Pak
11/7/05

$\mathcal{B}_n$ = poset of $P([n])$

Thm (Sperner, 1928)
Every antichain has size $\leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$

Pf (from Jukna): $\mathcal{F} = \{A_i \mid A_i \subseteq [n]\}$ antichain
Pick $A_i \in \mathcal{F}$ + look at all maxl chains in $[n]$ containing A_i. $|A_i| = k \Rightarrow$ # chains is $k! \cdot (n-k)!$
Also none of the maxl chains in $\mathcal{B}_n$ meet $\mathcal{F}$ more than once, so $n! \geq \sum |A_i|! (n - |A_i|)! \geq |\mathcal{F}| \lfloor \frac{n}{2} \rfloor! \lceil \frac{n}{2} \rceil!$
$\Rightarrow |\mathcal{F}| \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$ ✓

Thm (Mahtel, 1907)
$|V(G)| = 2n \quad |E(G)| \geq n^2+1$
$\Rightarrow \Delta \geq G$
Pf: (from Jukna Ch 4) induction base $n=1$ ✓
step: $2n+2$ vertices, pick $(xy) \in E$, $H = G - x - y$
$|E(H)| \geq n^2+1 \Rightarrow$ ✓, $|E(H)| < n^2+1 \Rightarrow$
$|N(x)|_{\not\sim y} + |N(y)|_{\not\sim x} \geq 2n+1 \Rightarrow N(x) \cap N(y) \neq \emptyset$ ✓

Second proof is more lively:
$A \subset V(G)$ largest indep set (i.e. A indep, $|A| = \alpha(G)$)
$B = V - A \quad \forall x \in B \quad N(x) \cap A \neq \emptyset$ and
$|N(x)| \leq \alpha(G) = |A|$ (c/w $\exists$ ~~Δ~~ uxv Δ)
$|E| \leq \sum_{x \in B} \deg x \leq \sum_{x \in B} |A| = |B| \cdot |A| \leq \left(\frac{|A|+|B|}{2}\right)^2 = n^2$ ✓
(↘ b/c every edge has at least one vx in B)

Thm (Graham-Kleitman 1973)
$\alpha : E(K_m) \leftrightarrow [\binom{n}{2}]$ labelling of edges
Then the longest increasing trail has length $\geq (n-1)$
(trail: oriented path that can go through the same vx more than once)

Pf: Let w_x = length of longest inc. trail ending at x

ETST $\sum_x w_x \geq n(n-1)$

Start: $\sum_x w_x = 0$ $\quad i^{th}$ step: look at $e_i = xy$

this ~~increases w_x by w_y~~ makes

$w_x' = \max\{w_x, w_y + 1\}$ $\quad w_y' = \max\{w_x + 1, w_y\}$

$\Rightarrow$ every edge addition increases weightsum by 2

$\Rightarrow \sum w_x \geq \binom{n}{2} 2$ ✓

■