Course 18.312: Algebraic Combinatorics

In-Class Exam # 2

April 17, 2009

Open notes. Closed Friends and Enemies. No calculators, computers, I-pods, or Zunes. Please explain your reasoning or method, even for computational problems. You may do the problems in any order. There is a total of 100 points. Good Luck.

0) (5 points) Please state a tentative title of your final project or a one-two sentence description.

1) Consider the full binary tree T containing 4 leaves (seven vertices in all). Consider coloring of T up to isomorphism. The symmetry group of T is a group of order 8.

   (15 points) a) What is the cycle-index polynomial of G acting on the vertices of T?

   (10 points) b) In how many ways can the vertices of T be colored in n colors up to reflective symmetry?

2) (20 points) In how many ways can we begin with the empty partition $\emptyset$, then **add** $2n$ squares one at a time (always keeping a partition), then **remove** n squares at a time, then **add** n squares at a time, and finally **remove** $2n$ squares one at a time, ending up at $\emptyset$?

3) Let G be a regular loopless (undirected) graph of degree d with p vertices and q edges.

   (5 points) a) Find a simple relation between p, q, and d.

   (5 points) b) Express the biggest eigenvalue of the adjacency matrix A of G in terms of p, q, and d.

   (You may use the fact from matrix theory that if M is a $p \times p$ matrix whose entries are nonnegative real numbers, and if z is a column vector of positive real numbers such that $Mz = \lambda z$, then λ is the largest (in absolute value) eigenvalue of M.)

(5 points) c) Suppose that G has no multiple edges. Express the number of closed walks in G of length two in terms of p, q, and d.

Suppose that G has no multiple edges and that the number of closed walks in G of length ℓ is given by

$$4^\ell + 5(-2)^\ell + 3 \cdot 2^\ell.$$

(10 points) d) Find the number $\kappa(G)$ of spanning trees of G. (Don't forget that A may have some eigenvalues equal to 0.) For full credit, give a purely numerical answer, not involving p, q, or d, but leaving exponents in your expression is okay.

4) Let $f(n)$ denote the number of permutations in the symmetric group S_n, all of whose cycles have length divisible by three.

(15 points) a) Let

$$F(x) = \sum_{n=0}^{\infty} f(n) \frac{x^n}{n!}.$$

Find a simple expression for $F(x)$. For full credit, your answer should not involve any summation symbols (or their equivalent), logarithms, or the function e^x.

(Bonus 5 points) b) Use part (a) to find a formula for $f(n)$. The answer should be expressed in terms of a binomial coefficient $\binom{r}{s}$ where r need not be an integer, but s is a nonnegative intger.

5) (5 points) a) Write the elementary symmetric function e_{41} as a sum of h_λ's.

(5 points) b) Write the Schur function $s_2 - s_{11}$ as a symmetric polynomial in the variables $\{x_1, x_2, x_3\}$.

(Bonus 10 points) c) For $k \geq 2$, write the Schur function $s_k - s_{k-1,1}$ as a symmetric polynomial in the variables $\{x_1, x_2\}$.

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