Problem Set Number 7, 18.385j/2.036j
MIT (Fall 2014)

Rodolfo R. Rosales (MIT, Math. Dept., Cambridge, MA 02139)

October 27, 2014

Due Fri., November 7, 2014.

1   Problem 141002 - Computer test of two timing

Statement for problem 141002

Consider the equation

$$\frac{d^2x}{dt^2} + \epsilon\, x^2\, \frac{dx}{dt} + x = 0, \tag{1.1}$$

where $0 < \epsilon \ll 1$.

a. Derive the averaged equations.

b. Given the initial conditions $x(0) = a$ and $\dot{x}(0) = 0$, solve the averaged equations and thereby find an approximate formula for $x = x(t, \epsilon)$.

c. Solve equation (1.1 numerically for $a = 1$, $\epsilon = 2$, and $0 \leq t \leq 50$. Plot the results in the same graph as your part **b** answer. *Note how good the agreement is, even though ϵ is not small.* In fact, most of the error is due to a (constant) phase shift. After correcting for it, the *agreement is impressive* **(show this).**   **Challenge: can you explain why the agreement is so good?**

2   Problem 08.02.06 - Strogatz (Hopf bifurcation using a computer)

Statement for problem 08.02.06

For the following system

$$\frac{dx}{dt} = \mu\, x + y - x^3 \qquad \text{and} \qquad \frac{dy}{dt} = -x + \mu\, y + 2y^3, \tag{2.1}$$

a Hopf bifurcation occurs at the origin when $\mu = 0$. Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. For small values of μ, verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius R scales with μ as predicted by theory.

3 Problem 08.07.x1 - Forced overdamped linear system

Statement for problem 08.07.x1

Strogatz' problem 8.7.3 mentions "an overdamped linear oscillator (or an RC-circuit) forced by a square wave", yet the equation considered is

$$\dot{x} + x = F(t), \tag{3.1}$$

while in fact it should be

$$\epsilon \ddot{x} + \dot{x} + x = F(t), \tag{3.2}$$

where $0 < \epsilon \ll 1$. One could argue that, because ϵ is small, the term $\epsilon \ddot{x}$ can be neglected. Yet the problem is concerned with the long term behavior of the system, where even small effects can become important (if they accumulate).

The purpose of this problem is to show that, while neglecting the term $\epsilon \ddot{x}$ can cause small changes, it does not alter the qualitative conclusions of the problem. **Show that (3.2) is equivalent to TWO systems of the form (3.1)**

$$\dot{x_1} + r_2\, x_1 = F(t) \quad \text{and} \quad \dot{x_2} + r_1\, x_2 = F(t), \tag{3.3}$$

for appropriate choices of the constants $r_j > 0$ and functions x_j.
Hint. Write the equation in 2-vector form and project the solution along the eigendirections.

4 A linear center that is a nonlinear spiral.

Statement: A linear center that is a nonlinear spiral

Consider the system
$$\dot{x} = -y - x^3 \text{ and } \dot{y} = x. \tag{4.1}$$

For this system linearization predicts a center at the origin. However, by transforming the system into polar coordinates, it is easy to see that the origin is actually a nonlinear spiral. **Task #1: do this.**

Task #2. Implement a two-time expansion for solutions near the critical point,[1] and calculate the rate of approach of the solutions towards the critical point as $t \to \infty$. Is it exponential or algebraic? What is the rate (if exponential) exponent (if algebraic)?

THE END.

---

[1] The small parameter here is the radius ϵ of the neighborhood considered.

18.385J / 2.036J Nonlinear Dynamics and Chaos
Fall 2014