Twogrid Method:

$$\begin{array}{ccc} & \text{Fine:} \quad A_h \cdot u = b_h & \\ R_h^{2h} \downarrow & & \uparrow I_{2h}^h \\ & \text{Coarse:} \quad A_{2h} \cdot v = b_{2h} & \end{array}$$

(1) Iterate $A_h \cdot u = b_h \quad (\nu_1 \times \text{GS}) \rightsquigarrow u_h$
(2) Restrict residual $r_h = b_h - A_h u_h$ by $r_{2h} = R_h^{2h} \cdot r_h$
(3) Solve for coarse error: $A_{2h} \cdot e_{2h} = r_{2h}$
(4) Interpolate error: $e_h = I_{2h}^h \cdot e_{2h}$
Update $\tilde{u}_h = u_h + e_h$
(5) Iterate $A_h \cdot u = b_h \quad (\nu_2 \times \text{GS})$ starting with $\tilde{u}_h$

v-cycle:

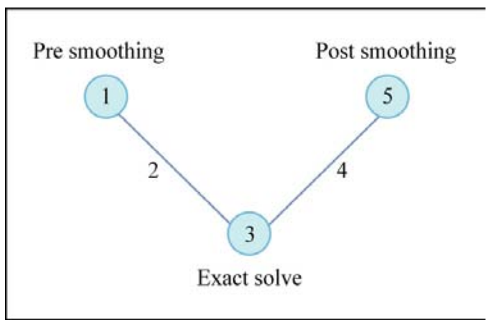
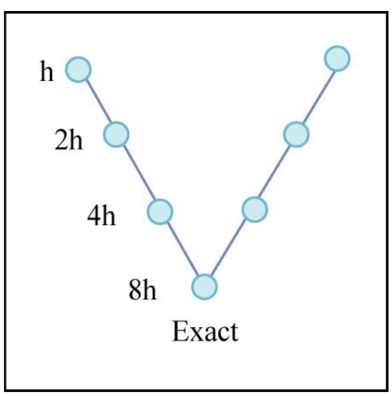


Image by MIT OpenCourseWare.

Multigrid:

- Use twogrid recursively in (3)

V-Cycle:



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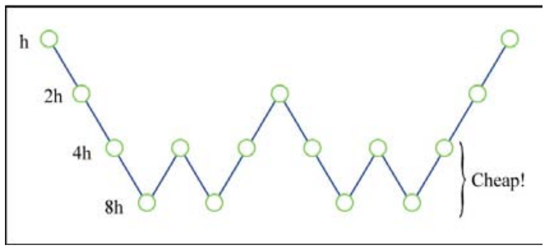
W-Cycle: Apply (3) twice



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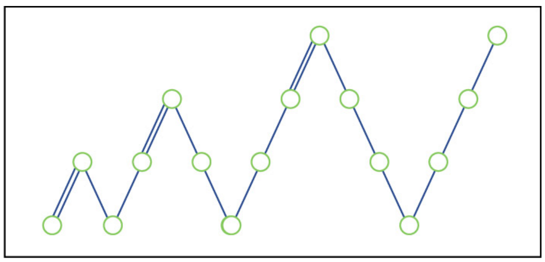
FMG (full multigrid):



Optimal: Cost = O(n). Image by MIT OpenCourseWare.

Krylov Methods

Consider $A \cdot x = b$ already preconditioned.
Iterative scheme: $x^{(k+1)} = x^{(k)} + (b - Ax^{(k)})$
$x^{(0)} = 0$
$x^{(1)} = b$
$x^{(2)} = 2b - Ab$
$x^{(3)} = 3b - 3Ab + A^2b$

Observe: $\boxed{x^{(k)} \in K_k}$

where $\left| \begin{array}{l} K_k = \text{span}\{b, Ab, \dots, A^{k-1}b\} \\ \text{Krylov subspace} \end{array} \right|$

$$\begin{array}{ccccccc} K_1 & \subset & K_2 & \subset & K_3 & \subset & \dots \\ \Cup & & \Cup & & \Cup & & \\ x^{(1)} & & x^{(2)} & & x^{(3)} & & \end{array}$$

Find sequence $x^{(k)} \in K_k$ which converges fast to $x = A^{-1} \cdot b$.
$\oplus$ Only requirement: Apply A (can be blackbox).

Examples of Krylov Methods:

Choose $x^{(k)} \in K_k$, such that
(1) $r_k = b - Ax_k \perp K_k \rightarrow$ conjugate gradients (CG)
(2) $||r_k||_2$ minimal $\rightarrow$ GRMES & MINRES
(3) $r_k \perp K_k(A^T) \rightarrow$ BiCG
(4) $||e_k||_2$ minimal $\rightarrow$ SYMMLQ

Conjugate Gradient Method

A symmetric positive definite
Enforce orthogonal residuals: $r_k \perp K_k$
Know $x_k \in K_k \Rightarrow r_k = b - Ax_k \in K_{k+1} \Rightarrow r_k = \gamma_k q_{k+1} \quad (\gamma_k \in \mathbb{R})$
where $q_1, q_2, q_3, \ldots$ orthonormal, and $q_k \in K_k$.
$\Rightarrow {r_i}^T \cdot r_k = 0 \ \forall i < k$.

$$\left.\begin{array}{rcl} \Delta r_k & = & r_k - r_{k-1} \perp K_i \\ \Delta x_i & = & x_i - x_{i-1} \in K_i \end{array}\right\} \Rightarrow {\Delta x_i}^T \cdot \Delta r_k = 0 \ \forall i < k$$

Also: $\Delta r_k = (b - Ax_k) - (b - Ax_{k-1}) = -A \cdot \Delta x_k$

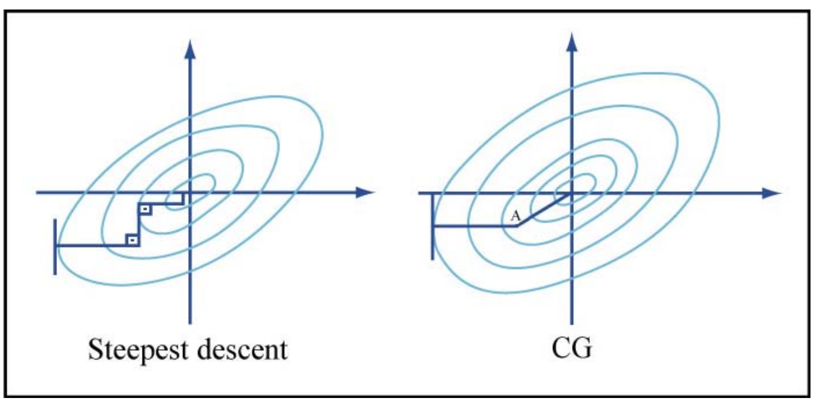
$$\Rightarrow \boxed{{\Delta x_i}^T \cdot A \cdot \Delta x_k = 0 \ \forall i < k}$$

Updates (= directions) are "A-orthogonal" or "conjugate";
Scalar product $(\Delta x_i, \Delta x_k)_A := {\Delta x_i}^T \cdot A \cdot \Delta x_k$.
Search direction: d_{k-1}
Update solution: $x_k = x_{k-1} + \alpha_k d_{k-1}$
New direction: $d_k = r_k + \beta_k d_{k-1}$

$\alpha_k = \dfrac{{||r_{k-1}||_2}^2}{d_{k-1}^T \cdot A \cdot d_{k-1}}$, so that error in direction d_{k-1} minimal

$\beta_k = \dfrac{{||r_k||_2}^2}{{||r_{k-1}||_2}^2}$, so that $(d_k, d_{k-1})_A = 0$

CG finds unique minimizer of $E(x) = \frac{1}{2} x^T \cdot A \cdot x - x^T \cdot b \quad (x_{\min} = A^{-1} \cdot b)$
using conjugate directions after at most n steps.



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In practice <u>much</u> faster than n steps.