Course 18.312: Algebraic Combinatorics

Problem Set # 4

Due Wednesday March 4th, 2009

You may discuss the homework with other students in the class, but please write the names of your collaborators at the top of your assignment. Please be advised that you should not just obtain the solution from another source. Please explain your reasoning to receive full credit, even for computational questions.

1) (10 points) Fix some positive integer $k \geq 2$. Give an algebraic or a bijective proof that the number of partitions of n in which every part appears at most $(k-1)$ times equals the number of partitions where every part is not divisible by k.

   (**Bonus 5 points**) Give both an algebraic and a bijective proof of this result.

2) A partition λ is **self-conjugate** if $\lambda^T = \lambda$. Let $sc(n)$ denote the number of partitions of n which are self-conjugate.

   (10 points) Deduce and prove an expression for the generating function of the sequence $\{sc(n)\}_{n\geq 1}$ as a simple infinite sum of rational expressions.

   Hint: Enumerate partitions by the size of their **Durfee square**, which is defined to be the largest square in the northwest quadrant of a partition. For example, the partition $[5,5,4,3,2]$ has a Durfee square of size 3-by-3.

3) Let $r(n)$ denote the number of partitions of n whose parts differ by at least 2 and $s(n)$ denote the number of those where, in addition, the part 1 does not appear.

   (10 points) Prove that $\sum_{n\geq 0} r(n)x^n = \sum_{k\geq 0} \frac{x^{k^2}}{\prod_{i=1}^{k}(1-x^i)}$.

   (**Bonus 5 points**) Prove that $\sum_{n\geq 0} r(n)x^n = \prod_{j=0}^{\infty} \frac{1}{(1-x^{5j+1})(1-x^{5j+4})}$.

   (10 points) Prove that $\sum_{n\geq 0} s(n)x^n = \sum_{k\geq 0} \frac{x^{k^2+k}}{\prod_{i=1}^{k}(1-x^i)}$.

   (**Bonus 5 points**) Prove that $\sum_{n\geq 0} s(n)x^n = \prod_{j=0}^{\infty} \frac{1}{(1-x^{5j+2})(1-x^{5j+3})}$.

4) (5 points) Give an example of a finite graded poset P with the Sperner property, together with a group G acting on P, such that the quotient poset P/G is not Sperner. (**Hint:** By Theorem 5.9, P cannot be a boolean poset.)

5) Let q be a prime power, and let $\mathbb{F}_q$ denote the finite field with q elements. Let $V_n(q) = \mathbb{F}_q^n$, the n-dimensional vector space over $\mathbb{F}_q$ of n-tuples of elements $\mathbb{F}_q$. Let $B_n(q)$ denote the poset of all subspaces of $V_n(q)$, ordered by inclusion. It's easy to see that $B_n(q)$ is graded of rank n, the rank of a subspace of $V_n(q)$ being its dimension.

(5 points) Draw the Hasse diagram of $B_3(2)$. (**Hint:** It has 16 elements.)

(5 points) Compute the Möbius function $\mu(\{0\}, V)$ for each element $V \in B_3(2)$.

(10 points) Show that the number of elements of $B_n(q)$ of rank k is given by the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q^n-1)(q^{n-1}-1)\cdots(q^{n-k+1}-1)}{(q^k-1)(q^{k-1}-1)\cdots(q-1)}.$$

(**Hint:** One way to do this is to count in two ways the number of k-tuples $(v_1, \ldots, v_k)$ of linearly independent elements from $\mathbb{F}_q^n$. (1) First choose v_1, then v_2, etc. and (2) First choose the subspace W spanned by $v_1, \ldots, v_k$, and then choose v_1, v_2, etc.)

(5 points) Show that $B_n(q)$ is rank-symmetric. (**Hint:** Use part (iii).)

(5 points) Show that every element $x \in B_n(q)_k$ covers $[k] = 1 + q + \cdots + q^{k-1}$ elements and is covered by $[n-k] = 1 + q + \cdots + q^{n-k-1}$ elements.

(5 points) Show that if V has dimension k ($0 \le k \le n$), then the interval $[0, V]$ in $B_n(q)$ is isomorphic to the poset $B_k(q)$.

(**Bonus 5 points**) Deduce and prove a formula for the values of the Möbius function $\mu(\{0\}, V)$ for general $B_n(q)$.

Define operators $U_i : \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i+1}$ and $D_i : \mathbb{R}B_n(q)_i \to \mathbb{R}B_n(q)_{i-1}$ by

$$U_i(x) = \sum_{y \in B_n(q)_{i+1}, y > x} y \quad \text{and} \quad D_i(x) = \sum_{z \in B_n(q)_{i-1}, z < x} z.$$

(10 points) Show that $D_{i+1}U_i - U_{i-1}D_i = ([n-i] - [i])I_i$.

(10 points) Deduce that $B_n(q)$ is rank-unimodal and Sperner.

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Spring 2009