Chapter 14

14.1 Krylov Subspace Methods

Idea: Use the space $\mathcal{K}_k = \text{span}\{x, Ax, \cdots, A^{k-1}x\}$, write $\mathcal{K}_k = [x, Ax, \cdots, A^{k-1}x]$ matrix for eigenvalue problems: Project A onto $\mathcal{K}_k$ and find the eigenvalues of the projection $\mathcal{K}_k = QR$, where Q is a basis of $\mathcal{K}_k$. Write $H = Q^T AQ$, find eigenvalues of H (non-symmetric called Arnoldi, symmetric called Lanczoz).

For $Ax = b$, find best $x \in \mathcal{K}_n$. Two possibilities for best:

- MINRES (non-symmetric case)

$$\min_{x\in\mathcal{K}_k} \|Ax - b\|_2 = \min_{x\in\mathcal{K}_k} \|r\|_2 \tag{14.1}$$

- Conjugate gradients

$$\min_{x\in\mathcal{K}_k} \|r\|_{A^{-1}} \tag{14.2}$$

How it works:
Consider

$$\mathcal{K}_n = \text{span}\{x, Ax, \cdots, A^{n-1}x\} \tag{14.3}$$

Let $K_n = [x, Ax, \cdots, A^{n-1}x]$
Write

$$y_1 = x_1 \tag{14.4}$$

$$y_2 = Ax_1 \tag{14.5}$$

$$\vdots$$

$$\begin{aligned} y_n &= A^{n-1}x && (14.6) \\ AK &= [y_2, \cdots, y_n, A^n y_1] && (14.7) \end{aligned}$$

Write

$$\begin{aligned} c &= -K^{-1}A^n y_1 && (14.8) \\ AK &= K[e_2, e_3, \cdots, e_n, -c] \\ &\equiv K \cdot C && (14.9) \end{aligned}$$

where, $C = \begin{bmatrix} 0 & & & -c_1 \\ 1 & & & \vdots \\ & \ddots & & \vdots \\ & & 1 & -c_n \end{bmatrix}$, Hessenberg.

Write

$$\begin{aligned} K &= QR && (14.10) \\ K^{-1}AK &= C \\ (QR)^{-1}AQR &= \\ R^{-1}Q^T AQR &= && (14.11) \\ Q^T AQ &= RCR^{-1} \\ &= H - \text{Hessenberg} && (14.12) \\ \text{Partition} \quad Q &= [Q_1|Q_2] && (14.13) \\ ([Q_1|Q_2])^T A[Q_1|Q_2] &= H \\ \left[\begin{array}{c|c} Q_1^T AQ_1 & Q_1^T AQ_2 \\ \hline Q_2^T AQ_1 & Q_2^T AQ_2 \end{array}\right] &= H. && (14.14) \end{aligned}$$

where, $Q_1^T AQ_1$ still Hessenberg, if $A^T = A \Rightarrow$ tridiag.
How do we compute Q? One column at a time.

$$\begin{aligned} Q^T AQ &= H && (14.15) \\ Aq_i &= \sum_{i=1}^{j+1} h_{ij} q_i && (14.16) \\ q_m^T Aq_j &= h_{mj}, \quad 1 \leq m \leq j && (14.17) \\ h_{j+1,j} q_{j+1} &= Aq_j - \sum_{i=1}^{j} h_{ij} q_i && (14.18) \end{aligned}$$

14.2 Arnoldi Method

```
q_1 = x/||x||_2;
for j = 1:k (k = number of columns Q to compute)
   z = A q_j
   for i = 1:j
      h_ij = q_i^T z
      z = z - h_ij q_i
   end
   h_{i+1,j} = ||z||_2
   q_{j+1} = z/h_{i+1,j}
end
```

$q_1 = \frac{x}{\|x\|_2}$;
for $j = 1 : k$ ($k =$ number of columns Q to compute)
  $z = Aq_j$
  for $i = 1 : j$
    $h_{ij} = q_i^T z$
    $z = z - h_{ij} q_i$
  end
  $h_{i+1,j} = \|z\|_2$
  $q_{j+1} = \frac{z}{h_{i+1,j}}$
end