Course 18.312: Algebraic Combinatorics

Lecture Notes # 6 Addendum by Gregg Musiker

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1 Partially Ordered Sets II: Dilworth's Theorem

Definition. An **anitchain** A of a poset P is a subset of elements of P such that for all $x, y \in A$, $x \not\leq y$ and $y \not\leq x$.

We denote the levels of a graded poset P as P_i where $P_i = \{x \in P : \text{ rank}(x) = i\}$.

Remark. Observe that each P_i is an antichain, thus we have the inequality

$$\max\{|A| : A \text{ is an antichain of } P\} \;\geq\; \{|P_i| : 0 \leq i \leq \text{rank}(P)\}.$$

We say that the poset P has the **Sperner Property** if this inequality is acutally an equality. A combinatorial condition to check if P has the Sperner property is discussed in Section 4 of Richard Stanley's notes "Topics in Algebraic Combinatorics".

Definition. A **chain cover** of a poset P is a collection of chains whose union is P.

(Robert) Dilworth's Theorem 1950. In any finite poset, the minimum size of a chain cover *equals* the maximum size of an antichain.

Related Proposition. In any finite poset, the minimum size of an antichain cover equals the maximum size of a chain.

Proof. If C is a chain and $\mathcal{A}$ is an antichain cover, then no antichain in $\mathcal{A}$ can contain more than one element of C, i.e. $\mathcal{A} = \{A_1, A_2, \ldots, A_k\}$ and $|A_i \cap C| \leq 1$, so $|\mathcal{A}| \geq |C| =$ the maximum size of a chain.

On the other hand, we can define the following antichain cover to show that equality is actually achieved:

$$A_i = \{x \in P \text{ s.t. the longest chain with maximal element } x \text{ has length } i\}.$$

These A_i's are like the levels P_i however we do not need to assume that P is graded. Consequently, $|\mathcal{A}|$ is the size of the longest chain, by construction.

Proof of Dilworth's Theorem. Turning to the other situation, notice that if we pick a chain cover $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ where these chains are pairwise disjoint, then for any antichain A, we have $|A \cap C_i| \leq 1$, and thus $|\mathcal{C}| \geq |A|$. Thus the maximum size of a chain cover is *greater than or equal to* the maximum sixe of an antichain. However, unlike the related proposition, this time the equality is not as easy to see. We follow an inductive proof based on a proof of Fred Galvin (1999).

Firstly, Dilworth's theorem is trivial if P is empty, so we may assume that P contains at least one element and we therefore let a denote a maximal element of P. By induction, we have that Dilworth's theorem holds for $P' = P \setminus \{a\}$, thus there exist k disjoint chains $C_1, \ldots, C_k$ and at least one antichain A_0 of size k such that $A_0 \cap C_i \neq \emptyset$ for $i = 1, 2, \ldots, k$. (If there were to exist a C_i such that $A_0 \cap C_i = \emptyset$, then we would have $|A_0| < |\mathcal{C}|$, going against our assumption that $|A_0| = k = |\mathcal{C}|$.)

We then use A_0's existence to build another antichain: let x_i be the maximal element in C_i that belongs to an antichain of length k in P'. (Note that if A_0 was the only antichain of size k then $\{x_1, \ldots, x_k\}$ would be A_0, but otherwise $A := \{x_1, \ldots, x_k\}$ could be a different set.)

Claim. This set that we just built, A, is an antichain.

Proof. If $x_j \geq x_i$, then the maximal element of C_i belonging to an antichain of length k would be comparable to the maximal element of C_j belonging to an antichain of length k. However, then the set A' which contains x_j and is an antichain of length k would satisfy $A' \cap C_i \neq \emptyset$ so there would exist $y \in A' \cap C_i$. But then, y would be in an antichain of length k and $x_i \geq y$ by x_i's maximality, and we get a contradiction since x_j and y were supposed to both be part of the antichain A'.

We finish the proof of Dilworth's Theorem by reducing to two cases: (1) If P contains a chain cover of size $(k+1)$ (larger is impossible), then by the maximality of a, we must have that $\{a\}$ is a chain. Otherwise, $P \setminus \{a\}$ would also contain a chain cover of size $(k+1)$ and we would have a contradiction. In this case, $\{a\} \cup A$ is an antichain of size $(k+1)$.

(2) On the other hand, if $\{a\} \cup C_i$ is a chain of P, then $a \geq x_i$ where $x_i \in A$ was defined to be the maximal element of chain C_i (in $P \setminus \{a\}$) contained in an antichain of size k. We let $C' = \{a\} \cup \{y \in C_i : y \leq x_i\}$. The poset $P \setminus C'$ cannot contain an antichain of size k, only one of size $(k-1)$. Therefore by induction and Dilworth's Theorem, $P \setminus C'$ can be covered by $(k-1)$, but not k, disjoint chains. Therefore, in this case, P's largest antichain is of size k and largest chain cover is also of size k.