Course 18.312: Algebraic Combinatorics

Lecture Notes # 27 Addendum by Gregg Musiker

April 15th, 2009

Today's lecture notes cover the Oriented Matrix Theorem, which is discussed in Sections 9 and 10 of Richard Stanley's "Topics in Algebraic Combinatorics" lecture notes. The proof presented in class more closely resembles the bijective proof in Section 4.4 of "Constructive Combinatorics" by Dennis Stanton and Dennis White. I also thank Adriano Garsia for introducing me to this proof.

1 A Combinatorial Proof of the Oriented Matrix Theorem

We begin with a method to turn an oriented rooted tree into a word:

$$w(T) = a_{1,fin(e_1)}a_{2,fin(e_2)}a_{3,fin(e_3)}\ldots,\widehat{a_{r,fin(e_r)}},\ldots,a_{n,fin(e_n)}.$$

Here, we have encoded each edge $(init(e), fin(e))$ as a variable $a_{init(e),fin(e)}$. Since T is an oriented and rooted tree, this means that every vertex besides the root r has precisely one edge eminating from it. (So in particular, the tree has exactly $|V|-1$ edges and each tree encodes a monomial of degree $(|V|-1)$.) We denote the edge having $init(e) = k$ as e_k. Writing the $a_{k,fin(e_k)}$ in order, excluding the root yields the above expression for $w(T)$. **Exercise:** Build the unique tree (up to isomorphism) with word $w(T) = a_{12}a_{25}a_{35}a_{43}a_{65}a_{73}a_{8,10}a_{9,10}a_{10,2}a_{11,3}$. Hint: it has vertex 5 as a root.

Example. There are 16 Cayley trees on 4 nodes (i.e. spanning trees of K_4), four of which are rooted at vertex 4. The words associated to these four trees are

$$a_{12}a_{23}a_{34},\quad a_{12}a_{24}a_{31},\quad a_{13}a_{23}a_{34},\ \text{and}\ \ a_{13}a_{24}a_{34}.$$

Since every vertex has exactly one outgoing edge, the words for these four trees are a subset of the monomials obtained by multiplying out the expression

$$(a_{12} + a_{13} + a_{14})(a_{21} + a_{23} + a_{24})(a_{31} + a_{32} + a_{34}).$$

However, some of the resulting terms, such as $a_{12}a_{23}a_{31}$, do not correspond to trees, but instead correspond to a digraph containing a cycle.

Theorem. For any complete graph K_{n+1}, then the sum of all monomials corresponding to oriented spanning trees of K_{n+1} which are rooted at vertex $(n+1)$ equals the determinant of the matrix $L(K_{n+1})$:

$$\begin{bmatrix} a_{12}+a_{13}+\cdots+a_{1,n+1} & -a_{12} & -a_{13} & \cdots & -a_{1n} \\ -a_{21} & a_{21}+a_{23}+\cdots+a_{2,n+1} & -a_{23} & \cdots & -a_{2n} \\ -a_{31} & -a_{32} & a_{31}+a_{32}+\cdots+a_{3,n+1} & \cdots & -a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & -a_{n3} & \cdots & a_{n1}+a_{n2}+\cdots+a_{n,n+1} \end{bmatrix}$$

Note that this determinant can also be restated as the alternating sum

$$\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) b_{1,\sigma(1)} b_{2,\sigma(2)} \cdots b_{n,\sigma(n)}$$

where

$$b_{ij} = \begin{cases} a_{i1} + a_{i2} + \cdots + a_{i,n+1} \text{ if } i = j \\ -a_{i,j} \text{ otherwise} \end{cases}.$$

Remark. This Theorem is equivalent to the oriented matrix tree theorem for a general multigraph with directed edges. In particular, if one replaces each instance of the variable a_{ij} with the number of directed edges from vertex i to vertex j (which can possibly be zero) then the determinant of this matrix exactly equals the number of oriented spanning trees rooted at vertex $(n+1)$.

We now work towards the proof of this theorem for the case of the digraph K_{n+1}. We first introudce the notation $R_i = a_{i1} + a_{i2} + \cdots + a_{i,n+1}$, $\delta_{i,j} = 1$ if $i = j$ and 0 otherwise, and the convention that $a_{ii} = 0$ for all i. This allows us to rewrite the above determinant as

$$\begin{aligned} & \sum_{\sigma \in S_n} \prod_{i=1}^{n} (R_i \delta_{i,\sigma(i)} - a_{i,\sigma(i)}) \\ = & \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \sum_{S \subseteq \{1,2,\ldots,n\}} \prod_{i \in S} (-a_{i,\sigma(i)}) \prod_{i \in \{1,2,\ldots,n\} \setminus S} R_i \delta_{i,\sigma(i)}. \end{aligned}$$

Rearranging the sum, and letting $\overline{S}$ denote $\{1, 2 \ldots, n\} \setminus S$, yields

$$\begin{aligned}
\det L(K_{n+1}) &= \sum_{S \subseteq \{1,2,\ldots,n\}} \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i \in S} (-a_{i,\sigma(i)}) \prod_{i \in \overline{S}} R_i \delta_{i,\sigma(i)} \\
&= \sum_{S \subseteq \{1,2,\ldots,n\}} \sum_{\substack{\sigma \in S_n, \sigma(i)=i \text{ for all } i \in \overline{S} \\ \sigma(i) \neq i \text{ for any } i \in S}} \operatorname{sgn}(\sigma)(-1)^{|S|} \prod_{i \in S} a_{i,\sigma(i)} \prod_{i \in \overline{S}} R_i.
\end{aligned}$$

Note that this sum now involves only permutations σ with no fixed points in set S. We call such a permutation a **derangement** on S and denote this set as $D(S)$ for shorthand. We also let $a(\sigma)$ be shorthand for $\prod_{i \in S} a_{i,\sigma(i)}$.

Claim 1. The terms in the expansion of $\prod_{i \in \overline{S}} R_i$ correspond to fixed-point-free maps from $\overline{S}$ to $\{1, 2, \ldots, n+1\}$.

Proof. Each monomial in this expansion has the form $a_{i_1,j_1} a_{i_2,j_2} \cdots a_{i_k,j_k}$ where $\{i_1, i_2, \ldots, i_k\}$ denotes the elements of $\overline{S}$ in order. Since we assume that $a_{i,i} = 0$ for all i, the only terms appearing in this expansion are those where $j_d \neq i_d$ for all $1 \leq d \leq k$. Since each i_d appears as the first index exactly once, we let this monomial denote the function f such that $f(i_d) = j_d$ for all $1 \leq d \leq k$. Such an f is therefore a fixed-point-free map with the correct domain and range.

Claim 2. The sums

$$\sum_{S \subseteq \{1,2,\ldots,n\}} \sum_{\sigma \in D(S)} \sum_{f \text{ is a fixed-point-free map } : \overline{S} \to \{1,2,\ldots,n+1\}} (-1)^{\#\text{cycles in } \sigma} a(\sigma)\, a(f), \quad (1)$$

$$\text{and} \sum_{f \text{ is a cycle free map } : \{1,2,\ldots,n\} \to \{1,2,\ldots,n+1\}} a(f) \quad (2)$$

are equal.

We defer the proof of Claim 2 for the moment to observe that together Claims 1 and 2 imply the Oriented Matrix Tree Theorem.

Remark. By definition, $\operatorname{sgn}(\sigma) = (-1)^{(\#\text{cycles in } \sigma) - |S|}$ since $\sigma(i) = i$ for all $i \in \overline{S}$.

It follows then by Claim 1, the notation $a(f) = \prod_{i \in \overline{S}} a_{i,f(i)}$, and the definition of $D(S)$ that $\det L(K_{n+1})$ equals the expression (1). Claim 2 then implies that $\det L(K_{n+1})$ also equals (2). Lastly, each cycle-free map f uniquely encodes an oriented rooted spanning tree using $a(f) = w(T)$, the word of the tree defined in the introduction.

We have therefore reduced the proof to the verification of Claim 2. We proceed using a technique known as a **sign-reversing involution**. This technique will be used again in lectures 36 and 37 when we count non-intersecting lattice paths in a directed acyclic graph (using Gessel-Viennot theory, also known as Lindström's Lemma).

Proof of Claim 2. The triple sum in expression (1) sums over the possible choices of maps $f : \{1, 2, \dots, n\} \to \{1, 2, \dots, n+1\}$ with no fixed points. The domain of such a map can be broken up into two sets, S and $\overline{S}$ (with S possibly empty), so that the restriction of f to S is a permutation σ with no fixed points, and the image of $f(\overline{S})$ is entirely contained in $\overline{S}$.

Example: The reader may find it helpful to draw the digraph corresponding to the word

$$a_{12}a_{24}a_{32}a_{47}a_{56}a_{65}a_{73},$$

and attempt to decompose this in at least one way into a disjoint set of digraphs on S and $\overline{S}$. **Hint:** For example, one may let $S = \{5, 6\}$ so that $a(\sigma) = a_{56}a_{65}$, and $a(f) = a_{12}a_{24}a_{32}a_{47}a_{73}$. Another decomposition would have $S = \{2, 3, 4, 5, 6, 7\}$, $a(\sigma) = a_{24}a_{47}a_{73}a_{32}a_{56}a_{65}$, and $a(f) = a_{12}$.

By convention, we color the digraph on set S, corresponding to $a(\sigma)$, **red** and the digraph on set $\overline{S}$, corresponding to $a(f)$, **blue**. Since f has no fixed points, it follows that if set S is non-empty, then it contains at least two elements and the digraph corresponding to the restriction $f|_S$ contains at least one cycle. **Thus the red digraph contains at least one cycle unless S is empty.**

We define a map on the space of these bicolored digraphs that takes the cycle containing the vertex with the lowest index (with respect to the ordering $\{1, 2, 3 \dots, n+1\}$) and switches the **color** of this cycle. Observe that changing this color still results in an $(S, \overline{S})$-decomposition, i.e. a (σ, f)-decomposition, that is still valid. The resulting word of this bicolored digraph is unchanged however the exponent of the sign $(-1)^{\#\text{cycles in } \sigma}$ is changed by precisely one since we only enumerate cycles in the set S. Thus, in the total sum, these two words **cancel** out each other and the entire sum reduces to the sum of contributions which do not pair-off in this way. This results in only the fixed points, or the elements which are not changed by this map. In this case, these terms exactly correspond to the **blue acyclic digraphs**, i.e. the digraphs where $S = \emptyset$ and we obtain expression (2).

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