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18.409 The Behavior of Algorithms in Practice
5/2/2002
Lecture 19
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\section*{Where We Are in the Proof}

We are trying to bound the quantity:
\[
\operatorname{Pr}_{b_{1}, \ldots, b_{d}}\left[\operatorname{dist}\left(0, \partial\left(\triangle\left(b_{1} \ldots b_{d}\right)\right)\right)<\epsilon\right]
\]

Where the probability is over the distribution with density
\[
\left(\prod_{i=1}^{d} \mu_{i}\left(b_{i}\right)\right)\left[0 \in \triangle\left(b_{1} \ldots b_{d}\right)\right] \operatorname{vol}\left(\triangle\left(b_{1} \ldots b_{d}\right)\right)
\]

Also recall that the \(\mu_{i}\) are Gaussian with variance \(\sigma^{2} \leq 1\) and have centers of norm \(\leq \Gamma \leq\) \(1+4 \sqrt{d \log (n)}\).

\section*{Last Time We Proved}
\[
\operatorname{Pr}_{b_{1}, \ldots, b_{d}}\left[\operatorname{dist}\left(b_{1}, \operatorname{dist}\left(b_{2}, \ldots, b_{d}\right)\right)<\epsilon\right] \leq e^{2}\left(\frac{d(1+\Gamma) \epsilon}{\sigma^{2}}\right)
\]

Where \(\operatorname{dist}\left(b_{2}, \ldots, b_{d}\right)\) denotes the affine span of \(b_{2}, \ldots, b_{d}\).

\section*{Today We Will Prove}
\[
\begin{equation*}
\underset{b_{1}, \ldots, b_{d}}{\operatorname{Pr}}\left[\operatorname{dist}\left(0, \operatorname{dist}\left(b_{2}, \ldots, b_{d}\right)\right)<\epsilon \operatorname{dist}\left(b_{1}, \operatorname{dist}\left(b_{2}, \ldots, b_{d}\right)\right)\right] \leq\left(\frac{10 d \Gamma^{2} \epsilon}{\sigma^{2}}\right) \tag{1}
\end{equation*}
\]

\section*{How Smooth are Gaussians}

We will now prove a utility lemma which will be useful later.

Lemma 1 Let \(\mu\) is a Guassian distribution centered at 0 with variance \(\sigma^{2}\). If \(X\) and \(Y\) are points such that \(\|X\| \leq T\) and \(\|X-Y\|<\epsilon \leq T\), then
\[
\frac{\mu(Y)}{\mu(X)} \geq e^{\frac{-3 \epsilon T}{2 \sigma^{2}}}
\]

Proof The worst case is when \(X=T\) and \(Y=T+\epsilon\). In this case we have:
\[
\frac{\mu(Y)}{\mu(X)}=\frac{e^{\frac{-\left(T+\epsilon^{2}\right)}{2 \sigma^{2}}}}{e^{\frac{-T^{2}}{2 s i g a^{2}}}}=e^{\frac{-2 T \epsilon-\epsilon^{2}}{2 \sigma^{2}}}
\]

Note: If \(\epsilon<T / \sigma^{2}\), this Lemma gives us that
\[
\frac{\mu(Y)}{\mu(X)} \geq e^{\frac{-3}{2}}
\]

\section*{Proof of Today's Bound}

Goal: Prove the origin is not much closer to \(\operatorname{dist}\left(b_{1}, \ldots, b_{d}\right)\) than \(b_{1}\) is.
Note: Since Gaussians are smooth, the chance of any nearby configuration is about the same.

Idea: Fix the shape of the simplex and shift it a little towards \(b_{1}\). The resulting configuration is just as likely and has the origin farther from \(\operatorname{dist}\left(b_{1}, \ldots, b_{d}\right)\).

\section*{Change of Variables}

It will be easier to fix the shape of the simplex if we do another change of variables.
Our handle on the simplex will be its center of gravity:
\[
x=\frac{1}{d} \sum_{i=1}^{d} b_{i}
\]

The shape of the simplex will be determined by the values:
\[
\delta_{i}=x-b_{i}
\]

For \(i=2 \ldots d\).
Additionally, we set
\[
\sum_{i=1}^{d} \delta_{i}=0
\]
which defines \(\delta_{1}\).

Observe that
\[
\triangle\left(b_{1} \ldots b_{d}\right)=\triangle\left(x-\delta_{1} \ldots x-\delta_{d}\right)=x+\triangle\left(-\delta_{1} \ldots-\delta_{d}\right)
\]

Therefore,
\[
0 \in \triangle\left(b_{1} \ldots b_{d}\right) \Leftrightarrow x \in \triangle\left(-\delta_{1} \ldots-\delta_{d}\right)
\]

Similarly,
\[
\operatorname{dist}\left(0, \operatorname{dist}\left(b_{2}, \ldots, b_{d}\right)=\operatorname{dist}\left(0, \operatorname{dist}\left(\delta_{2}, \ldots, \delta_{d}\right)\right.\right.
\]

Note: This change of variables is just a linear transformation and so the Jacobian of the transformation is constant.

\section*{Defining the Contraction Map}

We observe that:
\[
\begin{aligned}
& \operatorname{Pr}_{b_{1}, \ldots, b_{d}}^{\operatorname{Pr}}\left[\operatorname{dist}\left(b_{1}, \operatorname{dist}\left(b_{2}, \ldots, b_{d}\right)\right)<\epsilon\right] \leq \\
& \qquad \max _{\delta_{1}, \ldots, \delta_{d}} \operatorname{Pr}\left[\operatorname{dist}\left(x, \operatorname{dist}\left(\delta_{2}, \ldots, \delta_{d}\right)\right)<\epsilon \operatorname{dist}\left(\delta_{1}, \operatorname{dist}\left(\delta_{2}, \ldots, \delta_{d}\right)\right)\right]
\end{aligned}
\]

Subject to the condition that \(\left\|\delta_{1}-\delta_{i}\right\| \leq 2 \Gamma\) for all \(i\) and where \(x\) is has distribution:
\[
\nu(x)=\left(\prod_{i=1}^{d} \mu_{i}\left(b_{i}\right)\right)\left[x \in \triangle\left(\delta_{1} \ldots \delta_{d}\right)\right]
\]

Let \(S\) be \(\triangle\left(\delta_{1} \ldots \delta_{d}\right)\). Let \(S_{\epsilon}\) be obtained by contracting \(S\) at \(\delta_{1}\) by a factor of \((1-\epsilon)\). That is, \(S_{\epsilon}\) is the set of points \(y\) such that
\[
\operatorname{dist}\left(y, \operatorname{dist}\left(\delta_{2}, \ldots, \delta_{d}\right)\right) \geq \epsilon \operatorname{dist}\left(\delta_{1}, \operatorname{dist}\left(\delta_{2}, \ldots, \delta_{d}\right)\right)
\]

We observe that
\[
\operatorname{Pr}_{x}\left[\operatorname{dist}\left(x, \operatorname{dist}\left(\delta_{2}, \ldots, \delta_{d}\right)\right)<\epsilon \operatorname{dist}\left(\delta_{1}, \operatorname{dist}\left(\delta_{2}, \ldots, \delta_{d}\right)\right)\right]=\frac{\nu(S)-\nu\left(S_{\epsilon}\right)}{\nu(S)}
\]

To show this quantity is small, we just need to show that the probability measures don't change much.

Let \(\rho_{\epsilon}\) be the contraction map specified above.

Bounding \(\nu\left(S_{\epsilon}\right) / \nu(S)\)
We observe that \(\rho_{\epsilon}\) moves points by at most a distance of \(\epsilon 2 \Gamma\). Also, we know that \(x\) is at a distance of at most \(3 \Gamma\) from the center of its distribution.

Therefore, from Lemma 1, we know that under the map, \(\rho_{\epsilon}\), the product of the \(\mu\) 's changes my at most a factor of:
\[
e^{\frac{-3-6 \epsilon 2^{2} d}{2 \sigma^{2}}}
\]

Which is at least:
\[
\left(1-\frac{\epsilon 9 \Gamma^{2} d}{\sigma^{2}}\right)
\]

Additionally, are contraction map has a Jacobian of \((1-\epsilon)^{d}\) at every point.
Note: \((1-\epsilon)^{d} \leq(1-\epsilon d)\)
This implies that
\[
\frac{\nu\left(S_{\epsilon}\right)}{\nu(S)} \leq\left(1-\frac{\epsilon 9 \Gamma^{2} d}{\sigma^{2}}\right)(1-\epsilon d) \leq\left(1-\frac{10 d \epsilon \Gamma^{2}}{\sigma^{2}}\right)
\]

Therefore,
\[
\frac{\nu(S)-\nu\left(S_{\epsilon}\right)}{\nu(S)} \leq \frac{10 d \epsilon \Gamma^{2}}{\sigma^{2}}
\]

\section*{Completing the Bound}

Using Combination Lemma 2 which we will prove in the next section, we get that:
\[
\operatorname{Pr}\left[\operatorname{dist}\left(0, \operatorname{dist}\left(b_{2}, \ldots, b_{d}\right)\right)<\epsilon\right] \leq \epsilon \frac{10 e d^{2} \Gamma^{2}(1+\Gamma)}{\sigma^{4}}
\]

Therefore, since the simplex has \(d\) symmetric faces, we get that:
\[
\operatorname{Pr}\left[\operatorname{dist}\left(0, \partial\left(b_{1}, \ldots, b_{d}\right)\right)<\epsilon\right] \leq \epsilon \frac{10 e d^{3} \Gamma^{2}(1+\Gamma)}{\sigma^{4}}
\]

\section*{The Combination Lemma}

Lemma 2 Let \(a, b\) have some distribution such that
1. \(\operatorname{Pr}[f(a)<\epsilon] \leq \alpha \epsilon\)
2. \(\forall a, \operatorname{Pr}[g(a, b) \leq \epsilon] \leq(\beta \epsilon)^{2}\)

Then \(\operatorname{Pr}[f(a) g(a, b)<\epsilon] \leq 5 \alpha \beta \epsilon\)

Note: To really apply this to the previous circumstances, we would need to do this in the \(\delta\) world where
- \(a=\left\{\delta_{1}, \ldots \delta_{2}\right\}\)
- \(b=x\)

\section*{Proof}
\[
\begin{aligned}
\operatorname{Pr}[f(a) g(a, b)<\epsilon] \leq \operatorname{Pr}[f(a)<\beta \epsilon]+ & \sum_{i \geq 1} \operatorname{Pr}\left[f(a)<\beta \epsilon 2^{i} \text { and } g(a, b)<\frac{2^{i+1}}{\beta}\right. \\
& \leq \alpha \beta \epsilon+\sum_{i \geq 1} \alpha \beta \epsilon 2^{i}\left(2^{-i+1}\right)^{2}=\alpha \beta \epsilon\left(1+\sum_{i \geq 2} 2^{-i+2}\right)
\end{aligned}
\]

\section*{Back to the Original Problem}

Recall that the quantity which we were originally interested in was:
\[
\operatorname{Pr}_{\omega, r, b_{\pi_{1}}, \ldots, b_{\pi_{d}}}\left[\frac{\operatorname{dist}\left(z^{\omega, r}, \partial \triangle\left(b_{\pi_{1}} \ldots b_{\pi_{d}}\right)\right)<\omega, z>}{3 \Gamma}<\epsilon\right]
\]

Where the distribution of \(\omega, r\), and \(b_{i}\) was:
\[
\begin{aligned}
\left(\prod_{j \notin\left\{\pi_{1}, \ldots, \pi_{d}\right\}} \int_{a_{j}}\left[<\omega, a_{j}>\leq r\right] \mu_{i}\left(a_{j}\right)\right) \prod_{i=1}^{d} \mu_{\pi_{i}}\left(a_{\pi_{i}}\right) \\
\quad\left[z^{\omega, r} \in \triangle\left(b_{\pi_{1}} \ldots b_{\pi_{d}}\right)\right] \operatorname{Vol}\left(\triangle\left(b_{\pi_{1}} \ldots b_{\pi_{d}}\right)\right)
\end{aligned}
\]

We have succeeded in bounding the probability that \(z^{\omega, r}\) is close to the boundary of the triangle. Therefore, all that remains is to show it is unlikely that the angle is small. (That is, it is unlikely that \(\langle\omega, z\rangle\) is small).

Idea: Rotate the plane specified by \(\omega\) and \(r\) while preserving the intersection of the plane with the ray \(z\).

\section*{Another Change of Variables}

Previously, we had been specifying the plane with the variables \(\omega\) and \(r\). Now we will instead specify the plane by the variables \(\omega\) and \(t\) where \(t z\) is the point where the ray \(z\) intersects the plane.

Note: \(r=t\langle\omega, z\rangle\)
Jacobian \(\frac{\partial r}{\partial t}=\langle\omega, z\rangle\)
Also, for convenience we choose \(t z=z^{\omega, r}\) to be the origin of the plane containing the \(b_{\pi_{i}}\) 's.
We will now bound the quantity:
\[
\max _{b_{\pi_{1}}, \ldots, b_{\pi_{d}}, t} \operatorname{Pr}_{\omega}[<\omega, r><\epsilon]
\]

Subject to the constraint that \(\left\|b_{\pi_{i}}\right\| \leq \Gamma\) and \(|t| \leq \Gamma\). Where the distribution for \(\omega\) is:
\[
<\omega, z>\left(\prod_{j \notin\left\{\pi_{1}, \ldots, \pi_{d}\right\}} \int_{a_{j}}\left[<\omega, a_{j}>\leq t<\omega, z>\right] \mu_{i}\left(a_{j}\right)\right) \prod_{i=1}^{d} \mu_{\pi_{i}}\left(a_{\pi_{i}}\right)
\]

Observe that in the above expression everything except the term \(\langle\omega, z\rangle\) is like a constant for small changes in \(\omega\). We will write this distribution function as \(\langle\omega, z\rangle f(\omega)\).

\section*{"Longitude and Latitude"}

We will now change to "Longitude and Latitude" where we express the unit vector \(\omega\) as an angle, \(\theta\) (latitude) and a point, \(\phi\) on the unit sphere of \(d-1\) dimensions.

That is, \(\omega \in S^{d}\) becomes \(\theta \in[0, \pi / 2]\) and \(\phi \in S^{d-1}\). Where \(\cos (\theta)=\langle\omega, z\rangle\).
The Jacobian of this transformation is \([\sin (\theta)]^{d-1}\) which is like a constant for \(\theta \rightarrow \pi / 2\).
Therefore, we get that:
\[
\max _{b_{\pi_{1}}, \ldots, b_{\pi_{d}}, t} \operatorname{Pr}_{\omega}[<\omega, r><\epsilon] \leq \max _{b_{\pi_{1}}, \ldots, b_{\pi_{d}}, t, \phi} \operatorname{Pr}_{\theta}[\cos (\theta)<\epsilon]
\]

Where the density is the same as before except \(\langle\omega, z\rangle\) is replaced by \(\cos (\theta)\). That is, the density is \(f(\theta) \cos (\theta)\).

Note: We could derive a \(\theta_{0}\) such that \(\theta_{0}=\operatorname{poly}(n, d, 1 / \sigma)\) and \(0 \leq \theta \leq \theta^{\prime} \leq \theta_{0}\)
\[
\frac{f(\theta)}{f\left(\theta^{\prime}\right)} \leq e
\]
(This is what was meant earlier by the comment that \(f\) is almost constant for small changes in \(\omega\) )

Therefore, we could show that:
\[
\operatorname{Pr}_{\theta}[\cos (\theta)<\epsilon]<\operatorname{poly}\left(n, d, \frac{1}{\sigma}\right) \epsilon^{2}
\]

\section*{Putting it All Together}

We now have bounds showing it is unlikely the distance to the boundary is small and it is unlikely that the angle is small. Therefore, we can apply a Combination Lemma to get the following bound:
\[
\operatorname{Pr}_{\omega, r, b_{\pi_{1}}, \ldots, b_{\pi_{d}}}\left[\frac{\operatorname{dist}\left(z^{\omega, r}, \partial \triangle\left(b_{\pi_{1}} \ldots b_{\pi_{d}}\right)\right)<\omega, z>}{3 \Gamma}<\epsilon\right]<\epsilon \operatorname{poly}\left(n, d, \frac{1}{\sigma}\right)
\]

This bound implies that the expected size of the shadow of the polytope is polynomial.```

