Limit Theorems

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Dr. Kempthorne

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Outline

1. Limit Theorems
   - Weak Laws of Large Numbers
   - Limit Theorems

Weak Laws of Large Numbers

Bernoulli's Weak Law of Large Numbers

- $X_1, X_2, \ldots$ iid $Bernoulli(\theta)$.
- $S_n = \sum_{i=1}^{n} X_i \sim Binomial(n, \theta)$.

$$\frac{S_n}{n} \xrightarrow{P} \theta.$$

Proof: Apply Chebychev's Inequality,

Khintchin's Weak Law of Large Numbers

- $X_1, X_2, \ldots$ iid
- $E[X_1] = \mu$, finite
- $S_n = \dfrac{1}{n} \sum_{i=1}^{n} X_i$

Then

$$\frac{S_n}{n} \xrightarrow{P} \mu.$$

Khintchin's WLLN

Proof:

- If $Var(X_1) < \infty$, apply Chebychev's Inequality.
- If $Var(X_1) = \infty$, apply Levy-Continuity Theorem for characteristic functions.

$$\begin{aligned}\phi_{\overline{X}_n}(t) &= E[e^{it\overline{X}_n}] = \prod_{i=1}^{n} E[e^{i\frac{t}{n}X_i}] \\ &= \prod_{i=1}^{n} \phi_X(\tfrac{t}{n}) = [\phi_X(\tfrac{t}{n})]^n \\ &= [1 + \tfrac{i\mu t}{n} + o(\tfrac{t}{n})]^n \\ &\xrightarrow{n\to\infty} e^{it\mu}.\end{aligned}$$

(characteristic function of constant random variable μ)

So

$$\overline{X}_n \xrightarrow{\mathcal{D}} \mu, \text{ which implies } \overline{X}_n \xrightarrow{\mathcal{P}} \mu.$$

Limiting Moment-Generating Functions

Continuity Theorem. Suppose

- $X_1, \ldots, X_n$ and X are random variables
- $F_1(t), \ldots, F_n(t)$, and $F(t)$ are the corresponding sequence of cumulative distribution functions
- $M_{X_1}(t), \ldots, M_{X_n}(t)$ and $M_X(t)$ are the corresponding sequence of moment generating functions.

Then if $M_n(t) \to M(t)$ for all t in an open interval containing zero, then

$$F_n(x) \to F(x), \text{ at all continuity points of } F.$$

i.e.,

$$X_n \xrightarrow{\mathcal{L}} X$$

Limiting Characteristic Functions

Levy Continuity Theorem. Suppose

- $X_1, \ldots, X_n$ and X are random variables and
- $\phi_t(t), \ldots, \phi_n(t)$ and $\phi_X(t)$ are the corresponding sequence of characteristic functions.

Then

$$X_n \xrightarrow{\mathcal{L}} X$$

if and only if

$$\lim_{n\to\infty} \phi_n(t) = \phi_X(t), \text{ for all } t \in R.$$

Proof: See http://wiki.math.toronto.edu/TorontoMathWiki/images/0/00/MAT1000DanielRuedt.pdf

Outline

Limit Theorems

De Moivre-Laplace Theorem If $\{S_n\}$ is a sequence of *Binomial*(n, θ) random variables, $(0 < \theta < 1)$, then

$$\frac{S_n - n\theta}{\sqrt{n\theta(1-\theta)}} \xrightarrow{\mathcal{L}} Z,$$

where Z has a standard normal distribution.

Applying the "Continuity Correction":

$$\begin{aligned} P[k \le S_n \le m] &= P\left[k - \tfrac{1}{2} \le S_n \le m + \tfrac{1}{2}\right] \\ &= P\left[\frac{k - \frac{1}{2} - n\theta}{\sqrt{n\theta(1-\theta)}} \le \frac{S_n - n\theta}{\sqrt{n\theta(1-\theta)}} \le \frac{m + \frac{1}{2} - n\theta}{\sqrt{n\theta(1-\theta)}}\right] \\ &\xrightarrow{n\to\infty} \Phi\left(\frac{m + \frac{1}{2} - n\theta}{\sqrt{n\theta(1-\theta)}}\right) - \Phi\left(\frac{k - \frac{1}{2} - n\theta}{\sqrt{n\theta(1-\theta)}}\right) \end{aligned}$$

Limit Theorems

Central Limit Theorem

- $X_1, X_2, \ldots$ iid
- $E[X_1] = \mu$, and $Var[X_1] = \sigma^2$, both finite ($\sigma^2 > 0$).
- $S_n = \sum_{i=1}^{n} X_i$

Then

$$\frac{S_n - n\mu}{\sigma\sqrt{n}} \xrightarrow{\mathcal{L}} Z,$$

where Z has a standard normal distribution.

Equivalently:

$$\frac{(\frac{1}{n}S_n - \mu)}{\sqrt{\sigma^2/n}} \xrightarrow{\mathcal{L}} Z.$$

Limit Theorems: Central Limit Theorem

Limiting Distribution of $\overline{X}_n$

- $X_1, \ldots, X_n$ iid with $\mu = E[X]$, and mgf $M_X(t)$.
- The mgf of $\overline{X} = \frac{\sum_{i=1}^n X_i}{n}$ is

$$\begin{aligned} M_{\overline{X}}(t) &= E[e^{t\overline{X}}] = E[e^{\sum_1^n \frac{t}{n} X_i}] \\ &= \prod_1^n E[e^{\frac{t}{n} X_i}] = \prod_1^n M_X(\tfrac{t}{n}) \\ &= [M_X(\tfrac{t}{n})]^n \end{aligned}$$

- Applying Taylor's expansion, there exists $t_1 : 0 < t_1 < t/n$:

$$\begin{aligned} M_X(\tfrac{t}{n}) &= M_X(0) + M'_X(t_1)\tfrac{t}{n} \\ &= 1 + \tfrac{\mu t}{n} + \frac{[M'_X(t_1) - M'(0)]t}{n} \end{aligned}$$

- So

$$\begin{aligned} \lim_{n\to\infty} [M_X(\frac{t}{n})]^n &= \lim_{n\to\infty} [1 + \frac{\mu t}{n} + \frac{[M'_X(t_1) - M'(0)]t}{n}]^n \\ &= e^{\mu t} \end{aligned}$$

So $\overline{X}_n \xrightarrow{P} \mu$.

Limit Theorems: Central Limit Theorem

- Define $Z_n = \dfrac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} = \dfrac{\sqrt{n}(\overline{X}_n - \mu)}{\sigma} = \dfrac{S_n - n\mu}{\sqrt{n}\sigma}$
  where $S_n = \quad {}_1^n X_i$.
- Note: Z_n: $E[Z_n] \equiv 0$ and $Var[Z_n] \equiv 1$.

$$\begin{aligned} M_{Z_n}(t) = E[e^{tZ_n}] &= E\left(exp\{\frac{t}{\sqrt{n}} \quad {}_{i=1}^{n} \frac{X_i - \mu}{\sigma}\}\right) \\ &= \prod_{i=1}^{n} M_Y(\frac{t}{\sqrt{n}}) \end{aligned}$$

where $Y \stackrel{\mathcal{D}}{=} \frac{X_i - \mu}{\sigma}$, $E[Y] = 0 = M_Y'(0)$, and $E[Y^2] = 1 = M_Y''(0)$, and $M_Y(t) = E[e^{tY}]$. By Taylor's expansion, $\exists t_1 \in (0, t/\sqrt{n})$:

$$\begin{aligned} M_Y(\frac{t}{\sqrt{n}}) &= M_Y(0) + M_Y'(0)(\frac{t}{\sqrt{n}}) + \frac{1}{2}M_Y''(t_1)(\frac{t}{\sqrt{n}})^2 \\ &= 1 + \frac{t^2}{2n} + \frac{[M_Y''(t_1) - 1]t^2}{2n} \end{aligned}$$

Limit Theorems: Central Limit Theorem

- Since $\lim_{n\to\infty}[M''_Y(t_1) - 1] = 1 - 1 = 0$,

$$\begin{aligned}
\lim_{n\to\infty} M_{Z_n}(t) &= \lim_{n\to\infty} [1 + \frac{t^2}{2n} + \frac{[M''_Y(t_1) - 1]t^2}{2n}]^n \\
&= e^{+\frac{t^2}{2}}, \text{ the mgf of a } N(0,1) \text{ so } Z_n \xrightarrow{\mathcal{L}} N(0,1).
\end{aligned}$$

Classic Limit Theorem Examples

Poisson Limit of Binomials

$\{X_n\}$: $X_n \sim Binomial(n, \theta_n)$ where

- $\lim_{n \to \infty} \theta_n = 0$
- $\lim_{n \to \infty} n\theta_n = \lambda$, with $0 < \lambda < \infty$

$$X_n \xrightarrow{\mathcal{L}} Poisson(\lambda).$$

Classic Limit Theorem Examples

Sample Mean of Cauchy Distribution

- $X_1, \ldots, X_n$ i.i.d. $Cauchy(\mu, \gamma)$ r.v.s; $\mu \in R$, $\gamma > 0$

$$f(x \mid \theta) = \frac{1}{\pi\gamma}\left(\frac{1}{1+(\frac{x-\mu}{\gamma})^2}\right), -\infty < x < \infty.$$

- The characteristic function of the Cauchy is

$$\phi_X(t) = exp\{i\mu t - \gamma|t|\}$$

- The characteristic function of the sample mean is:

$$\begin{aligned} \phi_{\overline{X}_n}(t) &= E[e^{i\frac{t}{n}\sum_1^n X_i}] \\ &= \prod_{i=1}^n \phi_X(\tfrac{t}{n}) = [exp\{i\mu\tfrac{t}{n} - \gamma|\tfrac{t}{n}|\}]^n \\ &= exp\{i\mu t - \gamma|t|\} \end{aligned}$$

So $\overline{X}_n \stackrel{\mathcal{D}}{=} X_1$ for every**(!)** n.

Limit Theorems

Berry-Esseen Theorem If $X_1, \ldots, X_n$ iid with mean μ and variance $\sigma^2 > 0$, then for all n,

$$\sup_t \left| P\left(\frac{S_n - n\mu}{\sqrt{n}\sigma} \leq t \right) - \Phi(t) \right| \leq C^* \frac{E[|X_1 - \mu|^3]}{\sqrt{n}\sigma^3}.$$

- $C^* = \frac{33}{4}$: B&D, A.15.12.
- $C^* = 0.4748$: http://en.wikipedia.org/wiki/Berry-Esseen_theorem, Shevtsova (2011)

Asymptotic Order Notation

$$U_n = o_P(1) \quad \textit{iff} \quad U_n \xrightarrow{P} 0.$$

$$U_n = O_P(1) \quad \textit{iff} \quad \forall \epsilon > 0, \exists M < \infty, \text{ such that } \forall n$$
$$P[|U_n| \geq M] \leq \epsilon$$

$$\mathbf{U}_n = o_P(\mathbf{V}_n) \quad \textit{iff} \quad \frac{|\mathbf{U}_n|}{|\mathbf{V}_n|} = o_P(1)$$

$$\mathbf{U}_n = O_P(\mathbf{V}_n) \quad \textit{iff} \quad \frac{|\mathbf{U}_n|}{|\mathbf{V}_n|} = O_P(1)$$

Asymptotic Order Notation

Example: $Z_1, \ldots, Z_n$ are iid as Z

- If $E[|Z|] < \infty$, then by WLLN:
  $$\overline{Z}_n = \mu + o_P(1). \text{ where } \mu = E[Z].$$
- If $E[|Z|^2] < \infty$, then by CLT:
  $$\overline{Z}_n = \mu + O_P(\tfrac{1}{\sqrt{n}}).$$

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