Lecture 14

14.1 Estimates of parameters of normal distribution.

Let us consider a sample

$$X_1, \ldots, X_n \sim N(\alpha, \sigma^2)$$

from normal distribution with mean α and variance σ^2. Using different methods (for example, maximum likelihood) we showed that one can take $\bar{X}$ as an estimate of mean α and $\bar{X^2} - (\bar{X})^2$ as an estimate of variance σ^2. The question is: how close are these estimates to actual values of unknown parameters? By LLN we know that these estimates converge to α and σ^2,

$$\bar{X} \to \alpha, \bar{X^2} - (\bar{X})^2 \to \sigma^2, n \to \infty,$$

but we will try to describe precisely how close $\bar{X}$ and $\bar{X^2} - (\bar{X})^2$ are to α and σ^2.

We will start by studying the following

Question: What is the joint distribution of $(\bar{X}, \bar{X^2} - (\bar{X})^2)$ when the sample

$$X_1, \ldots, X_n \sim N(0, 1)$$

has standard normal distribution.

Orthogonal transformations.

The student well familiar with orthogonal transformations may skip to the beginning of next lecture. Right now we will repeat some very basic discussion from linear algebra and recall some properties and geometric meaning of orthogonal transormations. To make our discussion as easy as possible we will consider the case of 3-dimensional space $\mathbb{R}^3$.

Let us consider an orthonormal basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ as shown in figure 14.1, i.e. they are orthogonal to each other and each has length one. Then any vector $\vec{X}$ can be represented as

$$\vec{X} = X_1\vec{e}_1 + X_2\vec{e}_2 + X_3\vec{e}_3,$$

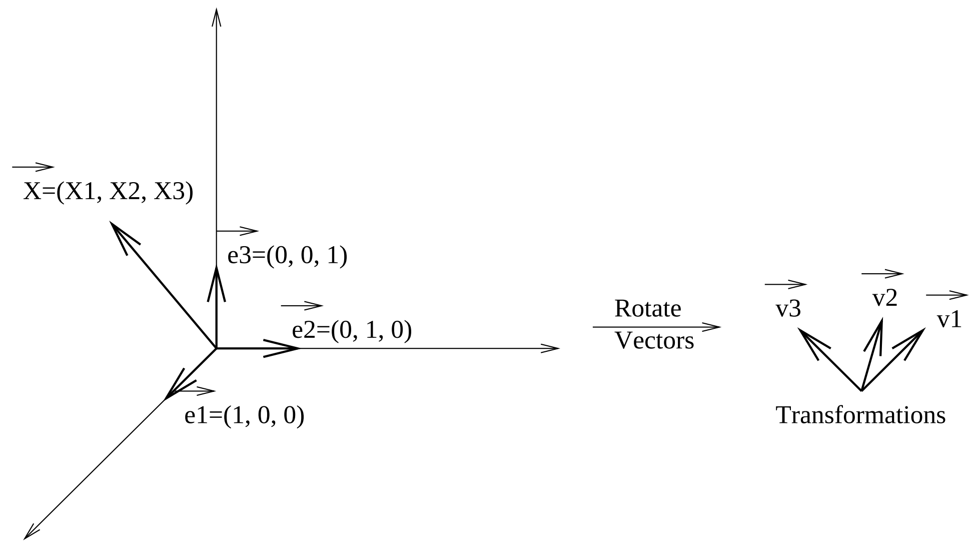

Figure 14.1: Unit Vectors Transformation.

where (X_1, X_2, X_3) are the coordinates of vector $\vec{X}$

Suppose now that we make a rotation (and, maybe, reflection) such that the vectors $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ go to another orthonormal basis $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$, i.e.

$$|\vec{v}_1| = |\vec{v}_2| = |\vec{v}_3| = 1, \vec{v}_1 \perp \vec{v}_2 \perp \vec{v}_3 \perp \vec{v}_1.$$

Let us denote the coordinates of vector $\vec{v}_i = (v_{i1}, v_{i2}, v_{i3})$ for $i = 1, 2, 3$. Then vector $\vec{X}$ is rotated to vector

$$\begin{aligned} & \vec{X} = X_1\vec{e}_1 + X_2\vec{e}_2 + X_3\vec{e}_3 \to X_1\vec{v}_1 + X_2\vec{v}_2 + X_3\vec{v}_3 \\ = \ & X_1(v_{11}, v_{12}, v_{13}) + X_2(v_{21}, v_{22}, v_{23}) + X_3(v_{31}, v_{32}, v_{33}) \\ = \ & (X_1, X_2, X_3) \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix} = \vec{X}V, \end{aligned}$$

where V is the matrix with elements v_{ij}.

If we want to make inverse rotation so that vectors $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ rotate back to $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$, we need to multiply vector $\vec{X}$ by the transpose V^T:

$$\vec{X} \to \vec{X}V^T = (X_1, X_2, X_3) \begin{pmatrix} v_{11} & v_{21} & v_{31} \\ v_{12} & v_{22} & v_{32} \\ v_{13} & v_{23} & v_{33} \end{pmatrix}.$$

Let us check that transpose V^T defines inverse rotation. For example, let us check that vector $\vec{v}_1 = (v_{11}, v_{12}, v_{13})$ goes to $\vec{e}_1 = (1, 0, 0)$. We have,

$$\vec{v}_1 V^T = \left(v_{11}^2 + v_{12}^2 + v_{13}^2, \quad v_{11}v_{21} + v_{12}v_{22} + v_{13}v_{23}, \quad v_{11}v_{31} + v_{12}v_{32} + v_{13}v_{33} \right)$$

$$= \Big(\text{ (length of } \vec{v}_1)^2, \quad \vec{v}_1 \cdot \vec{v}_2, \quad \vec{v}_1 \cdot \vec{v}_3 \Big) = (1, 0, 0)$$

since $(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ is an orthonormal basis. Therefore, we have proven that $\vec{v}_1 \to \vec{e}_1$. Similarly, $\vec{v}_2 \to \vec{e}_2$ and $\vec{v}_3 \to \vec{e}_3$.

Note that this inverse rotation V^T will send the basis $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$ to

$$\begin{aligned} \vec{v}_1' &= (v_{11}, v_{21}, v_{31}) \\ \vec{v}_2' &= (v_{12}, v_{21}, v_{32}) \\ \vec{v}_3' &= (v_{13}, v_{21}, v_{33}), \end{aligned}$$

- the columns of matrix V, which is, therefore, again an orthonormal basis:

$$|\vec{v}_1'| = |\vec{v}_2'| = |\vec{v}_3'| = 1$$

$$\vec{v}_1' \perp \vec{v}_2' \perp \vec{v}_3' \perp \vec{v}_1'.$$

This means that both rows and columns of V forms an orthonormal basis.

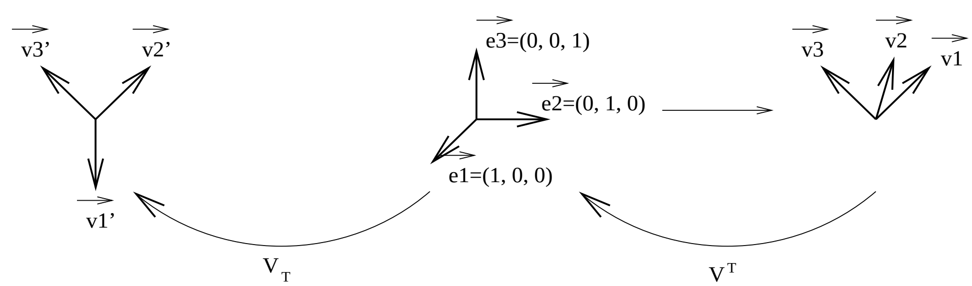

Figure 14.2: Unit Vectors Fact.