_{i,i+1} := m_i$ ($i \in \mathbb{Z}/r\mathbb{Z}$), and $m_{ij} = \infty$ otherwise. Suppose the corresponding Coxeter group is W. Then we can see that $\Gamma = W_+$. Notice that the algebra $\mathcal{H}_\tau(\Gamma, H)$ for genus 0 orbifolds is the algebra $A_+(W)$, i.e., we have $\mathcal{H}_\tau(\Gamma, H) = A_+(W)$.

The condition $\sum_j (1 - 1/m_j) \geq 2$ is equivalent to the condition that W has no finite parabolic subgroups of rank 3. From Theorem 7.26 (ii) and Theorem 7.15, we can see that $A_+(W)$ is free as a left module over R. We are done. ☐

7.15. **Notes.** Section 7.8 follows Section 6 of the paper [EOR]; Cherednik's definition of the double affine Hecke algebra of a root system is from Cherednik's book [Ch]; Sections 7.7 and 7.14 follow the paper [ER]; The other parts of this section follow the paper [E1].

8. Symplectic reflection algebras

8.1. **The definition of symplectic reflection algebras.** Rational Cherednik algebras for finite Coxeter groups are a special case of a wider class of algebras called symplectic reflection algebras. To define them, let V be a finite dimensional symplectic vector space over $\mathbb{C}$ with a symplectic form ω, and G be a finite group acting symplectically (linearly) on V. For simplicity let us assume that $(\wedge^2 V^*)^G = \mathbb{C}\omega$ (i.e., V is symplectically irreducible) and that G acts faithfully on V (these assumptions are not important, and essentially not restrictive).

Definition 8.1. A symplectic reflection in G is an element g such that the rank of the operator $1-g$ on V is 2.

If s is a symplectic reflection, then let $\omega_s(x,y)$ be the form ω applied to the projections of x,y to the image of $1-s$ along the kernel of $1-s$; thus ω_s is a skewsymmetric form of rank 2 on V.

Let $\mathcal{S} \subset G$ be the set of symplectic reflections, and $c : \mathcal{S} \to \mathbb{C}$ be a function which is invariant under the action of G. Let $t \in \mathbb{C}$.

Definition 8.2. The symplectic reflection algebra $\mathsf{H}_{t,c} = \mathsf{H}_{t,c}[G,V]$ is the quotient of the algebra $\mathbb{C}[G] \ltimes \mathbf{T}(V)$ by the ideal generated by the relation

$$[x,y] = t\omega(x,y) - 2\sum_{s\in\mathcal{S}} c_s\omega_s(x,y)s. \qquad (8.1)$$

Example 8.3. Let W be a finite Coxeter group with reflection representation $\mathfrak{h}$. We can set $V = \mathfrak{h} \oplus \mathfrak{h}^*$, $\omega(x,x') = \omega(y,y') = 0$, $\omega(y,x) = (y,x)$, for $x,x' \in \mathfrak{h}^*$ and $y,y' \in \mathfrak{h}$. In this case

(1) symplectic reflections are the usual reflections in W;
(2) $\omega_s(x,x') = \omega_s(y,y') = 0$, $\omega_s(y,x) = (y,\alpha_s)(\alpha_s^\vee,x)/2$.

Thus, $\mathsf{H}_{t,c}[G,\mathfrak{h}\oplus\mathfrak{h}^*]$ coincides with the rational Cherednik algebra $H_{t,c}(G,\mathfrak{h})$ defined in Section 3.

Example 8.4. Let Γ be a finite subgroup of $SL_2(\mathbb{C})$, and $V = \mathbb{C}^2$ be the tautological representation, with its standard symplectic form. Then all nontrivial elements of Γ are symplectic reflections, and for any symplectic reflection s, $\omega_s = \omega$. So the main commutation relation of $\mathsf{H}_{t,c}[\Gamma,V]$ takes the form

$$[y,x] = t - \sum_{g\in\Gamma, g\neq 1} 2c_g g.$$

Example 8.5. (Wreath products) Let Γ be as in the previous example, $G = \mathfrak{S}_n \ltimes \Gamma^n$, and $V = (\mathbb{C}^2)^n$. Then symplectic reflections are conjugates $(g,1,...,1)$, $g \in \Gamma$, $g \neq 1$, and also conmjugates of transpositions in $\mathfrak{S}_n$ (so there is one more conjugacy class of reflections than in the previous example).

Note also that for any V,G, $\mathsf{H}_{0,0}[G,V] = G \ltimes SV$, and $\mathsf{H}_{1,0}[G,V] = G \ltimes \mathrm{W}eyl(V)$, where $\mathrm{W}eyl(V)$ is the Weyl algebra of V, i.e. the quotient of the tensor algebra $\mathbf{T}(V)$ by the relation $xy - yx = \omega(x,y)$, $x,y \in V$.

8.2. **The PBW theorem for symplectic reflection algebras.** To ensure that the symplectic reflection algebras $\mathsf{H}_{t,c}$ have good properties, we need to prove a PBW theorem for them, which is an analog of Proposition 3.5. This is done in the following theorem, which also explains the special role played by symplectic reflections.

Theorem 8.6. *Let $\kappa : \wedge^2 V \to \mathbb{C}[G]$ be a linear G-equivariant function. Define the algebra H_κ to be the quotient of the algebra $\mathbb{C}[G] \ltimes \mathbf{T}(V)$ by the relation $[x, y] = \kappa(x, y)$, $x, y \in V$. Put an increasing filtration on H_κ by setting $\deg(V) = 1$, $\deg(G) = 0$, and define $\xi : \mathbb{C}G \ltimes SV \to \mathrm{gr}\mathsf{H}_\kappa$ to be the natural surjective homomorphism. Then ξ is an isomorphism if and only if κ has the form*

$$\kappa(x, y) = t\omega(x, y) - 2\sum_{s\in\mathcal{S}} c_s\omega_s(x, y)s,$$

for some $t \in \mathbb{C}$ and G-invariant function $c : \mathcal{S} \to \mathbb{C}$.

Unfortunately, for a general symplectic reflection algebra we don't have a Dunkl operator representation, so the proof of the more difficult "if" part of this Theorem is not as easy as the proof of Proposition 3.5. Instead of explicit computations with Dunkl operators, it makes use of the deformation theory of Koszul algebras, which we will now discuss.

8.3. **Koszul algebras.** Let R be a finite dimensional semisimple algebra (over $\mathbb{C}$). Let A be a $\mathbb{Z}_+$-graded algebra, such that $A[0] = R$, and assume that the graded components of A are finite dimensional.

Definition 8.7. (i) The algebra A is said to be quadratic if it is generated over R by $A[1]$, and has defining relations in degree 2.

(ii) A is Koszul if all elements of $\mathrm{Ext}^i(R, R)$ (where R is the augmentation module over A) have grade degree precisely i.

Remark 8.8. (1) Thus, in a quadratic algebra, $A[2] = A[1] \otimes_R A[1]/E$, where E is the subspace (R-subbimodule) of relations.

(2) It is easy to show that a Koszul algebra is quadratic, since the condition to be quadratic is just the Koszulity condition for $i = 1, 2$.

Now let A_0 be a quadratic algebra, $A_0[0] = R$. Let E_0 be the space of relations for A_0. Let $E \subset A_0[1] \otimes_R A_0[1][[\hbar]]$ be a free (over $\mathbb{C}[[\hbar]]$) R-subbimodule which reduces to E_0 modulo $\hbar$ ("deformation of the relations"). Let A be the ($\hbar$-adically complete) algebra generated over $R[[\hbar]]$ by $A[1] = A_0[1][[\hbar]]$ with the space of defining relations E. Thus A is a $\mathbb{Z}_+$-graded algebra.

The following very important theorem is due to Beilinson, Ginzburg, and Soergel, [BGS] (less general versions appeared earlier in the works of Drinfeld [Dr], Polishchuk-Positselski [PP], Braverman-Gaitsgory [BG]). We will not give its proof.

Theorem 8.9 (Koszul deformation principle)**.** *If A_0 is Koszul then A is a topologically free $\mathbb{C}[[\hbar]]$ module if and only if so is $A[3]$.*

Remark. Note that $A[i]$ for $i < 3$ are obviously topologically free.

We will now apply this theorem to the proof of Theorem 8.6.

8.4. **Proof of Theorem 8.6.** Let $\kappa : \wedge^2 V \to \mathbb{C}[G]$ be a linear G-equivariant map. We write $\kappa(x, y) = \sum_{g\in G} \kappa_g(x, y)g$, where $\kappa_g(x, y) \in \wedge^2 V^*$. To apply Theorem 8.9, let us homogenize our algebras. Namely, let $A_0 = (\mathbb{C}G \ltimes SV) \otimes \mathbb{C}[u]$ (where u has degree 1). Also let $\hbar$ be a formal parameter, and consider the deformation $A = \mathsf{H}_{\hbar u^2\kappa}$ of A_0. That is, A is the quotient of $G \ltimes \mathbf{T}(V)[u][[\hbar]]$ by the relations $[x, y] = \hbar u^2\kappa(x, y)$. This is a deformation of the type considered in Theorem 8.9, and it is easy to see that its flatness in $\hbar$ is equivalent to Theorem

8.6. Also, the algebra A_0 is Koszul, because the polynomial algebra SV is a Koszul algebra. Thus by Theorem 8.9, it suffices to show that A is flat in degree 3.

The flatness condition in degree 3 is "the Jacobi identity"

$$[\kappa(x,y),z]+[\kappa(y,z),x]+[\kappa(z,x),y]=0,$$

which must be satisfied in $\mathbb{C}G\ltimes V$. In components, this equation transforms into the system of equations

$$\kappa_g(x,y)(z-z^g)+\kappa_g(y,z)(x-x^g)+\kappa_g(z,x)(y-y^g)=0$$

for every $g\in G$ (here z^g denotes the result of the action of g on z).

This equation, in particular, implies that if x,y,g are such that $\kappa_g(x,y)\neq 0$ then for any $z\in V$ $z-z^g$ is a linear combination of $x-x^g$ and $y-y^g$. Thus $\kappa_g(x,y)$ is identically zero unless the rank of $(1-g)|_V$ is at most 2, i.e. $g=1$ or g is a symplectic reflection.

If $g=1$ then $\kappa_g(x,y)$ has to be G-invariant, so it must be of the form $t\omega(x,y)$, where $t\in\mathbb{C}$.

If g is a symplectic reflection, then $\kappa_g(x,y)$ must be zero for any x such that $x-x^g=0$. Indeed, if for such an x there had existed y with $\kappa_g(x,y)\neq 0$ then $z-z^g$ for any z would be a multiple of $y-y^g$, which is impossible since $\mathrm{Im}(1-g)|_V$ is 2-dimensional. This implies that $\kappa_g(x,y)=2c_g\omega_g(x,y)$, and c_g must be invariant.

Thus we have shown that if A is flat (in degree 3) then κ must have the form given in Theorem 8.6. Conversely, it is easy to see that if κ does have such form, then the Jacobi identity holds. So Theorem 8.6 is proved.

8.5. **The spherical subalgebra of the symplectic reflection algebra.** The properties of symplectic reflection algebras are similar to the properties of rational Cherednik algebras we have studied before. The main difference is that we no longer have the Dunkl representation and localization results, so some proofs are based on different ideas and are more complicated.

The spherical subalgebra of the symplectic reflection algebra is defined in the same way as in the Cherednik algebra case. Namely, let $\mathsf{e}=|G|^{-1}\sum_{g\in G}g$, and $\mathsf{B}_{t,c}=\mathsf{e}\mathsf{H}_{t,c}\mathsf{e}$.

Proposition 8.10. *$\mathsf{B}_{t,c}$ is commutative if and only if $t=0$.*

Proof. Let A be a $\mathbb{Z}_+$-filtered algebra. If A is not commutative, then we can define a nonzero Poisson bracket on $\mathrm{gr}A$ in the following way. Let m be the minimum of $\deg(a)+\deg(b)-\deg([a,b])$ (over $a,b\in A$ such that $[a,b]\neq 0$). Then for homogeneous elements $a_0,b_0\in A_0$ of degrees p,q, we can define $\{a_0,b_0\}$ to be the image in $A_0[p+q-m]$ of $[a,b]$, where a,b are any lifts of a_0,b_0 to A. It is easy to check that $\{\cdot,\cdot\}$ is a Poisson bracket on A_0 of degree $-m$.

Let us now apply this construction to the filtered algebra $A=\mathsf{B}_{t,c}$. We have $\mathrm{gr}(A)=A_0=(SV)^G$.

Lemma 8.11. *A_0 has a unique, up to scaling, Poisson bracket of degree -2, and no nonzero Poisson brackets of degrees <-2.*

Proof. A Poisson bracket on $(SV)^G$ is the same thing as a Poisson bracket on the variety V^*/G. On the smooth part $(V^*/G)_s$ of V^*/G, it is simply a bivector field, and we can lift it to a bivector field on the preimage V_s^* of $(V^*/G)_s$ in V^*, which is the set of points in V with trivial stabilizers. But the codimension on $V^*\setminus V_s^*$ in V^* is 2 (as $V^*\setminus V_s^*$ is a union of symplectic subspaces), so the bivector on V_s^* extends to a regular bivector on V^*. So if this bivector is homogeneous, it must have degree ≥ -2, and if it has degree -2 then it must be with constant coefficients, so being G-invariant, it is a multiple of ω. The lemma is proved. $\square$

Now, for each t, c we have a natural Poisson bracket on A_0 of degree -2, which depends linearly on t, c. So by the lemma, this bracket has to be of the form $f(t,c)\Pi$, where Π is the unique up to scaling Poisson bracket of degree -2, and f a homogeneous linear function. Thus the algebra $A = \mathsf{B}_{t,c}$ is not commutative unless $f(t,c) = 0$. On the other hand, if $f(t,c) = 0$, and $\mathsf{B}_{t,c}$ is not commutative, then, as we've shown, A_0 has a nonzero Poisson bracket of degree < -2. But By Lemma 8.11, there is no such brackets. So $\mathsf{B}_{t,c}$ must be commutative if $f(t,c) = 0$.

It remains to show that $f(t,c)$ is in fact a nonzero multiple of t. First note that $f(1,0) \neq 0$, since $\mathsf{B}_{1,0}$ is definitely noncommutative. Next, let us take a point (t,c) such that $\mathsf{B}_{t,c}$ is commutative. Look at the $\mathsf{H}_{t,c}$-module $\mathsf{H}_{t,c}\mathsf{e}$, which has a commuting action of $\mathsf{B}_{t,c}$ from the right. Its associated graded is SV as an $(\mathbb{C}G \ltimes SV, (SV)^G)$-bimodule, which implies that the generic fiber of $\mathsf{H}_{t,c}\mathsf{e}$ as a $\mathsf{B}_{t,c}$-module is the regular representation of G. So we have a family of finite dimensional representations of $\mathsf{H}_{t,c}$ on the fibers of $\mathsf{H}_{t,c}\mathsf{e}$, all realized in the regular representation of G. Computing the trace of the main commutation relation (8.1) of $\mathsf{H}_{t,c}$ in this representation, we obtain that $t = 0$ (since $\mathrm{Tr}\,(s) = 0$ for any reflection s). The proposition is proved. $\square$

Note that $\mathsf{B}_{0,c}$ has no zero divisors, since its associated graded algebra $(SV)^G$ does not. Thus, like in the Cherednik algebra case, we can define a Poisson variety M_c, the spectrum of $\mathsf{B}_{0,c}$, called the Calogero-Moser space of G, V. Moreover, the algebra $\mathsf{B}_c := \mathsf{B}_{\hbar,c}$ over $\mathbb{C}[\hbar]$ is an algebraic quantization of M_c.

8.6. **The center of the symplectic reflection algebra $\mathsf{H}_{t,c}$.** Consider the bimodule $\mathsf{H}_{t,c}\mathsf{e}$, which has a left action of $\mathsf{H}_{t,c}$ and a right commuting action of $\mathsf{B}_{t,c}$. It is obvious that $\mathrm{End}_{\mathsf{H}_{t,c}}\mathsf{H}_{t,c}\mathsf{e} = \mathsf{B}_{t,c}$ (with opposite product). The following theorem shows that the bimodule $\mathsf{H}_{t,c}\mathsf{e}$ has the double centralizer property (i.e., $\mathrm{End}_{\mathsf{B}_{t,c}}\mathsf{H}_{t,c}\mathsf{e} = \mathsf{H}_{t,c}$).

Note that we have a natural map $\xi_{t,c} : \mathsf{H}_{t,c} \to \mathrm{End}_{\mathsf{B}_{t,c}}\mathsf{H}_{t,c}\mathsf{e}$.

Theorem 8.12. *$\xi_{t,c}$ is an isomorphism for any t, c.*

Proof. The complete proof is given [EG]. We will give the main ideas of the proof skipping straightforward technical details. The first step is to show that the result is true in the graded case, $(t,c) = (0,0)$. To do so, note the following easy lemma:

Lemma 8.13. *If X is an affine complex algebraic variety with algebra of functions $\mathcal{O}_X$ and G a finite group acting freely on X then the natural map $\xi_X : G \ltimes \mathcal{O}_X \to \mathrm{End}_{\mathcal{O}_X^G}\mathcal{O}_X$ is an isomorphism.*

Therefore, the map $\xi_{0,0} : G \ltimes SV \to \mathrm{End}_{(SV)^G}(SV)$ is injective, and moreover becomes an isomorphism after localization to the field of quotients $\mathbb{C}(V^*)^G$. To show it's surjective, take $a \in \mathrm{End}_{(SV)^G}(SV)$. There exists $a' \in G \ltimes \mathbb{C}(V^*)$ which maps to a. Moreover, by Lemma 8.13, a' can have poles only at fixed points of G on V^*. But these fixed points form a subset of codimension ≥ 2, so there can be no poles and we are done in the case $(t,c) = (0,0)$.

Now note that the algebra $\mathrm{End}_{\mathsf{B}_{t,c}}\mathsf{H}_{t,c}\mathsf{e}$ has an increasing integer filtration (bounded below) induced by the filtration on $\mathsf{H}_{t,c}$. This is due to the fact that $\mathsf{H}_{t,c}\mathsf{e}$ is a finitely generated

$\mathsf{e}\mathsf{H}_{t,c}\mathsf{e}$-module (since it is true in the associated graded situation, by Hilbert's theorem about invariants). Also, the natural map $\mathrm{grEnd}_{\mathsf{B}_{t,c}}\mathsf{H}_{t,c}\mathsf{e} \to \mathrm{End}_{\mathrm{gr}\mathsf{B}_{t,c}}\mathrm{gr}\mathsf{H}_{t,c}\mathsf{e}$ is clearly injective. Therefore, our result in the case $(t,c) = (0,0)$ implies that this map is actually an isomorphism (as so is its composition with the associated graded of $\xi_{t,c}$). Identifying the two algebras by this isomorphism, we find that $\mathrm{gr}(\xi_{t,c}) = \xi_{0,0}$. Since $\xi_{0,0}$ is an isomorphism, $\xi_{t,c}$ is an isomorphism for all t, c, as desired. [2] □

Denote by $\mathsf{Z}_{t,c}$ the center of the symplectic reflection algebra $\mathsf{H}_{t,c}$. We have the following theorem.

Theorem 8.14. *If $t \neq 0$, the center of $\mathsf{H}_{t,c}$ is trivial. If $t = 0$, we have $\mathrm{gr}\mathsf{Z}_{0,c} = \mathsf{Z}_{0,0}$. In particular, $\mathsf{H}_{0,c}$ is finitely generated over its center.*

Proof. The $t \neq 0$ case was proved by Brown and Gordon [BGo] as follows. If $t \neq 0$, we have $\mathrm{gr}\mathsf{Z}_{t,c} \subseteq \mathsf{Z}_{0,0} = (SV)^G$. Also, we have a map

$$\tau_{t,c} : \mathsf{Z}_{t,c} \to \mathsf{B}_{t,c} = \mathsf{e}\mathsf{H}_{t,c}\mathsf{e}, \quad z \mapsto z\mathsf{e} = \mathsf{e}z\mathsf{e}.$$

The map $\tau_{t,c}$ is injective since $\mathrm{gr}(\tau_{t,c})$ is injective. In particular, the image of $\mathrm{gr}(\tau_{t,c})$ is contained in $Z(\mathsf{B}_{t,c})$, the center of $\mathsf{B}_{t,c}$. Thus it is enough to show that $Z(\mathsf{B}_{t,c})$ is trivial. To show this, note that $\mathrm{gr}Z(\mathsf{B}_{t,c})$ is contained in the Poisson center of $\mathsf{B}_{0,0}$ which is trivial. So $Z(\mathsf{B}_{t,c})$ is trivial.

Now suppose $t = 0$. We need to show that $\mathrm{gr}(\tau_{0,c}) : \mathrm{gr}(Z_{0,c}) \to Z_{0,0}$ is an isomorphism. It suffices to show that $\tau_{0,c}$ is an isomorphism. To show this, we construct $\tau_{0,c}^{-1} : \mathsf{B}_{0,c} \to \mathsf{Z}_{0,c}$ as follows.

For any $b \in \mathsf{B}_{0,c}$, since $\mathsf{B}_{0,c}$ is commutative, we have an element $\tilde{b} \in \mathrm{End}_{\mathsf{B}_{0,c}}(\mathsf{H}_{0,c}\mathsf{e})$ which is defined as the right multiplication by b. From Theorem 8.12, $\tilde{b} \in \mathsf{H}_{0,c}$. Moreover, $\tilde{b} \in \mathsf{Z}_{0,c}$ since it commutes with $\mathsf{H}_{0,c}$ as a linear operator on the faithful $\mathsf{H}_{0,c}$-module $\mathsf{H}_{0,c}\mathsf{e}$. So $\tilde{b} \in \mathsf{Z}_{0,c}$. It is easy to see that $\tilde{b}\mathsf{e} = b$. So we can set $\tilde{b} = \tau_{0,c}^{-1}(b)$ which defines the inverse map to $\tau_{0,c}$. □

8.7. **A review of deformation theory.** Now we would like to explain that symplectic reflection algebras are the most general deformations of algebras of the from $G \ltimes \mathrm{W}eyl(V)$. Before we do so, we give a brief review of classical deformation theory of associative algebras.

8.7.1. *Formal deformations of associative algebras.* Let A_0 be an associative algebra with unit over $\mathbb{C}$. Denote by μ_0 the multiplication in A_0.

Definition 8.15. A (flat) formal n-parameter deformation of A_0 is an algebra A over $\mathbb{C}[[\hbar]] = \mathbb{C}[[\hbar_1, \ldots, \hbar_n]]$ which is topologically free as a $\mathbb{C}[[\hbar]]$-module, together with an algebra isomorphism $\eta_0 : A/\mathfrak{m}A \to A_0$ where $\mathfrak{m} = \langle \hbar_1, \ldots, \hbar_n \rangle$ is the maximal ideal in $\mathbb{C}[[\hbar]]$.

When no confusion is possible, we will call A a deformation of A_0 (omitting "formal").

Let us restrict ourselves to one-parameter deformations with parameter $\hbar$. Let us choose an identification $\eta : A \to A_0[[\hbar]]$ as $\mathbb{C}[[\hbar]]$-modules, such that $\eta = \eta_0$ modulo $\hbar$. Then the

[2] Here we use the fact that the filtration is bounded from below. In the case of an unbounded filtration, it is possible for a map not to be an isomorphism if its associated graded is an isomorphism. An example of this is the operator of multiplication by $1 + t^{-1}$ in the space of Laurent polynomials in t, filtered by degree.

product in A is completely determined by the product of elements of A_0, which has the form of a "star-product"

$$\mu(a,b) = a * b = \mu_0(a,b) + \hbar\mu_1(a,b) + \hbar^2\mu_2(a,b) + \cdots,$$

where $\mu_i : A_0 \otimes A_0 \to A_0$ are linear maps, and $\mu_0(a,b) = ab$.

8.7.2. *Hochschild cohomology.* The main tool in deformation theory of associative algebras is Hochschild cohomology. Let us recall its definition.

Let A be an associative algebra. Let M be a bimodule over A. A Hochschild n-cochain of A with coefficients in M is a linear map $A^{\otimes n} \to M$. The space of such cochains is denoted by $C^n(A,M)$. The differential $\mathrm{d} : C^n(A,M) \to C^{n+1}(A,M)$ is defined by the formula

$$\begin{aligned} \mathrm{d}f(a_1,\ldots,a_{n+1}) &= f(a_1,\ldots,a_n)a_{n+1} - f(a_1,\ldots,a_n a_{n+1}) + f(a_1,\ldots,a_{n-1}a_n,a_{n+1}) \\ &\quad - \cdots + (-1)^n f(a_1a_2,\ldots,a_{n+1}) + (-1)^{n+1} a_1 f(a_2,\ldots,a_{n+1}). \end{aligned}$$

It is easy to show that $\mathrm{d}^2 = 0$.

Definition 8.16. The Hochschild cohomology $\mathrm{HH}^\bullet(A,M)$ is defined to be the cohomology of the complex $(C^\bullet(A,M),\mathrm{d})$.

Proposition 8.17. *One has a natural isomorphism*

$$\mathrm{HH}^i(A,M) \to \mathrm{Ext}^i_{A-\mathrm{bimod}}(A,M),$$

where $A-\mathrm{bimod}$ *denotes the category of* A*-bimodules.*

Proof. The proof is obtained immediately by considering the bar resolution of the bimodule A:

$$\cdots \to A \otimes A \otimes A \to A \otimes A \to A,$$

where the bimodule structure on $A^{\otimes n}$ is given by

$$b(a_1 \otimes a_2 \otimes \cdots \otimes a_n)c = ba_1 \otimes a_2 \otimes \cdots \otimes a_n c,$$

and the map $\partial_n : A^{\otimes n} \to A^{\otimes n-1}$ is given by the formula

$$\partial_n(a_1 \otimes a_2 \otimes ... \otimes a_n) = a_1a_2 \otimes \cdots \otimes a_n - \cdots + (-1)^n a_1 \otimes \cdots \otimes a_{n-1}a_n.$$

∎

Note that we have the associative Yoneda product

$$\mathrm{HH}^i(A,M) \otimes \mathrm{HH}^j(A,N) \to \mathrm{HH}^{i+j}(A, M \otimes_A N),$$

induced by tensoring of cochains.

If $M = A$, the algebra itself, then we will denote $\mathrm{HH}^\bullet(A,M)$ by $\mathrm{HH}^\bullet(A)$. For example, $\mathrm{HH}^0(A)$ is the center of A, and $\mathrm{HH}^1(A)$ is the quotient of the Lie algebra of derivations of A by inner derivations. The Yoneda product induces a graded algebra structure on $\mathrm{HH}^\bullet(A)$; it can be shown that this algebra is supercommutative.

8.7.3. *Hochschild cohomology and deformations.* Let A_0 be an algebra, and let us look for 1-parameter deformations $A = A_0[[\hbar]]$ of A_0. Thus, we look for such series μ which satisfy the associativity equation, modulo the automorphisms of the $\mathbb{C}[[\hbar]]$-module $A_0[[\hbar]]$ which are the identity modulo $\hbar$. [3]

The associativity equation $\mu \circ (\mu \otimes \mathrm{Id}) = \mu \circ (\mathrm{Id} \otimes \mu)$ reduces to a hierarchy of linear equations:

$$\sum_{s=0}^{N} \mu_s(\mu_{N-s}(a,b),c) = \sum_{s=0}^{N} \mu_s(a, \mu_{N-s}(b,c)).$$

(These equations are linear in μ_N if μ_i, $i < N$, are known).

To study these equations, one can use Hochschild cohomology. Namely, we have the following standard facts (due to Gerstenhaber, [Ge]), which can be checked directly.

(1) The linear equation for μ_1 says that μ_1 is a Hochschild 2-cocycle. Thus algebra structures on $A_0[\hbar]/\hbar^2$ deforming μ_0 are parametrized by the space $Z^2(A_0)$ of Hochschild 2-cocycles of A_0 with values in $M = A_0$.

(2) If μ_1, μ_1' are two 2-cocycles such that $\mu_1 - \mu_1'$ is a coboundary, then the algebra structures on $A_0[\hbar]/\hbar^2$ corresponding to μ_1 and μ_1' are equivalent by a transformation of $A_0[\hbar]/\hbar^2$ that equals the identity modulo $\hbar$, and vice versa. Thus equivalence classes of multiplications on $A_0[\hbar]/\hbar^2$ deforming μ_0 are parametrized by the cohomology $\mathrm{HH}^2(A_0)$.

(3) The linear equation for μ_N says that $\mathrm{d}\mu_N$ is a certain quadratic expression b_N in $\mu_1, \ldots, \mu_{N-1}$. This expression is always a Hochschild 3-cocycle, and the equation is solvable if and only if it is a coboundary. Thus the cohomology class of b_N in $\mathrm{HH}^3(A_0)$ is the only obstruction to solving this equation.

8.7.4. *Universal deformation.* In particular, if $\mathrm{HH}^3(A_0) = 0$ then the equation for μ_n can be solved for all n, and for each n the freedom in choosing the solution, modulo equivalences, is the space $H = \mathrm{HH}^2(A_0)$. Thus there exists an algebra structure over $\mathbb{C}[[H]]$ on the space $A_u := A_0[[H]]$ of formal functions from H to A_0, $a, b \mapsto \mu_u(a,b) \in A_0[[H]]$, $(a, b \in A_0)$, such that $\mu_u(a,b)(0) = ab \in A_0$, and every 1-parameter flat formal deformation A of A_0 is given by the formula $\mu(a,b)(\hbar) = \mu_u(a,b)(\gamma(\hbar))$ for a unique formal series $\gamma \in \hbar H[[\hbar]]$, with the property that $\gamma'(0)$ is the cohomology class of the cocycle μ_1.

Such an algebra A_u is called a *universal deformation* of A_0. It is unique up to an isomorphism (which may involve an automorphism of $\mathbb{C}[[H]]$). [4]

Thus in the case $\mathrm{HH}^3(A_0) = 0$, deformation theory allows us to completely classify 1-parameter flat formal deformations of A_0. In particular, we see that the "moduli space" parametrizing formal deformations of A_0 is a smooth space – it is the formal neighborhood of zero in H.

If $\mathrm{HH}^3(A_0)$ is nonzero then in general the universal deformation parametrized by H does not exist, as there are obstructions to deformations. In this case, the moduli space of

[3] Note that we don't have to worry about the existence of a unit in A since a formal deformation of an algebra with unit always has a unit.

[4] In spite of the universal property of A_u, it may happen that there is an isomorphism between the algebras A^1 and A^2 corresponding to different paths $\gamma_1(\hbar), \gamma_2(\hbar)$ (of course, reducing to a nontrivial automorphism of A_0 modulo $\hbar$). For this reason, sometimes A_u is called a *semiuniversal*, rather than universal, deformation of A_0.

deformations will be a closed subscheme of the formal neighborhood of zero in H, which is often singular. On the other hand, even when $\mathrm{HH}^3(A_0) \neq 0$, the universal deformation parametrized by (the formal neighborhood of zero in) H may exist (although its existence may be more difficult to prove than in the vanishing case). In this case one says that the deformations of A_0 are unobstructed (since all obstructions vanish even though the space of obstructions doesn't).

8.8. **Deformation-theoretic interpretation of symplectic reflection algebras.** Let V be a symplectic vector space (over $\mathbb{C}$) and $\mathrm{W}eyl(V)$ the Weyl algebra of V. Let G be a finite group acting symplectically on V. Then from the definition, we have

$$A_0 := \mathsf{H}_{1,0}[G, V] = G \ltimes \mathrm{W}eyl(V).$$

Let us calculate the Hochschild cohomology of this algebra.

Theorem 8.18 (Alev, Farinati, Lambre, Solotar, [AFLS])**.** *The cohomology space* $\mathrm{HH}^i(G \ltimes \mathrm{W}eyl(V))$ *is naturally isomorphic to the space of conjugation invariant functions on the set* S_i *of elements* $g \in G$ *such that* $\operatorname{rank}(1-g)|_V = i$.

Corollary 8.19. *The odd cohomology of* $G \ltimes \mathrm{W}eyl(V)$ *vanishes, and* $\mathrm{HH}^2(G \ltimes \mathrm{W}eyl(V))$ *is the space* $\mathbb{C}[\mathcal{S}]^G$ *of conjugation invariant functions on the set of symplectic reflections. In particular, there exists a universal deformation* A *of* $A_0 = G \ltimes \mathrm{W}eyl(V)$ *parametrized by* $\mathbb{C}[\mathcal{S}]^G$.

Proof. Directly from the theorem. □

Proof of Theorem 8.18.

Lemma 8.20. *Let* B *be a* $\mathbb{C}$*-algebra together with an action of a finite group* G*. Then*

$$\mathrm{HH}^*(G \ltimes B, G \ltimes B) = \Big(\bigoplus_{g\in G} \mathrm{HH}^*(B, Bg)\Big)^G,$$

where Bg *is the bimodule isomorphic to* B *as a space where the left action of* B *is the usual one and the right action is the usual action twisted by* g*.*

Proof. The algebra $G \ltimes B$ is a projective B-module. Therefore, using the Shapiro lemma, we get

$$\begin{aligned} & \mathrm{HH}^*(G \ltimes B, G \ltimes B) = \mathrm{Ext}^*_{(G\times G)\ltimes(B\otimes B^{op})}(G \ltimes B, G \ltimes B) \\ = \ & \mathrm{Ext}^*_{G_{diagonal}\ltimes(B\otimes B^{op})}(B, G \ltimes B) = \mathrm{Ext}^*_{B\otimes B^{op}}(B, G \ltimes B)^G \\ = \ & \Big(\bigoplus_{g\in G} \mathrm{Ext}^*_{B\otimes B^{op}}(B, Bg)\Big)^G = \Big(\bigoplus_{g\in G} \mathrm{HH}^*(B, Bg)\Big)^G, \end{aligned}$$

as desired. □

Now apply the lemma to $B = \mathrm{W}eyl(V)$. For this we need to calculate $\mathrm{HH}^*(B, Bg)$, where g is any element of G. We may assume that g is diagonal in some symplectic basis: $g = \mathrm{diag}(\lambda_1, \lambda_1^{-1}, \ldots, \lambda_n, \lambda_n^{-1})$. Then by the Künneth formula we find that

$$\mathrm{HH}^*(B, Bg) = \bigotimes_{i=1}^{n} \mathrm{HH}^*(\mathbb{A}_1, \mathbb{A}_1 g_i),$$

where $\mathbb{A}_1$ is the Weyl algebra of the 2-dimensional space, (generated by x, y with defining relation $xy - yx = 1$), and $g_i = \mathrm{diag}(\lambda_i, \lambda_i^{-1})$.

Thus we need to calculate $\mathrm{HH}^*(B, Bg)$, $B = \mathbb{A}_1$, $g = \mathrm{diag}(\lambda, \lambda^{-1})$.

Proposition 8.21. *$\mathrm{HH}^*(B, Bg)$ is 1-dimensional, concentrated in degree 0 if $\lambda = 1$ and in degree 2 otherwise.*

Proof. If $B = \mathbb{A}_1$ then B has the following Koszul resolution as a B-bimodule:

$$B \otimes B \to B \otimes \mathbb{C}^2 \otimes B \to B \otimes B \to B.$$

Here the first map is given by the formula

$$b_1 \otimes b_2 \quad \mapsto \quad b_1 \otimes x \otimes yb_2 - b_1 \otimes y \otimes xb_2 - b_1 y \otimes x \otimes b_2 + b_1 x \otimes y \otimes b_2,$$

the second map is given by

$$b_1 \otimes x \otimes b_2 \mapsto b_1 x \otimes b_2 - b_1 \otimes xb_2, \quad b_1 \otimes y \otimes b_2 \mapsto b_1 y \otimes b_2 - b_1 \otimes yb_2,$$

and the third map is the multiplication.

Thus the cohomology of B with coefficients in Bg can be computed by mapping this resolution into Bg and taking the cohomology. This yields the following complex $C^\bullet$:

$$0 \to Bg \to Bg \oplus Bg \to Bg \to 0, \tag{8.2}$$

where the first nontrivial map is given by $bg \mapsto [bg, y] \otimes x - [bg, x] \otimes y$, and the second nontrivial map is given by $bg \otimes x \mapsto [x, bg]$, $bg \otimes y \mapsto [y, bg]$.

Consider first the case $g = 1$. Equip the complex $C^\bullet$ with the Bernstein filtration ($\deg(x) = \deg(y) = 1$), starting with $0, 1, 2$, for C^0, C^1, C^2, respectively (this makes the differential preserve the filtration). Consider the associated graded complex $C^\bullet_{\mathrm{gr}}$. In this complex, brackets are replaced with Poisson brackets, and thus it is easy to see that $C^\bullet_{\mathrm{gr}}$ is the De Rham complex for the affine plane, so its cohomology is $\mathbb{C}$ in degree 0 and 0 in other degrees. Therefore, the cohomology of $C^\bullet$ is the same.

Now consider $g \neq 1$. In this case, declare that C^0, C^1, C^2 start in degrees 2,1,0 respectively (which makes the differential preserve the filtration), and again consider the graded complex $C^\bullet_{\mathrm{gr}}$. The graded Euler characteristic of this complex is $(t^2 - 2t + 1)(1 - t)^{-2} = 1$.

The cohomology in the C^0_{gr} term is the set of $b \in \mathbb{C}[x, y]$ such that $ab = ba^g$ for all a. This means that $\mathrm{HH}^0 = 0$.

The cohomology of the C^2_{gr} term is the quotient of $\mathbb{C}[x, y]$ by the ideal generated by $a - a^g$, $a \in \mathbb{C}[x, y]$. Thus the cohomology HH^2 of the rightmost term is 1-dimensional, in degree 0. By the Euler characteristic argument, this implies that $\mathrm{HH}^1 = 0$. The cohomology of the filtered complex $C^\bullet$ is therefore the same, and we are done. □

The proposition implies that in the n-dimensional case $\mathrm{HH}^*(B, Bg)$ is 1-dimensional, concentrated in degree $\mathrm{rank}\,(1 - g)$. It is not hard to check that the group G acts on the sum of these 1-dimensional spaces by simply permuting the basis vectors. Thus the theorem is proved. □

Remark 8.22. Another proof of Theorem 8.18 is given in [Pi].

Theorem 8.23. *The algebra $\mathsf{H}_{1,c}[G, V]$, with formal c, is the universal deformation of $\mathsf{H}_{1,0}[G, V] = G \ltimes \mathrm{W}eyl(V)$. More specifically, the map $f : \mathbb{C}[\mathcal{S}]^G \to \mathrm{HH}^2(G \ltimes \mathrm{W}eyl(V))$ induced by this deformation coincides with the isomorphism of Corollary 8.19.*

Proof. The proof (which we will not give) can be obtained by a direct computation with the Koszul resolution for $G \ltimes \mathrm{W}eyl(V)$. Such a proof is given in [Pi]. The paper [EG] proves a slightly weaker statement that the map f is an isomorphism, which suffices to show that $\mathsf{H}_{1,c}(G, V)$ is the universal deformation of $\mathsf{H}_{1,0}[G, V]$. ☐

8.9. **Finite dimensional representations of $\mathsf{H}_{0,c}$.** Let $\mathsf{M}_c = \mathrm{Spec}\mathsf{Z}_{0,c}$. We can regard $\mathsf{H}_{0,c} = \mathsf{H}_{0,c}[G, V]$ as a finitely generated module over $\mathsf{Z}_{0,c} = \mathcal{O}(\mathsf{M}_c)$. Let $\chi \in \mathsf{M}_c$ be a central character, $\chi : \mathsf{Z}_{0,c} \to \mathbb{C}$. Denote by $\langle\chi\rangle$ the ideal in $\mathsf{H}_{0,c}$ generated by the kernel of χ.

Proposition 8.24. *If χ is generic then $\mathsf{H}_{0,c}/\langle\chi\rangle$ is the matrix algebra of size $|G|$. In particular, $\mathsf{H}_{0,c}$ has a unique irreducible representation V_χ with central character χ. This representation is isomorphic to $\mathbb{C}G$ as a G-module.*

Proof. It is shown by a standard argument (which we will skip) that it is sufficient to check the statement in the associated graded case $c = 0$. In this case, for generic χ, $G \ltimes SV/\langle\chi\rangle = G \ltimes \mathrm{F}un(\mathcal{O}_\chi)$, where $\mathcal{O}_\chi$ is the (free) orbit of G consisting of the points of V^* that map to $\chi \in V^*/G$, and $\mathrm{F}un(\mathcal{O}_\chi)$ is the algebra of functions on $\mathcal{O}_\chi$. It is easy to see that this algebra is isomorphic to a matrix algebra, and has a unique irreducible representation, $\mathrm{F}un(\mathcal{O}_\chi)$, which is a regular representation of G. ☐

Corollary 8.25. *Any irreducible representation of $\mathsf{H}_{0,c}$ has dimension $\leq |G|$.*

Proof. We will use the following lemma.

Lemma 8.26 (The Amitsur-Levitzki identity)**.** *For any $N \times N$ matrices $X_1, \ldots, X_{2N}$ with entries in a commutative ring A,*

$$\sum_{\sigma \in \mathfrak{S}_{2n}} (-1)^\sigma X_{\sigma(1)} \cdots X_{\sigma(2N)} = 0.$$

Proof. Consider the ring $\mathrm{Mat}_N(A) \otimes \wedge(\xi_1, \ldots, \xi_{2n})$. Let $X = \sum_i X_i \xi_i \in R$. So we have

$$X^2 = \sum_{i<j} [X_i, X_j]\xi_i\xi_j \in \mathrm{Mat}_N(A \otimes \wedge^{\mathrm{even}}(\xi_1, \ldots, \xi_{2n})).$$

It is obvious that $\mathrm{Tr}\, X^2 = 0$. Similarly, one can easily show that $\mathrm{Tr}\, X^4 = 0, \ldots, \mathrm{Tr}\, X^{2N} = 0$. Since the ring $A \otimes \wedge^{\mathrm{even}}(\xi_1, \ldots, \xi_{2n})$ is commutative, from the Cayley-Hamilton theorem, we know that $X^{2N} = 0$ which implies the lemma. ☐

Since for generic χ the algebra $\mathsf{H}_{0,c}/\langle\chi\rangle$ is a matrix algebra, the algebra $\mathsf{H}_{0,c}$ satisfies the Amitsur-Levitzki identity. Next, note that since $\mathsf{H}_{0,c}$ is a finitely generated $\mathsf{Z}_{0,c}$-module (by passing to the associated graded and using Hilbert's theorem), every irreducible representation of $\mathsf{H}_{0,c}$ is finite dimensional. If $\mathsf{H}_{0,c}$ had an irreducible representation E of dimension $m > |G|$, then by the density theorem the matrix algebra Mat_m would be a quotient of $\mathsf{H}_{0,c}$. But one can show that the Amitsur-Levitzki identity of degree $|G|$ is not satisfied for matrices of bigger size than $|G|$. Contradiction. Thus, $\dim E \leq |G|$, as desired. ☐

In general, for special central characters there are representations of $\mathsf{H}_{0,c}$ of dimension less than $|G|$. However, in some cases one can show that all irreducible representations have dimension exactly $|G|$. For example, we have the following result.

Theorem 8.27. *Let $G = \mathfrak{S}_n$, $V = \mathfrak{h} \oplus \mathfrak{h}^*$, $\mathfrak{h} = \mathbb{C}^n$ (the rational Cherednik algebra for $\mathfrak{S}_n$). Then for $c \neq 0$, every irreducible representation of $\mathsf{H}_{0,c}$ has dimension $n!$ and is isomorphic to the regular representation of $\mathfrak{S}_n$.*

Proof. Let E be an irreducible representation of $\mathsf{H}_{0,c}$. Let us calculate the trace in E of any permutation $\sigma \neq 1$. Let j be an index such that $\sigma(j) = i \neq j$. Then $s_{ij}\sigma(j) = j$. Hence in $\mathsf{H}_{0,c}$ we have

$$[y_j, x_i s_{ij}\sigma] = [y_j, x_i]s_{ij}\sigma = cs_{ij}^2\sigma = c\sigma.$$

Hence $\mathrm{Tr}_E(\sigma) = 0$, and thus E is a multiple of the regular representation of $\mathfrak{S}_n$. But by Theorem 8.25, $\dim E \leq n!$, so we get that E is the regular representation, as desired. □

8.10. **Azumaya algebras.** Let Z be a finitely generated commutative algebra over $\mathbb{C}$, $M = \mathrm{Spec}Z$ the corresponding affine scheme, and A a finitely generated Z-algebra.

Definition 8.28. A is said to be an Azumaya algebra of degree N if the completion $\hat{A}_\chi$ of A at every maximal ideal χ in Z is the matrix algebra of size N over the completion $\hat{Z}_\chi$ of Z.

Thus, an Azumaya algebra should be thought of as a bundle of matrix algebras on M. [5] For example, if E is an algebraic vector bundle on M then $\mathrm{End}(E)$ is an Azumaya algebra. However, not all Azumaya algebras are of this form.

Example 8.29. For $q \in \mathbb{C}^*$, consider the quantum torus

$$\mathbb{T}_q = \mathbb{C}\langle X^{\pm 1}, Y^{\pm 1}\rangle / \langle XY - qYX\rangle.$$

If q is a root of unity of order N, then the center of $\mathbb{T}_q$ is $\langle X^{\pm N}, Y^{\pm N}\rangle = \mathbb{C}[M]$ where $M = (\mathbb{C}^*)^2$. It is not difficult to show that $\mathbb{T}_q$ is an Azumaya algebra of degree N, but $\mathbb{T}_q \otimes_{\mathbb{C}[M]} \mathbb{C}(M) \not\cong \mathrm{Mat}_N(\mathbb{C}(M))$, so $\mathbb{T}_q$ is not the endomorphism algebra of a vector bundle.

Example 8.30. Let X be a smooth irreducible variety over a field of characteristic p. Then $\mathcal{D}(X)$, the algebra of differential operators on X, is an Azumaya algebra with rank $p^{\dim X}$, which is not an endomorphism algebra of a vector bundle. Its center is $Z = \mathcal{O}(T^*X)^{\mathbb{F}}$, the Frobenius twisted functions on T^*X.

It is clear that if A is an Azumaya algebra (say, over $\mathbb{C}$) then for every central character χ of A, $A/\langle\chi\rangle$ is the algebra $\mathrm{Mat}_N(\mathbb{C})$ of complex N by N matrices, and every irreducible representation of A has dimension N.

The following important result is due to M. Artin.

Theorem 8.31. *Let A be a finitely generated (over $\mathbb{C}$) polynomial identity (PI) algebra of degree N (i.e. all the polynomial relations of the matrix algebra of size N are satisfied in A). Then A is an Azumaya algebra if and only if every irreducible representation of A has dimension exactly N.*

Proof. See [Ar] Theorem 8.3. □

[5] If M is not affine, one can define, in a standard manner, the notion of a sheaf of Azumaya algebras on M.

Thus, by Theorem 8.27, for $G = \mathfrak{S}_n$, the rational Cherednik algebra $H_{0,c}(\mathfrak{S}_n, \mathbb{C}^n)$ for $c \neq 0$ is an Azumaya algebra of degree $n!$. Indeed, this algebra is PI of degree $n!$ because the classical Dunkl representation embeds it into matrices of size $n!$ over $\mathbb{C}(x_1, \ldots, x_n, p_1, \ldots, p_n)^{\mathfrak{S}_n}$.

Let us say that $\chi \in M$ is an Azumaya point if for some affine neighborhood U of χ the localization of A to U is an Azumaya algebra. Obviously, the set $\mathrm{Az}(M)$ of Azumaya points of M is open.

Now we come back to the study the space M_c corresponding to a symplectic reflection algebra $\mathsf{H}_{0,c}$.

Theorem 8.32. *The set* $\mathrm{Az}(\mathsf{M}_c)$ *coincides with the set of smooth points of* M_c*.*

The proof of this theorem is given in the following two subsections.

Corollary 8.33. *If* $G = \mathfrak{S}_n$ *and* $V = \mathfrak{h} \oplus \mathfrak{h}^*$, $\mathfrak{h} = \mathbb{C}^n$ *(the rational Cherednik algebra case) then* M_c *is a smooth algebraic variety for* $c \neq 0$*.*

Proof. Directly from the above theorem. □

8.11. **Cohen-Macaulay property and homological dimension.** To prove Theorem 8.32, we will need some commutative algebra tools. Let Z be a finitely generated commutative algebra over $\mathbb{C}$ without zero divisors. By Noether's normalization lemma, there exist elements $z_1, \ldots, z_n \in Z$ which are algebraically independent, such that Z is a finitely generated module over $\mathbb{C}[z_1, \ldots, z_n]$.

Definition 8.34. The algebra Z (or the variety $\mathrm{Spec} Z$) is said to be Cohen-Macaulay if Z is a locally free (=projective) module over $\mathbb{C}[z_1, \ldots, z_n]$. [6]

Remark 8.35. It was shown by Serre that if Z is locally free over $\mathbb{C}[z_1, \ldots, z_n]$ for some choice of $z_1, \ldots, z_n$, then it happens for any choice of them (such that Z is finitely generated as a module over $\mathbb{C}[z_1, \ldots, z_n]$).

Remark 8.36. Another definition of the Cohen-Macaulay property is that the dualizing complex $\omega_Z^\bullet$ of Z is concentrated in degree zero. We will not discuss this definition here.

It can be shown that the Cohen-Macaulay property is stable under localization. Therefore, it makes sense to make the following definition.

Definition 8.37. An algebraic variety X is Cohen-Macaulay if the algebra of functions on every affine open set in X is Cohen-Macaulay.

Let Z be a finitely generated commutative algebra over $\mathbb{C}$ without zero divisors, and let M be a finitely generated module over Z.

Definition 8.38. M is said to be Cohen-Macaulay if for some algebraically independent $z_1, \ldots, z_n \in Z$ such that Z is finitely generated over $\mathbb{C}[z_1, \ldots, z_n]$, M is locally free over $\mathbb{C}[z_1, \ldots, z_n]$.

Again, if this happens for some $z_1, \ldots, z_n$, then it happens for any of them. We also note that M can be Cohen-Macaulay without Z being Cohen-Macaulay, and that Z is a Cohen-Macaulay algebra iff it is a Cohen-Macaulay module over itself.

We will need the following standard properties of Cohen-Macaulay algebras and modules.

[6] It was proved by Quillen that a locally free module over a polynomial algebra is free; this is a difficult theorem, which will not be needed here.

Theorem 8.39. (i) *Let $Z_1 \subset Z_2$ be a finite extension of finitely generated commutative $\mathbb{C}$-algebras, without zero divisors, and M be a finitely generated module over Z_2. Then M is Cohen-Macaulay over Z_2 iff it is Cohen-Macaulay over Z_1.*

(ii) *Suppose that Z is the algebra of functions on a smooth affine variety. Then a Z-module M is Cohen-Macaulay if and only if it is projective.*

Proof. The proof can be found in the text book [Ei]. □

In particular, this shows that the algebra of functions on a smooth affine variety is Cohen-Macaulay. Algebras of functions on many singular varieties are also Cohen-Macaulay.

Example 8.40. The algebra of regular functions on the cone $xy = z^2$ is Cohen-Macaulay. This algebra can be identified as $\mathbb{C}[a, b]^{\mathbb{Z}_2}$ by letting $x = a^2, y = b^2$ and $z = ab$, where the $\mathbb{Z}_2$ action is defined by $a \mapsto -a$, $b \mapsto -b$. It contains a subalgebra $\mathbb{C}[a^2, b^2]$, and as a module over this subalgebra, it is free of rank 2 with generators 1, ab.

Example 8.41. Any irreducible affine algebraic curve is Cohen-Macaulay. For example, the algebra of regular functions on $y^2 = x^3$ is isomorphic to the subalgebra of $\mathbb{C}[t]$ spanned by $1, t^2, t^3, \ldots$. It contains a subalgebra $\mathbb{C}[t^2]$ and as a module over this subalgebra, it is free of rank 2 with generators 1, t^3.

Example 8.42. Consider the subalgebra in $\mathbb{C}[x, y]$ spanned by 1 and $x^i y^j$ with $i+j \geq 2$. It is a finite generated module over $\mathbb{C}[x^2, y^2]$, but not free. So this algebra is not Cohen-Macaulay.

Another tool we will need is homological dimension. We will say that an algebra A has homological dimension $\leq d$ if any (left) A-module M has a projective resolution of length $\leq d$. The homological dimension of A is the smallest integer having this property. If such an integer does not exist, A is said to have infinite homological dimension.

It is easy to show that the homological dimension of A is $\leq d$ if and only if for any A-modules M, N one has $\mathrm{Ext}^i(M, N) = 0$ for $i > d$. Also, the homological dimension clearly does not decrease under taking associated graded of the algebra under a positive filtration (this is clear from considering the spectral sequence attached to the filtration).

It follows immediately from this definition that homological dimension is Morita invariant. Namely, recall that a Morita equivalence between algebras A and B is an equivalence of categories A-mod $\to$ B-mod. Such an equivalence maps projective modules to projective ones, since projectivity is a categorical property (P is projective if and only if the functor $\mathrm{Hom}(P, \cdot)$ is exact). This implies that if A and B are Morita equivalent then their homological dimensions are the same.

Then we have the following important theorem.

Theorem 8.43. *The homological dimension of a commutative finitely generated $\mathbb{C}$-algebra Z is finite if and only if Z is regular, i.e. is the algebra of functions on a smooth affine variety.*

8.12. **Proof of Theorem 8.32.** First let us show that any smooth point χ of M_c is an Azumaya point. Since $\mathsf{H}_{0,c} = \mathrm{End}_{\mathsf{B}_{0,c}} \mathsf{H}_{0,c}\mathsf{e} = \mathrm{End}_{\mathsf{Z}_{0,c}}(\mathsf{H}_{0,c}\mathsf{e})$, it is sufficient to show that the coherent sheaf on M_c corresponding to the module $\mathsf{H}_{0,c}\mathsf{e}$ is a vector bundle near χ. By Theorem 8.39 (ii), for this it suffices to show that $\mathsf{H}_{0,c}\mathsf{e}$ is a Cohen-Macaulay $\mathsf{Z}_{0,c}$-module.

To do so, first note that the statement is true for $c = 0$. Indeed, in this case the claim is that SV is a Cohen-Macaulay module over $(SV)^G$. But SV is a polynomial algebra, which is Cohen-Macaulay, so the result follows from Theorem 8.39, (i).

Now, we claim that if Z, M are positively filtered and $\mathrm{gr}M$ is a Cohen-Macaulay $\mathrm{gr}Z$-module then M is a Cohen-Macaulay Z-module. Indeed, let $z_1, \ldots, z_n$ be homogeneous algebraically independent elements of $\mathrm{gr}Z$ such that $\mathrm{gr}Z$ is a finite module over the subalgebra generated by them. Let $z_1', \ldots, z_n'$ be their liftings to Z. Then $z_1', \ldots, z_n'$ are algebraically independent, and the module M over $\mathbb{C}[z_1', \ldots, z_n']$ is finitely generated and (locally) free since so is the module $\mathrm{gr}M$ over $\mathbb{C}[z_1, \ldots, z_n]$.

Recall now that $\mathrm{gr}\mathsf{H}_{0,c}\mathsf{e} = SV$, $\mathrm{gr}\mathsf{Z}_{0,c} = (SV)^G$. Thus the $c = 0$ case implies the general case, and we are done.

Now let us show that any Azumaya point of M_c is smooth. Let U be an affine open set in M_c consisting of Azumaya points. Then the localization $\mathsf{H}_{0,c}(U) := \mathsf{H}_{0,c} \otimes_{\mathsf{Z}_{0,c}} \mathcal{O}_U$ is an Azumaya algebra. Moreover, for any $\chi \in U$, the unique irreducible representation of $\mathsf{H}_{0,c}(U)$ with central character χ is the regular representation of G (since this holds for generic χ by Proposition 8.24). This means that e is a rank 1 idempotent in $\mathsf{H}_{0,c}(U)/\langle\chi\rangle$ for all χ. In particular, $\mathsf{H}_{0,c}(U)\mathsf{e}$ is a vector bundle on U. Thus the functor $F : \mathcal{O}_U\text{-mod} \to \mathsf{H}_{0,c}(U)\text{-mod}$ given by the formula $F(Y) = \mathsf{H}_{0,c}(U)\mathsf{e} \otimes_{\mathcal{O}_U} Y$ is an equivalence of categories (the quasi-inverse functor is given by the formula $F^{-1}(N) = \mathsf{e}N$). Thus $\mathsf{H}_{0,c}(U)$ is Morita equivalent to $\mathcal{O}_U$, and therefore their homological dimensions are the same.

On the other hand, the homological dimension of $\mathsf{H}_{0,c}$ is finite (in fact, it equals to $\dim V$). To show this, note that by the Hilbert syzygies theorem, the homological dimension of SV is $\dim V$. Hence, so is the homological dimension of $G \ltimes SV$ (as $\mathrm{Ext}^*_{G\ltimes SV}(M, N) = \mathrm{Ext}^*_{SV}(M, N)^G)$. Thus, since $\mathrm{gr}\mathsf{H}_{0,c} = G \ltimes SV$, we get that $\mathsf{H}_{0,c}$ has homological dimension $\le \dim V$. Hence, the homological dimension of $\mathsf{H}_{0,c}(U)$ is also $\le \dim V$ (as the homological dimension clearly does not increase under the localization). But $\mathsf{H}_{0,c}(U)$ is Morita equivalent to $\mathcal{O}_U$, so $\mathcal{O}_U$ has a finite homological dimension. By Theorem 8.43, this implies that U consists of smooth points.

Corollary 8.44. $\mathrm{Az}(\mathsf{M}_c)$ *is also the set of points at which the Poisson structure of* M_c *is symplectic.*

Proof. The variety M_c is symplectic outside of a subset of codimension 2, because so is M_0. Thus the set S of smooth points of M_c where the top exterior power of the Poisson bivector vanishes is of codimension ≥ 2. Since the top exterior power of the Poisson bivector is locally a regular function, this implies that S is empty. Thus, every smooth point is symplectic, and the corollary follows from the theorem. ☐

8.13. **Notes.** Our exposition in this section follows Section 8 – Section 10 of [E4].

9. Calogero-Moser spaces

9.1. **Hamiltonian reduction along an orbit.** Let $\mathcal{M}$ be an affine algebraic variety and G a reductive algebraic group. Suppose $\mathcal{M}$ is Poisson and the action of G preserves the Poisson structure. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ the dual of $\mathfrak{g}$. Let $\mu : \mathcal{M} \to \mathfrak{g}^*$ be a moment map for this action (we assume it exists). It induces a map $\mu^* : S\mathfrak{g} \to \mathbb{C}[\mathcal{M}]$.

Let $\mathcal{O}$ be a closed coadjoint orbit of G, $I_{\mathcal{O}}$ be the ideal in $S\mathfrak{g}$ corresponding to $\mathcal{O}$, and let $J_{\mathcal{O}}$ be the ideal in $\mathbb{C}[\mathcal{M}]$ generated by $\mu^*(I_{\mathcal{O}})$. Then $J_{\mathcal{O}}^G$ is a Poisson ideal in $\mathbb{C}[\mathcal{M}]^G$, and $A = \mathbb{C}[\mathcal{M}]^G / J_{\mathcal{O}}^G$ is a Poisson algebra.

Geometrically, $\mathrm{Spec}(A) = \mu^{-1}(\mathcal{O})/G$ (categorical quotient). It can also be written as $\mu^{-1}(z)/G_z$, where $z \in \mathcal{O}$ and G_z is the stabilizer of z in G.

Definition 9.1. The scheme $\mu^{-1}(\mathcal{O})/G$ is called *the Hamiltonian reduction of $\mathcal{M}$ with respect to G along* $\mathcal{O}$. We will denote by $R(\mathcal{M}, G, \mathcal{O})$.

The following proposition is standard.

Proposition 9.2. *If $\mathcal{M}$ is a symplectic variety and the action of G on $\mu^{-1}(\mathcal{O})$ is free, then $R(\mathcal{M}, G, \mathcal{O})$ is a symplectic variety, of dimension* $\dim(\mathcal{M}) - 2\dim(G) + \dim(\mathcal{O})$.

9.2. **The Calogero-Moser space.** Let $\mathcal{M} = T^*\mathrm{M}at_n(\mathbb{C})$, and $G = \mathrm{PGL}_n(\mathbb{C})$ (so $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$). Using the trace form we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$, and $\mathcal{M}$ with $\mathrm{M}at_n(\mathbb{C}) \oplus \mathrm{M}at_n(\mathbb{C})$. Then a moment map is given by the formula $\mu(X, Y) = [X, Y]$, for $X, Y \in \mathrm{M}at_n(\mathbb{C})$.

Let $\mathcal{O}$ be the orbit of the matrix $\mathrm{diag}(-1, -1, \ldots, -1, n-1)$, i.e. the set of traceless matrices T such that $T + 1$ has rank 1.

Definition 9.3 (Kazhdan, Kostant, Sternberg, [KKS])**.** The scheme $\mathcal{C}_n := R(\mathcal{M}, G, \mathcal{O})$ is called *the Calogero-Moser space.*

Proposition 9.4. *The action of G on $\mu^{-1}(\mathcal{O})$ is free, and thus (by Proposition 9.2) $\mathcal{C}_n$ is a smooth symplectic variety (of dimension $2n$).*

Proof. It suffices to show that if X, Y are such that $XY - YX + 1$ has rank 1, then (X, Y) is an irreducible set of matrices. Indeed, in this case, by Schur's lemma, if $B \in \mathrm{GL}_n$ is such that $BX = XB$ and $BY = YB$ then B is a scalar, so the stabilizer of (X, Y) in PGL_n is trivial.

To show this, assume that $\mathcal{W} \neq 0$ is an invariant subspace of X, Y. In this case, the eigenvalues of $[X, Y]$ on $\mathcal{W}$ are a subcollection of the collection of $n-1$ copies of -1 and one copy of $n-1$. The sum of the elements of this subcollection must be zero, since it is the trace of $[X, Y]$ on $\mathcal{W}$. But then the subcollection must be the entire collection, so $\mathcal{W} = \mathbb{C}^n$, as desired. □

Thus, $\mathcal{C}_n$ is the space of conjugacy classes of pairs of $n \times n$ matrices (X, Y) such that the matrix $XY - YX + 1$ has rank 1.

In fact, one also has the following more complicated theorem.

Theorem 9.5 (G. Wilson, [Wi])**.** *The Calogero-Moser space is connected.*

We will give a proof of this theorem later, in Subsection 9.4.

9.3. **The Calogero-Moser integrable system.** Let $\mathcal{M}$ be a symplectic variety, and let $H_1, \ldots, H_n$ be regular functions on $\mathcal{M}$ such that $\{H_i, H_j\} = 0$ and H_i's are algebraically independent everywhere. Assume that $\mathcal{M}$ carries a symplectic action of a reductive algebraic group G with moment map $\mu : \mathcal{M} \to \mathfrak{g}^*$, which preserves the functions H_i, and let $\mathcal{O}$ be a coadjoint orbit of G. Assume that G acts freely on $\mu^{-1}(\mathcal{O})$, and so the Calogero-Moser space $R(\mathcal{M}, G, \mathcal{O})$ is symplectic. The functions H_i descend to $R(\mathcal{M}, G, \mathcal{O})$. If they are still algebraically independent and $n = \dim R(\mathcal{M}, G, \mathcal{O})/2$, then we get an integrable system on $R(\mathcal{M}, G, \mathcal{O})$.

A vivid example of this is the Kazhdan-Kostant-Sternberg construction of the Calogero-Moser system. In this case $\mathcal{M} = T^*\mathrm{Mat}_n(\mathbb{C})$ (regarded as the set of pairs of matrices (X, Y) as in Section 9.2), with the usual symplectic form $\omega = \mathrm{Tr}\,(\mathrm{d}Y \wedge \mathrm{d}X)$. Let $H_i = \mathrm{Tr}\,(Y^i)$, $i = 1, \ldots, n$. Let $G = \mathrm{PGL}_n(\mathbb{C})$ act on $\mathcal{M}$ by conjugation, and let $\mathcal{O}$ be the coadjoint orbit of G considered in Subsection 9.2. Then the system $H_1, \ldots, H_n$ descends to a system of functions in involution on $R(\mathcal{M}, G, \mathcal{O})$, which is the Calogero-Moser space $\mathcal{C}_n$. Since this space is $2n$-dimensional, $H_1, \ldots, H_n$ form an integrable system on $\mathcal{C}_n$. It is called *the (rational) Calogero-Moser system.*

The Calogero-Moser flow is, by definition, the Hamiltonian flow on $\mathcal{C}_n$ defined by the Hamiltonian $H = H_2 = \mathrm{Tr}\,(Y^2)$. Thus this flow is integrable, in the sense that it can be included in an integrable system. In particular, its solutions can be found in quadratures using the inductive procedure of reduction of order. However (as often happens with systems obtained by reduction), solutions can also be found by a much simpler procedure, since they can be found already on the "non-reduced" space $\mathcal{M}$: indeed, on $\mathcal{M}$ the Calogero-Moser flow is just the motion of a free particle in the space of matrices, so it has the form $g_t(X, Y) = (X + 2Yt, Y)$. The same formula is valid on $\mathcal{C}_n$. In fact, we can use the same method to compute the flows corresponding to all the Hamiltonians $H_i = \mathrm{Tr}\,(Y^i)$, $i \in \mathbb{N}$: these flows are given by the formulas

$$g_t^{(i)}(X, Y) = (X + iY^{i-1}t, Y).$$

Let us write the Calogero-Moser system explicitly in coordinates. To do so, we first need to introduce local coordinates on the Calogero-Moser space $\mathcal{C}_n$.

To this end, let us restrict our attention to the open set $U_n \subset \mathcal{C}_n$ which consists of conjugacy classes of those pairs (X, Y) for which the matrix X is diagonalizable, with distinct eigenvalues; by Wilson's Theorem 9.5, this open set is dense in $\mathcal{C}_n$.

A point $P \in U_n$ may be represented by a pair (X, Y) such that $X = \mathrm{diag}(x_1, \ldots, x_n)$, $x_i \neq x_j$. In this case, the entries of $T := XY - YX$ are $(x_i - x_j)y_{ij}$. In particular, the diagonal entries are zero. Since the matrix $T + 1$ has rank 1, its entries κ_{ij} have the form $a_i b_j$ for some numbers a_i, b_j. On the other hand, $\kappa_{ii} = 1$, so $b_j = a_j^{-1}$ and hence $\kappa_{ij} = a_i a_j^{-1}$. By conjugating (X, Y) by the matrix $\mathrm{diag}(a_1, \ldots, a_n)$, we can reduce to the situation when $a_i = 1$, so $\kappa_{ij} = 1$. Hence the matrix T has entries $1 - \delta_{ij}$ (zeros on the diagonal, ones off the diagonal). Moreover, the representative of P with diagonal X and T as above is unique up to the action of the symmetric group $\mathfrak{S}_n$. Finally, we have $(x_i - x_j)y_{ij} = 1$ for $i \neq j$, so the entries of the matrix Y are $y_{ij} = 1/(x_i - x_j)$ if $i \neq j$. On the other hand, the diagonal entries y_{ii} of Y are unconstrained. Thus we have obtained the following result.

Proposition 9.6. *Let $\mathbb{C}^n_{\mathrm{reg}}$ be the open set of $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that $x_i \neq x_j$ for $i \neq j$. Then there exists an isomorphism of algebraic varieties $\xi : T^*(\mathbb{C}^n_{\mathrm{reg}}/\mathfrak{S}_n) \to U_n$ given by the* *formula*

$$(x_1, \ldots, x_n, p_1, \ldots, p_n) \mapsto (X, Y),$$

where $X = \mathrm{diag}(x_1, \ldots, x_n)$, *and* $Y = Y(\mathbf{x}, \mathbf{p}) := (y_{ij})$,

$$y_{ij} = \frac{1}{x_i - x_j}, i \neq j, \ y_{ii} = p_i.$$

In fact, we have a stronger result:

Proposition 9.7. *ξ is an isomorphism of symplectic varieties (where the cotangent bundle is equipped with the usual symplectic structure).*

For the proof of Proposition 9.7, we will need the following general and important but easy theorem.

Theorem 9.8 (The necklace bracket formula)**.** *Let $a_1, \ldots, a_r$ and $b_1, \ldots, b_s$ be either X or Y. Then on $\mathcal{M}$ we have*

$$\begin{aligned}\{\mathrm{Tr}\,(a_1 \cdots a_r), \mathrm{Tr}\,(b_1 \cdots b_s)\} \quad = \quad & \sum_{(i,j): a_i = Y, b_j = X} \mathrm{Tr}\,(a_{i+1} \cdots a_r a_1 \cdots a_{i-1} b_{j+1} \cdots b_s b_1 \cdots b_{j-1}) - \\ & \sum_{(i,j): a_i = X, b_j = Y} \mathrm{Tr}\,(a_{i+1} \cdots a_r a_1 \cdots a_{i-1} b_{j+1} \cdots b_s b_1 \cdots b_{j-1}).\end{aligned}$$

Proof of Proposition 9.7. Let $a_k = \mathrm{Tr}\,(X^k)$, $b_k = \mathrm{Tr}\,(X^k Y)$. It is easy to check using the necklace bracket formula that on $\mathcal{M}$ we have

$$\{a_m, a_k\} = 0, \quad \{b_m, a_k\} = k a_{m+k-1}, \quad \{b_m, b_k\} = (k-m) b_{m+k-1}.$$

On the other hand, $\xi^* a_k = \sum x_i^k$, $\xi^* b_k = \sum x_i^k p_i$. Thus we see that

$$\{f, g\} = \{\xi^* f, \xi^* g\},$$

where f, g are either a_k or b_k. But the functions a_k, b_k, $k = 0, \ldots, n-1$, form a local coordinate system near a generic point of U_n, so we are done. $\square$

Now let us write the Hamiltonian of the Calogero-Moser system in coordinates. It has the form

$$H = \mathrm{Tr}\,(Y(\mathbf{x}, \mathbf{p})^2) = \sum_i p_i^2 - \sum_{i \neq j} \frac{1}{(x_i - x_j)^2}. \tag{9.1}$$

Thus the Calogero-Moser Hamiltonian describes the motion of a system of n particles on the line with interaction potential $-1/x^2$, which we considered in Section 2.

Now we finally see the usefulness of the Hamiltonian reduction procedure. The point is that it is not clear at all from formula (9.1) why the Calogero-Moser Hamiltonian should be completely integrable. However, our reduction procedure implies the complete integrability of H, and gives an explicit formula for the first integrals: [7]

$$H_i = \mathrm{Tr}\,(Y(\mathbf{x}, \mathbf{p})^i).$$

Moreover, this procedure immediately gives us an explicit solution of the system. Namely, assume that $\mathbf{x}(t), \mathbf{p}(t)$ is the solution with initial condition $\mathbf{x}(0), \mathbf{p}(0)$. Let $(X_0, Y_0) =$

[7]Thus, for type A we have two methods of proving the integrability of the Calogero-Moser system - one using Dunkl operators and one using Hamiltonian reduction.

$\xi(\mathbf{x}(0), \mathbf{p}(0))$. Then $x_i(t)$ are the eigenvalues of the matrix $X_t := X_0 + 2tY_0$, and $p_i(t) = x_i'(t)/2$.

9.4. **Proof of Wilson's theorem.** Let us now give a proof of Theorem 9.5.

We have already shown that all components of $\mathcal{C}_n$ are smooth and have dimension $2n$. Also, we know that there is at least one component (the closure of U_n), and that the other components, if they exist, do not contain pairs (X, Y) in which X is regular semisimple. This means that these components are contained in the hypersurface $\Delta(X) = 0$, where $\Delta(X)$ stands for the discriminant of X (i.e., $\Delta(X) := \prod_{i \neq j}(x_i - x_j)$, where x_i are the eigenvalues of X).

Thus, to show that such additional components don't in fact exist, it suffices to show that the dimension of the subscheme $\mathcal{C}_n(0)$ cut out in $\mathcal{C}_n$ by the equation $\Delta(X) = 0$ is $2n - 1$.

To do so, first notice that the condition $\mathrm{rank}\,([X, Y] + 1) = 1$ is equivalent to the equation $\wedge^2([X, Y] + 1) = 0$; thus, the latter can be used as the equation defining $\mathcal{C}_n$ inside $T^*\mathrm{Mat}_n/PGL_n$.

Define $\mathcal{C}_n^0 := \mathrm{Spec}(\mathrm{gr}\mathcal{O}(\mathcal{C}_n))$ to be the degeneration of $\mathcal{C}_n$ (we use the filtration on $\mathcal{O}(\mathcal{C}_n)$ defined by $\deg(X) = 0$, $\deg(Y) = 1$). Then $\mathcal{C}_n^0$ is a closed subscheme in the scheme $\widetilde{\mathcal{C}}_n^0$ cut out by the equations $\wedge^2([X, Y]) = 0$ in $T^*\mathrm{M}at_n/\mathrm{PGL}_n$.

Let $(\widetilde{\mathcal{C}}_n^0)_{red}$ be the reduced part of $\widetilde{\mathcal{C}}_n^0$. Then $(\widetilde{\mathcal{C}}_n^0)_{\mathrm{red}}$ coincides with the categorical quotient $\{(X, Y) | \mathrm{rank}\,([X, Y]) \leq 1\}/\mathrm{PGL}_n$.

Our proof is based on the following proposition.

Proposition 9.9. *The categorical quotient* $\{(X, Y) | \mathrm{rank}\,([X, Y]) \leq 1\}/\mathrm{PGL}_n$ *coincides with the categorical quotient* $\{(X, Y) | [X, Y] = 0\}/\mathrm{PGL}_n$.

Proof. It is clear that $\{(X, Y) | [X, Y] = 0\}/\mathrm{PGL}_n$ is contained in $\{(X, Y) | \mathrm{rank}\,([X, Y]) \leq 1\}/\mathrm{PGL}_n$. For the proof of the opposite inclusion we need to show that any regular function on $\{(X, Y) | \mathrm{rank}\,([X, Y]) \leq 1\}/\mathrm{PGL}_n$ is completely determined by its values on the subvariety $\{(X, Y) | [X, Y] = 0\}/\mathrm{PGL}_n$, i.e. that any invariant polynomial on the set of pairs of matrices with commutator of rank at most 1 is completely determined by its values on pairs of commuting matrices. To this end, we need the following lemma from linear algebra.

Lemma 9.10. *If A, B are square matrices such that $[A, B]$ has rank ≤ 1, then there exists a basis in which both A, B are upper triangular.*

Proof. Without loss of generality, we can assume $\ker A \neq 0$ (by replacing A with $A - \lambda$ if needed) and that $A \neq 0$. It suffices to show that there exists a proper nonzero subspace invariant under A, B; then the statement will follow by induction in dimension.

Let $C = [A, B]$ and suppose $\mathrm{rank}\, C = 1$ (since the case $\mathrm{rank}\, C = 0$ is trivial). If $\ker A \subset \ker C$, then $\ker A$ is B-invariant: if $Av = 0$ then $ABv = BAv + Cv = 0$. Thus $\ker A$ is the required subspace. If $\ker A \not\subseteq \ker C$, then there exists a vector v such that $Av = 0$ but $Cv \neq 0$. So $ABv = Cv \neq 0$. Thus $\mathrm{Im}C \subset \mathrm{Im}A$. So $\mathrm{Im}A$ is B-invariant: $BAv = ABv + Cv \in \mathrm{Im}A$. So $\mathrm{Im}A$ is the required subspace.

This proves the lemma. □

Now we are ready to prove Proposition 9.9. By the fundamental theorem of invariant theory, the ring of invariants of X and Y is generated by traces of words of X and Y: $\mathrm{Tr}(w(X, Y))$. If X and Y are upper triangular with eigenvalues x_i, y_i, then any such trace has the form $\sum x_i^m y_i^r$, i.e. coincides with the value of the corresponding invariant on the diagonal parts $X_{\rm diag}, Y_{\rm diag}$ of X and Y, which commute. The proposition is proved. □

We will also need the following proposition:

Proposition 9.11. *The categorical quotient $\{(X,Y)|[X,Y]=0\}/\mathrm{PGL}_n$ is isomorphic to $(\mathbb{C}^n\times\mathbb{C}^n)/\mathfrak{S}_n$, i.e. its function algebra is $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{\mathfrak{S}_n}$.*

Proof. Restriction to diagonal matrices defines a homomorphism

$$\xi:\mathcal{O}(\{(X,Y)|[X,Y]=0\}/\mathrm{PGL}_n)\to\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{\mathfrak{S}_n}.$$

Since (as explained in the proof of Proposition 9.9), any invariant polynomial of entries of commuting matrices is determined by its values on diagonal matrices, this map is injective. Also, $\xi(\mathrm{Tr}\,(X^mY^r))=\sum x_i^m y_i^r$, where x_i,y_i are the eigenvalues of X and Y.

Now we use the following well known theorem of H. Weyl (from his book "Classical groups").

Theorem 9.12. *Let B be an algebra over $\mathbb{C}$. Then the algebra S^nB is generated by elements of the form*

$$b\otimes 1\otimes\cdots\otimes 1+1\otimes b\otimes\cdots\otimes 1+\cdots+1\otimes 1\otimes\cdots\otimes b.$$

Proof. Since S^nB is linearly spanned by elements of the form $x\otimes\cdots\otimes x$, $x\in B$, it suffices to prove the theorem in the special case $B=\mathbb{C}[x]$. In this case, the result is simply the fact that the ring of symmetric functions is generated by power sums, which is well known. □

Corollary 9.13. *The ring $\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{\mathfrak{S}_n}$ is generated by the polynomials $\sum x_i^m y_i^r$ for $m,r\ge 0$, $m+r>0$.*

Proof. Apply Theorem 9.12 in the case $B=\mathbb{C}[x,y]$. □

Corollary 9.13 implies that ξ is surjective. Proposition 9.11 is proved. □

Now we are ready to prove Wilson's theorem. Let $\mathcal{C}_n(0)^0$ be the degeneration of $\mathcal{C}_n(0)$, i.e. the subscheme of $\mathcal{C}_n^0$ cut out by the equation $\Delta(X)=0$. According to Propositions 9.9 and 9.11, the reduced part $(\mathcal{C}_n(0)^0)_{\rm red}$ is contained in the hypersurface in $(\mathbb{C}^n\times\mathbb{C}^n)/\mathfrak{S}_n$ cut out by the equation $\prod_{i<j}(x_i-x_j)=0$. This hypersurface has dimension $2n-1$, so we are done.

9.5. **The Gan-Ginzburg theorem.** Let $\mathrm{Comm}(n)$ be *the commuting scheme* defined in $T^*\mathrm{Mat}_n=\mathrm{Mat}_n\times\mathrm{Mat}_n$ by the equations $[X,Y]=0$, $X,Y\in\mathrm{Mat}_n$. Let $G=\mathrm{PGL}_n$, and consider the categorical quotient $\mathrm{Comm}(n)/G$ (i.e., the Hamiltonian reduction $\mu^{-1}(0)/G$ of $T^*\mathrm{Mat}_n$ by the action of G), whose algebra of regular functions is $A=\mathbb{C}[\mathrm{Comm}(n)]^G$.

It is not known whether the commuting scheme $\mathrm{Comm}(n)$ is reduced (i.e. whether the corresponding ideal is a radical ideal); this is a well known open problem. The underlying variety is irreducible (as was shown by Gerstenhaber [Ge1]), but very singular, and has a very complicated structure. However, we have the following result.

Theorem 9.14 (Gan, Ginzburg, [GG])**.** *$\mathrm{Comm}(n)/G$ is reduced, and isomorphic to $\mathbb{C}^{2n}/\mathfrak{S}_n$. Thus $A=\mathbb{C}[x_1,\dots,x_n,y_1,\dots,y_n]^{\mathfrak{S}_n}$. The Poisson bracket on this algebra is induced from the standard symplectic structure on $\mathbb{C}^{2n}$.*

Sketch of the proof. Look at the *almost commuting variety* $\mathcal{M}_n \subset \mathfrak{gl}_n \times \mathfrak{gl}_n \times \mathbb{C}^n \times (\mathbb{C}^n)^*$ defined by

$$\mathcal{M}_n = \{(X, Y, \mathbf{v}, \mathbf{f}) | [X, Y] + \mathbf{v} \otimes \mathbf{f} = 0\}.$$

Gan and Ginzburg proved the following result.

Theorem 9.15. *$\mathcal{M}_n$ is a complete intersection. It has $n+1$ irreducible components denoted by $\mathcal{M}_n^i$, labeled by $i = \dim \mathbb{C}\langle X, Y\rangle \mathbf{v}$. Also, $\mathcal{M}_n$ is generically reduced.*

Since $\mathcal{M}_n$ is generically reduced and is a complete intersection, by a standard result of commutative algebra it is reduced. Thus $\mathbb{C}[\mathcal{M}_n]$ has no nonzero nilpotents. This implies $\mathbb{C}[\mathcal{M}_n]^G$ has no nonzero nilpotents.

However, it is easy to show that the algebra $\mathbb{C}[\mathcal{M}_n]^G$ is isomorphic to the algebra of invariant polynomials of entries of X and Y modulo the "rank 1" relation $\wedge^2[X, Y] = 0$. By a scheme-theoretic version of Proposition 9.9 (proved in [EG]), the latter is isomorphic to A. This implies the theorem (the statement about Poisson structures is checked directly in coordinates on the open part where X is regular semisimple). □

9.6. **The space M_c for $\mathfrak{S}_n$ and the Calogero-Moser space.** Let $\mathsf{H}_{0,c} = \mathsf{H}_{0,c}[\mathfrak{S}_n, V]$ be the symplectic reflection algebra of the symmetric group $\mathfrak{S}_n$ and space $V = \mathfrak{h} \oplus \mathfrak{h}^*$, where $\mathfrak{h} = \mathbb{C}^n$ (i.e., the rational Cherednik algebra $H_{0,c}(\mathfrak{S}_n, \mathfrak{h})$). Let $\mathsf{M}_c = \operatorname{Spec} \mathsf{B}_{0,c}[\mathfrak{S}_n, V]$ be the Calogero-Moser space defined in Section 8.5. It is a symplectic variety for $c \neq 0$.

Theorem 9.16. *For $c \neq 0$ the space M_c is isomorphic to the Calogero-Moser space $\mathcal{C}_n$ as a symplectic variety.*

Proof. To prove the theorem, we will first construct a map $f : \mathsf{M}_c \to \mathcal{C}_n$, and then prove that f is an isomorphism.

Without loss of generality, we may assume that $c = 1$. As we have shown before, the algebra $\mathsf{H}_{0,c}$ is an Azumaya algebra. Therefore, M_c can be regarded as the moduli space of irreducible representations of $\mathsf{H}_{0,c}$.

Let $E \in \mathsf{M}_c$ be an irreducible representation of $\mathsf{H}_{0,c}$. We have seen before that E has dimension $n!$ and is isomorphic to the regular representation as a representation of $\mathfrak{S}_n$. Let $\mathfrak{S}_{n-1} \subset \mathfrak{S}_n$ be the subgroup which preserves the element 1. Then the space of invariants $E^{\mathfrak{S}_{n-1}}$ has dimension n. On this space we have operators $X, Y : E^{\mathfrak{S}_{n-1}} \to E^{\mathfrak{S}_{n-1}}$ obtained by restriction of the operators x_1, y_1 on E to the subspace of invariants. We have

$$[X, Y] = T := \sum_{i=2}^{n} s_{1i}.$$

Let us now calculate the right hand side of this equation explicitly. Let e be the symmetrizer of $\mathfrak{S}_{n-1}$. Let us realize the regular representation E of $\mathfrak{S}_n$ as $\mathbb{C}[\mathfrak{S}_n]$ with action of $\mathfrak{S}_n$ by left multiplication. Then $v_1 = \mathsf{e}, v_2 = \mathsf{e}s_{12}, \ldots, v_n = \mathsf{e}s_{1n}$ is a basis of $E^{\mathfrak{S}_{n-1}}$. The element T commutes with e, so we have

$$T v_i = \sum_{j \neq i} v_j.$$

This means that $T+1$ has rank 1, and hence the pair (X, Y) defines a point on the Calogero-Moser space $\mathcal{C}_n$. [8]

[8]Note that the pair (X, Y) is well defined only up to conjugation, because the representation E is well defined only up to an isomorphism.

We now set $(X,Y)=f(E)$. It is clear that $f:\mathsf{M}_c\to\mathcal{C}_n$ is a regular map. So it remains to show that f is an isomorphism. This is equivalent to showing that the corresponding map of function algebras $f^*:\mathcal{O}(\mathcal{C}_n)\to\mathsf{B}_{0,c}$ is an isomorphism.

Let us calculate f and f^* more explicitly. To do so, consider the open set U in M_c consisting of representations in which x_i-x_j acts invertibly. These are exactly the representations that are obtained by restricting representations of $\mathfrak{S}_n\ltimes\mathbb{C}[x_1,\dots,x_n,p_1,\dots,p_n,\delta(\mathbf{x})^{-1}]$ using the classical Dunkl embedding. Thus representations $E\in\mathsf{U}$ are of the form $E=E_{\lambda,\mu}$ ($\lambda,\mu\in\mathbb{C}^n$, and λ has distinct coordinates), where $E_{\lambda,\mu}$ is the space of complex valued functions on the orbit $\mathcal{O}_{\lambda,\mu}\subset\mathbb{C}^{2n}$, with the following action of $\mathsf{H}_{0,c}$:

$$(x_iF)(\mathbf{a},\mathbf{b})=a_iF(\mathbf{a},\mathbf{b}),\quad (y_iF)(\mathbf{a},\mathbf{b})=b_iF(\mathbf{a},\mathbf{b})+\sum_{j\neq i}\frac{(s_{ij}F)(\mathbf{a},\mathbf{b})}{a_i-a_j}.$$

(the group $\mathfrak{S}_n$ acts by permutations).

Now let us consider the space $E_{\lambda,\mu}^{\mathfrak{S}_{n-1}}$. A basis of this space is formed by characteristic functions of $\mathfrak{S}_{n-1}$-orbits on $\mathcal{O}_{\lambda,\mu}$. Using the above presentation, it is straightforward to calculate the matrices of the operators X and Y in this basis:

$$X=\mathrm{diag}(\lambda_1,\dots,\lambda_n),$$

and

$$Y_{ij}=\mu_i \text{ if } j=i,\ Y_{ij}=\frac{1}{\lambda_i-\lambda_j}\text{ if } j\neq i.$$

This shows that f induces an isomorphism $f|_{\mathsf{U}}:\mathsf{U}\to U_n$, where U_n is the subset of $\mathcal{C}_n$ consisting of pairs (X,Y) for which X has distinct eigenvalues.

From this presentation, it is straigtforward that $f^*(\mathrm{Tr}\,(X^p))=x_1^p+\dots+x_n^p$ for every positive integer p. Also, f commutes with the natural $\mathrm{SL}_2(\mathbb{C})$-action on M_c and $\mathcal{C}_n$ (by $(X,Y)\to(aX+bY,cX+dY)$), so we also get $f^*(\mathrm{Tr}\,(Y^p))=y_1^p+\dots+y_n^p$, and

$$f^*(\mathrm{Tr}\,(X^pY))=\frac{1}{p+1}\sum_{m=0}^{p}\sum_i x_i^m y_i x_i^{p-m}.$$

Now, using the necklace bracket formula on $\mathcal{C}_n$ and the commutation relations of $\mathsf{H}_{0,c}$, we find, by a direct computation, that f^* preserves Poisson bracket on the elements $\mathrm{Tr}\,(X^p)$, $\mathrm{Tr}\,(X^qY)$. But these elements are a local coordinate system near a generic point, so it follows that f is a Poisson map. Since the algebra $\mathsf{B}_{0,c}$ is Poisson generated by $\sum x_i^p$ and $\sum y_i^p$ for all p, we get that f^* is a surjective map.

Also, f^* is injective. Indeed, by Wilson's theorem the Calogero-Moser space is connected, and hence the algebra $\mathcal{O}(\mathcal{C}_n)$ has no zero divisors, while $\mathcal{C}_n$ has the same dimension as M_c. This proves that f^* is an isomorphism, so f is an isomorphism. $\square$

9.7. **The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ and the Calogero-Moser space.** The Hilbert scheme $\mathrm{Hilb}_n(\mathbb{C}^2)$ is defined to be

$$\begin{aligned}&\mathrm{Hilb}_n(\mathbb{C}^2)=\{\text{ ideals } I\subset\mathbb{C}[x,y]|\mathrm{codim}I=n\}\\ =\ &\{(E,v)|E \text{ is a } \mathbb{C}[x,y]\text{-module of dimension } n,\ v \text{ is a cyclic vector of } E\}.\end{aligned}$$

The second equality can be easily seen from the short exact sequence

$$0\to I\to\mathbb{C}[x,y]\to E\to 0.$$

Let $S^{(n)}\mathbb{C}^2 = \underbrace{\mathbb{C}^2 \times \cdots \times \mathbb{C}^2}_{n \text{ times}}/\mathfrak{S}_n$, where $\mathfrak{S}_n$ acts by permutation. We have a natural map $\mathrm{Hilb}_n(\mathbb{C}^2) \to S^{(n)}\mathbb{C}^2$ which sends every ideal I to its zero set (with multiplicities). This map is called the *Hilbert-Chow map*.

Theorem 9.17 (Fogarty, [F])**.** (i) $\mathrm{Hilb}_n(\mathbb{C}^2)$ *is a smooth quasiprojective variety.*
(ii) *The Hilbert-Chow map* $\mathrm{Hilb}_n(\mathbb{C}^2) \to S^{(n)}\mathbb{C}^2$ *is projective. It is a resolution of singularities.*

Proof. Proof can be found in [Na]. □

Now consider the Calogero-Moser space $\mathcal{C}_n$ defined in Section 9.2.

Theorem 9.18 (see [Na])**.** *The Hilbert Scheme* $\mathrm{Hilb}_n(\mathbb{C}^2)$ *is* C^∞*-diffeomorphic to* $\mathcal{C}_n$*.*

Remark 9.19. More precisely there exists a family of algebraic varieties over $\mathbb{A}_1$, say X_t, $t \in \mathbb{A}_1$, such that X_t is isomorphic to $\mathcal{C}_n$ if $t \neq 0$ and X_0 is the Hilbert scheme; and also if we denote by $\overline{X_t}$ the deformation of $\mathbb{C}^{2n}/\mathfrak{S}_n$ into the Calogero-Moser space, then there exists a map $f_t : X_t \twoheadrightarrow \overline{X_t}$, such that for $t \neq 0$, f_t is an isomorphism and f_0 is the Hilbert-Chow map.

Remark 9.20. Consider the action of $G = \mathrm{PGL}_n$ on $T^*\mathrm{M}at_n$. As we have discussed, the corresponding moment map is $\mu(X,Y) = [X,Y]$, so $\mu^{-1}(0) = \{(X,Y)|[X,Y] = 0\}$ is the commuting variety. We can consider two kinds of quotient $\mu^{-1}(0)/G$ (i.e., of Hamiltonian reduction):

(1) The categorical quotient, i.e.,

$$\mathrm{Spec}(\mathbb{C}[x_{ij}, y_{ij}]/\langle [X,Y] = 0\rangle)^G \cong (\mathbb{C}^n \times \mathbb{C}^n)/\mathfrak{S}_n.$$

It is a reduced (by Gan-Ginzburg Theorem 9.14), affine but singular variety.

(2) The GIT quotient, in which the stability condition is that there exists a cyclic vector for X, Y. This quotient is $\mathrm{Hilb}_n(\mathbb{C}^2)$, which is smooth but not affine.

Both of these reductions are degenerations of the reduction along the orbit of matrices T such that $T+1$ has rank 1, which yields the space $\mathcal{C}_n$. This explains why Theorem 9.18 and the results mentioned in Remark 9.19 are natural to expect.

9.8. **The cohomology of $\mathcal{C}_n$.** We also have the following result describing the cohomology of $\mathcal{C}_n$ (and hence, by Theorem 9.18, of $\mathrm{Hilb}_n(\mathbb{C}^2)$). Define the *age filtration* for the symmetric group $\mathfrak{S}_n$ by setting

$$\mathrm{age}(\text{transposition}) = 1.$$

Then one can show that for any $\sigma \in \mathfrak{S}_n$, $\mathrm{age}(\sigma) = \mathrm{rank}\,(1-\sigma)|_{\text{reflection representation}}$. It is easy to see that $0 \leq \mathrm{age} \leq n-1$. Notice also that the age filtration can be defined for any Coxeter group.

Theorem 9.21 (Lehn-Sorger, Vasserot)**.** *The cohomology ring* $\mathrm{H}^*(\mathcal{C}_n, \mathbb{C})$ *lives in even degrees only and is isomorphic to* $\mathrm{gr}(\mathrm{C}enter(\mathbb{C}[\mathfrak{S}_n]))$ *under the age filtration (with doubled degrees).*

Proof. Let us sketch a noncommutative-algebraic proof of this theorem, given in [EG]. This proof is based on the following result.

Theorem 9.22 (Nest-Tsygan, [NT]). *If M is an affine symplectic variety and A is a quantization of M, then*

$$\mathrm{HH}^*(A[\hbar^{-1}], A[\hbar^{-1}]) \cong \mathrm{H}^*(M, \mathbb{C}((\hbar)))$$

as an algebra over $\mathbb{C}((\hbar))$.

Now, we know that the algebra $\mathsf{B}_{t,c}$ is a quantization of $\mathcal{C}_n$. Therefore by above theorem, the cohomology algebra of $\mathcal{C}_n$ is the cohomology of $\mathsf{B}_{t,c}$ (for generic t). But $\mathsf{B}_{t,c}$ is Morita equivalent to $\mathsf{H}_{t,c}$, so this cohomology is the same as the Hochschild cohomology of $\mathsf{H}_{t,c}$. However, the latter can be computed by using that $\mathsf{H}_{t,c}$ is given by generators and relations (by producing explicit representatives of cohomology classes and computing their product), which gives the result. □

9.9. **Notes.** Sections 9.1–9.6 follow Section 1, 2, 4 of [E4]; the parts about the Hilbert scheme and its relation to Calogero-Moser spaces follow the book [Na] (see also [GS]); the other parts follow the paper [EG].

10.1. **Quantum moment maps and quantum Hamiltonian reduction.** Now we would like to quantize the notion of a moment map. Let $\mathfrak{g}$ be a Lie algebra, and A be an associative algebra equipped with a $\mathfrak{g}$-action, i.e. a Lie algebra map $\phi : \mathfrak{g} \to \mathrm{D}erA$. A quantum moment map for (A, ϕ) is an associative algebra homomorphism $\mu : U(\mathfrak{g}) \to A$ such that for any $a \in \mathfrak{g}$, $b \in A$ one has $[\mu(a), b] = \phi(a)b$.

The space of $\mathfrak{g}$-invariants $A^{\mathfrak{g}}$, i.e. elements $b \in A$ such that $[\mu(a), b] = 0$ for all $a \in \mathfrak{g}$, is a subalgebra of A. Let $J \subset A$ be the left ideal generated by $\mu(a)$, $a \in \mathfrak{g}$. Then J is not a 2-sided ideal, but $J^{\mathfrak{g}} := J \cap A^{\mathfrak{g}}$ is a 2-sided ideal in $A^{\mathfrak{g}}$. Indeed, let $c \in A^{\mathfrak{g}}$, and $b \in J^{\mathfrak{g}}$, $b = \sum_i b_i\mu(a_i)$, $b_i \in A, a_i \in \mathfrak{g}$. Then $bc = \sum b_i\mu(a_i)c = \sum b_i c\mu(a_i) \in J^{\mathfrak{g}}$.

Thus, the algebra $A//\mathfrak{g} := A^{\mathfrak{g}}/J^{\mathfrak{g}}$ is an associative algebra, which is called the quantum Hamiltonian reduction of A with respect to the quantum moment map μ.

10.2. **The Levasseur-Stafford theorem.** In general, similarly to the classical case, it is rather difficult to compute the quantum reduction $A//\mathfrak{g}$. For example, in this subsection we will describe $A//\mathfrak{g}$ in the case when $A = \mathcal{D}(\mathfrak{g})$ is the algebra of differential operators on a reductive Lie algebra $\mathfrak{g}$, and $\mathfrak{g}$ acts on A through the adjoint action on itself. This description is a very nontrivial result of Levasseur and Stafford.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$, and W the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Let $\mathfrak{h}_{\mathrm{reg}}$ denote the set of regular points in $\mathfrak{h}$, i.e. the complement of the reflection hyperplanes. To describe $\mathcal{D}(\mathfrak{g})//\mathfrak{g}$, we will construct a homomorphism $\mathrm{HC} : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{D}(\mathfrak{h})^W$, called the Harish-Chandra homomorphism (as it was first constructed by Harish-Chandra). Recall that we have the classical Harish-Chandra isomorphism $\zeta : \mathbb{C}[\mathfrak{g}]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}]^W$, defined simply by restricting $\mathfrak{g}$-invariant functions on $\mathfrak{g}$ to the Cartan subalgebra $\mathfrak{h}$. Using this isomorphism, we can define an action of $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ on $\mathbb{C}[\mathfrak{h}]^W$, which is clearly given by W-invariant differential operators. However, these operators will, in general, have poles on the reflection hyperplanes. Thus we get a homomorphism $\mathrm{HC}' : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W$.

The homomorphism HC' is called the radial part homomorphism, as for example for $\mathfrak{g} = \mathfrak{su}(2)$ it computes the radial parts of rotationally invariant differential operators on $\mathbb{R}^3$ in spherical coordinates. This homomorphism is not yet what we want, since it does not actually land in $\mathcal{D}(\mathfrak{h})^W$ (the radial parts have poles).

Thus we define the Harish-Chandra homomorphism by twisting HC' by the discriminant $\delta(\mathbf{x}) = \prod_{\alpha>0}(\alpha, \mathbf{x})$ ($\mathbf{x} \in \mathfrak{h}$, and α runs over positive roots of $\mathfrak{g}$):

$$\mathrm{HC}(D) := \delta \circ \mathrm{HC}'(D) \circ \delta^{-1} \in \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W.$$

Theorem 10.1. (i) (*Harish-Chandra,* [HC]) *For any reductive* $\mathfrak{g}$*,* HC *lands in* $\mathcal{D}(\mathfrak{h})^W \subset \mathcal{D}(\mathfrak{h}_{\mathrm{reg}})^W$.

(ii) (*Levasseur-Stafford* [LS]) *The homomorphism* HC *defines an isomorphism* $\mathcal{D}(\mathfrak{g})//\mathfrak{g} = \mathcal{D}(\mathfrak{h})^W$.

Remark 10.2. (1) Part (i) of the theorem says that the poles magically disappear after conjugation by δ.

(2) Both parts of this theorem are quite nontrivial. The first part was proved by Harish-Chandra using analytic methods, and the second part by Levasseur and Stafford using the theory of $\mathcal{D}$-modules.

In the case $\mathfrak{g} = \mathfrak{gl}_n$, Theorem 10.1 is a quantum analog of Theorem 9.14. The remaining part of this subsection is devoted to the proof of Theorem 10.1 in this special case, using Theorem 9.14.

We start the proof with the following proposition, valid for any reductive Lie algebra.

Proposition 10.3. *If $D \in (S\mathfrak{g})^{\mathfrak{g}}$ is a differential operator with constant coefficients, then* $\mathrm{HC}(D)$ *is the W-invariant differential operator with constant coefficients on $\mathfrak{h}$, obtained from D via the classical Harish-Chandra isomorphism $\eta : (S\mathfrak{g})^{\mathfrak{g}} \to (S\mathfrak{h})^W$.*

Proof. Without loss of generality, we may assume that $\mathfrak{g}$ is simple.

Lemma 10.4. *Let D be the Laplacian $\Delta_{\mathfrak{g}}$ of $\mathfrak{g}$, corresponding to an invariant form. Then* $\mathrm{HC}(D)$ *is the Laplacian $\Delta_{\mathfrak{h}}$.*

Proof. Let us calculate $\mathrm{HC}'(D)$. We have

$$D = \sum_{i=1}^{r} \partial_{x_i}^2 + 2\sum_{\alpha>0} \partial_{f_\alpha}\partial_{e_\alpha},$$

where x_i is an orthonormal basis of $\mathfrak{h}$, and e_α, f_α are root elements such that $(e_\alpha, f_\alpha) = 1$. Thus if $F(\mathbf{x})$ is a $\mathfrak{g}$-invariant function on $\mathfrak{g}$, then we get

$$(DF)|_{\mathfrak{h}} = \sum_{i=1}^{r} \partial_{x_i}^2 (F|_{\mathfrak{h}}) + 2\sum_{\alpha>0} (\partial_{f_\alpha}\partial_{e_\alpha} F)|_{\mathfrak{h}}.$$

Now let $\mathbf{x} \in \mathfrak{h}$, and consider $(\partial_{f_\alpha}\partial_{e_\alpha} F)(\mathbf{x})$. We have

$$(\partial_{f_\alpha}\partial_{e_\alpha} F)(\mathbf{x}) = \partial_s\partial_t|_{s=t=0} F(\mathbf{x} + tf_\alpha + se_\alpha).$$

On the other hand, we have

$$\mathrm{Ad}(\mathbf{e}^{s\alpha(\mathbf{x})^{-1}e_\alpha})(\mathbf{x} + tf_\alpha + se_\alpha) = \mathbf{x} + tf_\alpha + ts\alpha(\mathbf{x})^{-1}h_\alpha + \cdots,$$

where $h_\alpha = [e_\alpha, f_\alpha]$. Hence, $\partial_s\partial_t|_{s=t=0} F(\mathbf{x}+tf_\alpha+se_\alpha) = \alpha(\mathbf{x})^{-1}(\partial_{h_\alpha}F)(\mathbf{x})$. This implies that

$$\mathrm{HC}'(D)F(\mathbf{x}) = \Delta_{\mathfrak{h}} F(\mathbf{x}) + 2\sum_{\alpha>0} \alpha(\mathbf{x})^{-1}\partial_{h_\alpha} F(\mathbf{x}).$$

Now the statement of the Lemma reduces to the identity $\delta^{-1}\circ\Delta_{\mathfrak{h}}\circ\delta = \Delta_{\mathfrak{h}} + 2\sum_{\alpha>0}\alpha(\mathbf{x})^{-1}\partial_{h_\alpha}$. This identity follows immediately from the identity $\Delta_{\mathfrak{h}}\delta = 0$. To prove the latter, it suffices to note that δ is the lowest degree nonzero polynomial on $\mathfrak{h}$, which is antisymmetric under the action of W. The lemma is proved. □

Now let D be any element of $(S\mathfrak{g})^{\mathfrak{g}} \subset \mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$ of degree d (operator with constant coefficients). It is obvious that the leading order part of the operator $\mathrm{HC}(D)$ is the operator $\eta(D)$ with constant coefficients, whose symbol is just the restriction of the symbol of D from $\mathfrak{g}^*$ to $\mathfrak{h}^*$. Our job is to show that in fact $\mathrm{HC}(D) = \eta(D)$. To do so, denote by Y the difference $\mathrm{HC}(D) - \eta(D)$. Assume $Y \neq 0$. By Lemma 10.4, the operator $\mathrm{HC}(D)$ commutes with $\Delta_{\mathfrak{h}}$. Therefore, so does Y. Also Y has homogeneity degree d but order $m \le d-1$. Let $S(\mathbf{x}, \mathbf{p})$ be the symbol of Y ($\mathbf{x} \in \mathfrak{h}, \mathbf{p} \in \mathfrak{h}^*$). Then S is a homogeneous function of homogeneity degree d under the transformations $\mathbf{x} \to t^{-1}\mathbf{x}$, $\mathbf{p} \to t\mathbf{p}$, polynomial in $\mathbf{p}$ of degree m. From these properties of S it is clear that S is not a polynomial (its degree in $\mathbf{x}$ is $m - d < 0$). On the other hand, since Y commutes with $\Delta_{\mathfrak{h}}$, the Poisson bracket of S with $\mathbf{p}^2$ is zero. Thus Proposition 10.3 follows from Lemma 2.22. □

Now we continue the proof of Theorem 10.1. Consider the filtration on $\mathcal{D}(\mathfrak{g})$ in which $\deg(\mathfrak{g}) = 1 \deg(\mathfrak{g}^*) = 0$ (the order filtration), and the associated graded map $\mathrm{grHC} : \mathbb{C}[\mathfrak{g} \times \mathfrak{g}^*]^{\mathfrak{g}} \to \mathbb{C}[\mathfrak{h}_{\mathrm{reg}} \times \mathfrak{h}^*]^W$, which attaches to every differential operator the symbol of its radial part. It is easy to see that this map is just the restriction map to $\mathfrak{h} \oplus \mathfrak{h}^* \subset \mathfrak{g} \oplus \mathfrak{g}^*$, so it actually lands in $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$.

Moreover, grHC is a map **onto** $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. Indeed, grHC is a Poisson map, so the surjectivity follows from the following Lemma.

Lemma 10.5. *$\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ is generated as a Poisson algebra by $\mathbb{C}[\mathfrak{h}]^W$ and $\mathbb{C}[\mathfrak{h}^*]^W$, i.e. by functions $f_m = \sum x_i^m$ and $f_m^* = \sum p_i^m$, $m \geq 1$.*

Proof. We have $\{f_m^*, f_r\} = mr \sum x_i^{r-1} p_i^{m-1}$. Thus the result follows from Corollary 9.13. □

Let K_0 be the kernel of grHC. Then by Theorem 9.14, K_0 is the ideal of the commuting scheme $\mathrm{Comm}(\mathfrak{g})/G$.

Now consider the kernel K of the homomorphism HC. It is easy to see that $K \supset J^{\mathfrak{g}}$, so $\mathrm{gr}(K) \supset \mathrm{gr}(J)^{\mathfrak{g}}$. On the other hand, since K_0 is the ideal of the commuting scheme, we clearly have $\mathrm{gr}(J)^{\mathfrak{g}} \supset K_0$, and $K_0 \supset \mathrm{gr}K$. This implies that $K_0 = \mathrm{gr}K = \mathrm{gr}(J)^{\mathfrak{g}}$, and $K = J^{\mathfrak{g}}$.

It remains to show that $\mathrm{Im}\,\mathrm{HC} = \mathcal{D}(\mathfrak{h})^W$. Since $\mathrm{gr}K = K_0$, we have $\mathrm{grIm}\,\mathrm{HC} = \mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$. Therefore, to finish the proof of the Harish-Chandra and Levasseur-Stafford theorems, it suffices to prove the following proposition.

Proposition 10.6. $\mathrm{Im}\,\mathrm{HC} \supset \mathcal{D}(\mathfrak{h})^W$.

Proof. We will use the following Lemma.

Lemma 10.7 (N. Wallach, [Wa]). *$\mathcal{D}(\mathfrak{h})^W$ is generated as an algebra by W-invariant functions and W-invariant differential operators with constant coefficients.*

Proof. The lemma follows by taking associated graded algebras from Lemma 10.5. □

Remark 10.8. Levasseur and Stafford showed [LS] that this lemma is valid for any finite group W acting on a finite dimensional vector space $\mathfrak{h}$. However, the above proof does not apply, since, as explained in [Wa], Lemma 10.5 fails for many groups W, including Weyl groups of exceptional Lie algebras E_6, E_7, E_8 (in fact it even fails for the cyclic group of order > 2 acting on a 1-dimensional space!). The general proof is more complicated and uses some results in noncommutative algebra.

Lemma 10.7 and Proposition 10.3 imply Proposition 10.6. □

Thus, Theorem 10.1 is proved.

10.3. **Corollaries of Theorem 10.1.** Let $\mathfrak{g}_{\mathbb{R}}$ be the compact form of $\mathfrak{g}$, and $\mathcal{O}$ a regular coadjoint orbit in $\mathfrak{g}_{\mathbb{R}}^*$. Consider the map

$$\psi_{\mathcal{O}} : \mathfrak{h} \to \mathbb{C}, \quad \psi_{\mathcal{O}}(\mathbf{x}) = \int_{\mathcal{O}} \mathbf{e}^{(\mathbf{b},\mathbf{x})} \mathrm{d}\mathbf{b}, \ \mathbf{x} \in \mathfrak{h},$$

where $\mathrm{d}\mathbf{b}$ is the measure on the orbit coming from the Kirillov-Kostant symplectic structure.

Theorem 10.9 (Harish-Chandra formula)**.** *For a regular element $\mathbf{x} \in \mathfrak{h}$, we have*

$$\psi_{\mathcal{O}}(\mathbf{x}) = \delta^{-1}(\mathbf{x}) \sum_{w \in W} (-1)^{\ell(w)} \mathbf{e}^{(w\lambda, \mathbf{x})},$$

where λ is the intersection of $\mathcal{O}$ with the dominant chamber in the dual Cartan subalgebra $\mathfrak{h}^*_{\mathbb{R}} \subset \mathfrak{g}^*_{\mathbb{R}}$, and $\ell(w)$ is the length of an element $w \in W$.

Proof. Take $D \in (S\mathfrak{g})^{\mathfrak{g}}$. Then $\delta(\mathbf{x})\psi_{\mathcal{O}}$ is an eigenfunction of $\mathrm{HC}(D) = \eta(D) \in (S\mathfrak{h})^W$ with eigenvalue $\chi_{\mathcal{O}}(D)$, where $\chi_{\mathcal{O}}(D)$ is the value of the invariant polynomial D at the orbit $\mathcal{O}$.

Since the solutions of the equation $\eta(D)\varphi = \chi_{\mathcal{O}}(D)\varphi$ have a basis $\mathbf{e}^{(w\lambda,\mathbf{x})}$ where $w \in W$, we have

$$\delta(\mathbf{x})\psi_{\mathcal{O}}(\mathbf{x}) = \sum_{w \in W} c_w \cdot \mathbf{e}^{(w\lambda, \mathbf{x})}.$$

Since it is antisymmetric, we have $c_w = c \cdot (-1)^{\ell(w)}$, where c is a constant. The fact that $c = 1$ can be shown by comparing the asymptotics of both sides as $\mathbf{x} \to \infty$ in the regular chamber (using the stationary phase approximation for the integral). ☐

From Theorem 10.9 and the Weyl Character formula, we have the following corollary.

Corollary 10.10 (Kirillov character formula for finite dimensional representations, [Ki])**.** *If λ is a dominant integral weight, and L_λ is the corresponding representation of G, then*

$$\mathrm{Tr}_{\,L_\lambda}(\mathbf{e}^{\mathbf{x}}) = \frac{\delta(\mathbf{x})}{\delta_{\mathrm{Tr}}(\mathbf{x})} \int_{\mathcal{O}_{\lambda+\rho}} \mathbf{e}^{(\mathbf{b},\mathbf{x})} d\mathbf{b},$$

where $\delta_{\mathrm{Tr}}(\mathbf{x})$ is the trigonometric version of $\delta(\mathbf{x})$, i.e. the Weyl denominator $\prod_{\alpha>0}(\mathbf{e}^{\alpha(\mathbf{x})/2} - \mathbf{e}^{-\alpha(\mathbf{x})/2})$, and $\mathcal{O}_\mu$ denotes the coadjoint orbit passing through μ.

10.4. **The deformed Harish-Chandra homomorphism.** Finally, we would like to explain how to quantize the Calogero-Moser space $\mathcal{C}_n$, using the procedure of quantum Hamiltonian reduction.

Let $\mathfrak{g} = \mathfrak{gl}_n$, $A = \mathcal{D}(\mathfrak{g})$ as above. Let k be a complex number, and W_k be the representation of $\mathfrak{sl}_n$ on the space of functions of the form $(x_1 \cdots x_n)^k f(x_1, \ldots, x_n)$, where f is a Laurent polynomial of degree 0. We regard W_k as a $\mathfrak{g}$-module by pulling it back to $\mathfrak{g}$ under the natural projection $\mathfrak{g} \to \mathfrak{sl}_n$. Let I_k be the annihilator of W_k in $U(\mathfrak{g})$. The ideal I_k is the quantum counterpart of the coadjoint orbit of matrices T such that $T + 1$ has rank 1.

Let $B_k = \mathcal{D}(\mathfrak{g})^{\mathfrak{g}}/(\mathcal{D}(\mathfrak{g})\mu(I_k))^{\mathfrak{g}}$ where $\mu : U(\mathfrak{g}) \to A$ is the quantum momentum map (the quantum Hamiltonian reduction with respect to the ideal I_k). Then B_k has a filtration induced from the order filtration of $\mathcal{D}(\mathfrak{g})^{\mathfrak{g}}$.

Let $\mathrm{HC}_k : \mathcal{D}(\mathfrak{g})^{\mathfrak{g}} \to B_k$ be the natural homomorphism, and $K(k)$ be the kernel of HC_k.

Theorem 10.11 (Etingof-Ginzburg, [EG])**.** (i) *$K(0) = K$, $B_0 = \mathcal{D}(\mathfrak{h})^W$*, $\mathrm{HC}_0 = \mathrm{HC}$.

(ii) $\mathrm{gr}K(k) = \mathrm{Ker}\,(\mathrm{grHC}_k) = K_0$ *for all complex k. Thus,* HC_k *is a flat family of homomorphisms.*

(iii) *The algebra $\mathrm{gr}B_k$ is commutative and isomorphic to $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]^W$ as a Poisson algebra.*

Because of this theorem, the homomorphism HC_k is called the deformed Harish-Chandra homomorphism.

Theorem 10.11 implies that B_k is a quantization of the Calogero-Moser space $\mathcal{C}_n$ (with deformation parameter $1/k$). But we already know one such quantization - the spherical

Cherednik algebra $B_{1,k}$ for the symmetric group. Therefore, the following theorem comes as no surprize.

Theorem 10.12 ([EG]). *The algebra B_k is isomorphic to the spherical rational Cherednik algebra $B_{1,k}(\mathfrak{S}_n, \mathbb{C}^n)$.*

Thus, quantum Hamiltonian reduction provides a Lie-theoretic construction of the spherical rational Cherednik algebra for the symmetric group. A similar (but more complicated) Lie theoretic construction exists for symplectic reflection algebras for wreath product groups defined in Example 8.5 (see [EGGO]).

10.5. **Notes.** Our exposition in this section follows Section 4, Section 5 of [E4].

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