# The 1-D Heat Equation

18.303 Linear Partial Differential Equations

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## 1 The 1-D Heat Equation

### 1.1 Physical derivation

Reference: Guenther & Lee §1.3-1.4, Myint-U & Debnath §2.1 and §2.5

[Sept. 8, 2006]

In a metal rod with non-uniform temperature, heat (thermal energy) is transferred from regions of higher temperature to regions of lower temperature. Three physical principles are used here.

1. Heat (or thermal) energy of a body with uniform properties:

$$\text{Heat energy } = cmu,$$

where $m$ is the body mass, $u$ is the temperature, $c$ is the specific heat, units $[c] = L^2T^{-2}U^{-1}$ (basic units are $M$ mass, $L$ length, $T$ time, $U$ temperature). $c$ is the energy required to raise a unit mass of the substance 1 unit in temperature.

2. Fourier's law of heat transfer: rate of heat transfer proportional to negative temperature gradient,

$$\frac{\text{Rate of heat transfer}}{\text{area}} = -K_0 \frac{\partial u}{\partial x} \tag{1}$$

where $K_0$ is the thermal conductivity, units $[K_0] = MLT^{-3}U^{-1}$. In other words, heat is transferred from areas of high temp to low temp.

3. Conservation of energy.

Consider a uniform rod of length $l$ with non-uniform temperature lying on the $x$-axis from $x = 0$ to $x = l$. By uniform rod, we mean the density $\rho$, specific heat $c$, thermal conductivity $K_0$, cross-sectional area $A$ are ALL constant. Assume the sides of the rod are insulated and only the ends may be exposed. Also assume there is no heat source within the rod. Consider an arbitrary thin slice of the rod of width $\Delta x$ between $x$ and $x + \Delta x$. The slice is so thin that the temperature throughout the slice is $u(x,t)$. Thus,

$$\text{Heat energy of segment} = c \times \rho A \Delta x \times u = c\rho A \Delta x u(x,t).$$

By conservation of energy,

$$\begin{array}{l}\text{change of}\\ \text{heat energy of}\\ \text{segment in time } \Delta t\end{array} = \begin{array}{l}\text{heat in from}\\ \text{left boundary}\end{array} - \begin{array}{l}\text{heat out from}\\ \text{right boundary}\end{array}.$$

From Fourier's Law (1),

$$c\rho A \Delta x u(x, t + \Delta t) - c\rho A \Delta x u(x,t) = \Delta t A \left(-K_0 \frac{\partial u}{\partial x}\right)_x - \Delta t A \left(-K_0 \frac{\partial u}{\partial x}\right)_{x+\Delta x}$$

Rearranging yields (recall $\rho$, $c$, $A$, $K_0$ are constant),

$$\frac{u(x, t + \Delta t) - u(x,t)}{\Delta t} = \frac{K_0}{c\rho} \left( \frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x} \right)$$

Taking the limit $\Delta t, \Delta x \to 0$ gives the Heat Equation,

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \tag{2}$$

where

$$\kappa = \frac{K_0}{c\rho} \tag{3}$$

is called the thermal diffusivity, units $[\kappa] = L^2/T$. Since the slice was chosen arbitrarily, the Heat Equation (2) applies throughout the rod.

## 1.2 Initial condition and boundary conditions

To make use of the Heat Equation, we need more information:

1. Initial Condition (IC): in this case, the initial temperature distribution in the rod $u(x,0)$.

2. Boundary Conditions (BC): in this case, the temperature of the rod is affected by what happens at the ends, $x = 0, l$. What happens to the temperature at the end of the rod must be specified. In reality, the BCs can be complicated. Here we consider three simple cases for the boundary at $x = 0$.

(I) Temperature prescribed at a boundary. For $t > 0$,

$$u(0,t) = u_1(t).$$

(II) Insulated boundary. The heat flow can be prescribed at the boundaries,

$$-K_0 \frac{\partial u}{\partial x}(0,t) = \phi_1(t)$$

(III) Mixed condition: an equation involving $u(0,t)$, $\partial u/\partial x(0,t)$, etc.

Example 1. Consider a rod of length $l$ with insulated sides is given an initial temperature distribution of $f(x)$ degree C, for $0 < x < l$. Find $u(x,t)$ at subsequent times $t > 0$ if end of rod are kept at $0^o$ C.

The Heat Eqn and corresponding IC and BCs are thus

$$\text{PDE:} \qquad u_t = \kappa u_{xx}, \qquad 0 < x < l, \tag{4}$$

$$\text{IC:} \qquad u(x,0) = f(x), \qquad 0 < x < l, \tag{5}$$

$$\text{BC:} \qquad u(0,t) = u(L,t) = 0, \qquad t > 0. \tag{6}$$

Physical intuition: we expect $u \to 0$ as $t \to \infty$.

## 1.3 Non-dimensionalization

Dimensional (or physical) terms in the PDE (2): $k$, $l$, $x$, $t$, $u$. Others could be introduced in IC and BCs. To make the solution more meaningful and simpler, we group as many physical constants together as possible. Let the characteristic length, time and temperature be $L_*$, $T_*$ and $U_*$, respectively, with dimensions $[L_*] = L$, $[T_*] = T$, $[U_*] = U$. Introduce dimensionless variables via

$$\hat{x} = \frac{x}{L_*}, \qquad \hat{t} = \frac{t}{T_*}, \qquad \hat{u}\left(\hat{x}, \hat{t}\right) = \frac{u(x,t)}{U_*}, \qquad \hat{f}(\hat{x}) = \frac{f(x)}{U_*}. \tag{7}$$

The variables $\hat{x}$, $\hat{t}$, $\hat{u}$ are dimensionless (i.e. no units, $[\hat{x}] = 1$). The sensible choice for the characteristic length is $L_* = l$, the length of the rod. While $x$ is in the range $0 < x < l$, $\hat{x}$ is in the range $0 < \hat{x} < 1$.

The choice of dimensionless variables is an ART. Sometimes the statement of the problem gives hints: e.g. the length $l$ of the rod (1 is nicer to deal with than $l$, an unspecified quantity). Often you have to solve the problem first, look at the solution, and try to simplify the notation.

From the chain rule,

$$\begin{aligned}
u_t &= \frac{\partial u}{\partial t} = U_* \frac{\partial \hat{u}}{\partial \hat{t}} \frac{\partial \hat{t}}{\partial t} = \frac{U_*}{T_*} \frac{\partial \hat{u}}{\partial \hat{t}}, \\
u_x &= \frac{\partial u}{\partial x} = U_* \frac{\partial \hat{u}}{\partial \hat{x}} \frac{\partial \hat{x}}{\partial x} = \frac{U_*}{L_*} \frac{\partial \hat{u}}{\partial \hat{x}} \\
u_{xx} &= \frac{U_*}{L_*^2} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}
\end{aligned}$$

Substituting these into the Heat Eqn (4) gives

$$u_t = \kappa u_{xx} \qquad \Rightarrow \qquad \frac{\partial \hat{u}}{\partial \hat{t}} = \frac{T_* \kappa}{L_*^2} \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}$$

To make the PDE simpler, we choose $T_* = L_*^2/\kappa = l^2/\kappa$, so that

$$\frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial^2 \hat{u}}{\partial \hat{x}^2}, \qquad 0 < \hat{x} < 1, \qquad \hat{t} > 0.$$

The characteristic (diffusive) time scale in the problem is $T_* = l^2/\kappa$. For different substances, this gives time scale over which diffusion takes place in the problem. The IC (5) and BC (6) must also be non-dimensionalized:

$$\begin{aligned}
&\text{IC:} \qquad \hat{u}(\hat{x}, 0) = \hat{f}(\hat{x}), \qquad && 0 < \hat{x} < 1, \\
&\text{BC:} \qquad \hat{u}\left(0, \hat{t}\right) = \hat{u}\left(1, \hat{t}\right) = 0, \qquad && \hat{t} > 0.
\end{aligned}$$

### 1.4 Dimensionless problem

Dropping hats, we have the dimensionless problem

$$\begin{aligned}
&\text{PDE:} \qquad u_t = u_{xx}, \qquad && 0 < x < 1, \qquad (8) \\
&\text{IC:} \qquad u(x, 0) = f(x), \qquad && 0 < x < 1, \qquad (9) \\
&\text{BC:} \qquad u(0, t) = u(1, t) = 0, \qquad && t > 0, \qquad (10)
\end{aligned}$$

where $x$, $t$ are dimensionless scalings of physical position and time.

## 2 Separation of variables

Ref: Guenther & Lee, §4.2 and 5.1, Myint-U & Debnath §6.4

[Sept 12, 2006]

We look for a solution to the dimensionless Heat Equation (8) – (10) of the form

$$u(x, t) = X(x)\, T(t) \qquad (11)$$

Take the relevant partial derivatives:

$$u_{xx} = X''(x)\,T(t), \qquad u_t = X(x)\,T'(t)$$

where primes denote differentiation of a single-variable function. The PDE (8), $u_t = u_{xx}$, becomes

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

The left hand side (l.h.s.) depends only on $t$ and the right hand side (r.h.s.) only depends on $x$. Hence if $t$ varies and $x$ is held fixed, the r.h.s. is constant, and hence $T'/T$ must also be constant, which we set to $-\lambda$ by convention:

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda, \qquad \lambda = \text{constant}. \tag{12}$$

The BCs become, for $t > 0$,

$$\begin{aligned} u(0,t) &= X(0)\,T(t) = 0 \\ u(1,t) &= X(1)\,T(t) = 0 \end{aligned}$$

Taking $T(t) = 0$ would give $u = 0$ for all time and space (called the trivial solution), from (11), which does not satisfy the IC unless $f(x) = 0$. If you are lucky and $f(x) = 0$, then $u = 0$ is the solution (this has to do with uniqueness of the solution, which we'll come back to). If $f(x)$ is not zero for all $0 < x < 1$, then $T(t)$ cannot be zero and hence the above equations are only satisfied if

$$X(0) = X(1) = 0. \tag{13}$$

## 2.1 Solving for $X(x)$

Ref: Guenther & Lee, §4.2 and 5.1 and 7.1, Myint-U & Debnath §7.1 – 7.3

We obtain a boundary value problem for $X(x)$, from (12) and (13),

$$X''(x) + \lambda X(x) = 0, \qquad 0 < x < 1, \tag{14}$$

$$X(0) = X(1) = 0. \tag{15}$$

This is an example of a Sturm-Liouville problem (from your ODEs class).

There are 3 cases: $\lambda > 0$, $\lambda < 0$ and $\lambda = 0$.

(i) $\lambda < 0$. Let $\lambda = -k^2 < 0$. Then the solution to (14) is

$$X = Ae^{kx} + Be^{-kx}$$

for integration constants $A$, $B$ found from imposing the BCs (15),

$$X(0) = A + B = 0, \qquad X(1) = Ae^{k} + Be^{-k} = 0.$$

The first gives $A = -B$, the second then gives $A\left(e^{2k} - 1\right) = 0$, and since $|k| > 0$ we have $A = B = u = 0$, which is the trivial solution. Thus we discard the case $\lambda < 0$.

(ii) $\lambda = 0$. Then $X(x) = Ax + B$ and the BCs imply $0 = X(0) = B$, $0 = X(1) = A$, so that $A = B = u = 0$. We discard this case also.

(iii) $\lambda > 0$. In this case, (14) is the simple harmonic equation whose solution is

$$X(x) = A\cos\left(\sqrt{\lambda}x\right) + B\sin\left(\sqrt{\lambda}x\right). \tag{16}$$

The BCs imply $0 = X(0) = A$, and $B\sin\sqrt{\lambda} = 0$. We don't want $B = 0$, since that would give the trivial solution $u = 0$, so we must have

$$\sin\sqrt{\lambda} = 0. \tag{17}$$

Thus $\sqrt{\lambda} = n\pi$, for any nonzero integer $n$ ($n = 1, 2, 3, ...$). We use subscripts to label the particular $n$-value. The values of $\lambda$ are called the eigenvalues of the Sturm-Liouville problem (14),

$$\lambda_n = n^2\pi^2, \qquad n = 1, 2, 3, ...$$

and the corresponding solutions of (14) are called the eigenfunctions of the Sturm-Liouville problem (14),

$$X_n(x) = b_n \sin(n\pi x), \qquad n = 1, 2, 3, ... \tag{18}$$

We have assumed that $n > 0$, since $n < 0$ gives the same solution as $n > 0$.

## 2.2 Solving for $T(t)$

When solving for $X(x)$, we found that non-trivial solutions arose for $\lambda = n^2\pi^2$ for all nonzero integers $n$. The equation for $T(t)$ is thus, from (12),

$$T'(t) = -n^2\pi^2 T(t)$$

and, for $n$, the solution is

$$T_n = c_n e^{-n^2\pi^2 t}, \qquad n = 1, 2, 3, ... \tag{19}$$

where the $c_n$'s are constants of integration.

### 2.3 Full solution $u(x,t)$

Ref: Myint-U & Debnath §6.4, Ch 5

Putting things together, we have, from (11), (18) and (19),

$$u_n(x,t) = B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}, \qquad n = 1, 2, 3, ... \tag{20}$$

where $B_n = c_n b_n$. Each function $u_n(x,t)$ is a solution to the PDE (8) and the BCs (10). But, in general, they will not individually satisfy the IC (9),

$$u_n(x,0) = B_n \sin(n\pi x) = f(x).$$

We now apply the principle of superposition: if $u_1$ and $u_2$ are two solutions to the PDE (8) and BC (10), then $c_1 u_1 + c_2 u_2$ is also a solution, for any constants $c_1$, $c_2$. This relies on the linearity of the PDE and BCs. We will, of course, soon make this more precise....

Since each $u_n(x,0)$ is a solution of the PDE, then the principle of superposition says any finite sum is also a solution. To solve the IC, we will probably need all the solutions $u_n$, and form the infinite sum (convergence properties to be checked),

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t). \tag{21}$$

$u(x,t)$ satisfies the BCs (10) since each $u_n(x,t)$ does. Assuming term-by-term differentiation holds (to be checked) for the infinite sum, then $u(x,t)$ also satisfies the PDE (8). To satisfy the IC, we need to find $B_n$'s such that

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} u_n(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x). \tag{22}$$

This is the Fourier Sine Series of $f(x)$.

To solve for the $B_n$'s, we use the orthogonality property for the eigenfunctions $\sin(n\pi x)$,

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\, dx = \begin{cases} 0 & m \neq n \\ 1/2 & m = n \end{cases} = \frac{1}{2}\delta_{mn} \tag{23}$$

where $\delta_{mn}$ is the kronecker delta,

$$\delta_{mn} = \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

The orthogonality relation (23) is derived by substituting

$$2\sin(m\pi x)\sin(n\pi x) = \cos((m-n)\pi x) - \cos((m+n)\pi x)$$

into the integral on the left hand side of (23) and noting

$$\int_0^1 \cos(m\pi x)\,dx = \delta_{m0}.$$

The orthogonality of the functions $\sin(n\pi x)$ is analogous to that of the unit vectors $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ in 2-space; the integral from 0 to 1 in (23) above is analogous to the dot product in 2-space.

To solve for the $B_n$'s, we multiply both sides of (22) by $\sin(m\pi x)$ and integrate from 0 to 1:

$$\int_0^1 \sin(m\pi x)\,f(x)\,dx = \sum_{n=1}^{\infty} B_n \int_0^1 \sin(n\pi x)\sin(m\pi x)\,dx$$

Substituting (23) into the right hand side yields

$$\int_0^1 \sin(m\pi x)\,f(x)\,dx = \sum_{n=1}^{\infty} B_n \frac{1}{2}\delta_{nm}$$

By definition of $\delta_{nm}$, the only term that is non-zero in the infinite sum is the one where $n = m$, thus

$$\int_0^1 \sin(m\pi x)\,f(x)\,dx = \frac{1}{2}B_m$$

Rearranging yields

$$B_m = 2\int_0^1 \sin(m\pi x)\,f(x)\,dx. \tag{24}$$

The full solution is, from (20) and (21),

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\,e^{-n^2\pi^2 t}, \tag{25}$$

where $B_n$ are given by (24).

To derive the solution (25) of the Heat Equation (8) and corresponding BCs (10) and IC (9), we used properties of linear operators and infinite series that need justification.

# 3 Example : Cooling of a rod from a constant initial temperature

Suppose the initial temperature distribution $f(x)$ in the rod is constant, i.e. $f(x) = u_0$. The solution for the temperature in the rod is (25),

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\,e^{-n^2\pi^2 t},$$

where, from (24), the Fourier coefficients are given by

$$B_n = 2\int_0^1 \sin(n\pi x) f(x)\, dx = 2u_0 \int_0^1 \sin(n\pi x)\, dx.$$

Calculating the integrals gives

$$B_n = 2u_0 \int_0^1 \sin(n\pi x)\, dx = -2u_0 \frac{\cos(n\pi) - 1}{n\pi} = -\frac{2u_0}{n\pi}\left((-1)^n - 1\right) = \begin{cases} 0 & n \text{ even} \\ \frac{4u_0}{n\pi} & n \text{ odd} \end{cases}$$

In other words,

$$B_{2n} = 0, \qquad B_{2n-1} = \frac{4u_0}{(2n-1)\pi}$$

and the solution becomes

$$u(x,t) = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{(2n-1)} \exp\left(-(2n-1)^2 \pi^2 t\right). \tag{26}$$

## 3.1 Approximate form of solution

The number of terms of the series (26) needed to get a good approximation for $u(x,t)$ depends on how close $t$ is to 0. The series is

$$u(x,t) = \frac{4u_0}{\pi}\left(\sin(\pi x)\, e^{-\pi^2 t} + \frac{\sin(3\pi x)}{3} e^{-9\pi^2 t} + \cdots\right).$$

The ratio of the first and second terms is

$$\begin{aligned} \frac{|\text{second term}|}{|\text{first term}|} &= \frac{e^{-8\pi^2 t}}{3} \frac{|\sin 3\pi x|}{|\sin \pi x|} \\ &\leq e^{-8\pi^2 t} \qquad \text{using} \qquad |\sin nx| \leq n\,|\sin x| \\ &\leq e^{-8} \qquad \text{for} \qquad t \geq \frac{1}{\pi^2} \\ &< 0.00034 \end{aligned}$$

It can also be shown that the first term dominates the sum of the rest of the terms, and hence

$$u(x,t) \approx \frac{4u_0}{\pi} \sin(\pi x)\, e^{-\pi^2 t}, \quad \text{for} \quad t \geq \frac{1}{\pi^2}. \tag{27}$$

What does $t = 1/\pi^2$ correspond to in physical time? In physical time $t' = l^2 t/\kappa$ (recall our scaling - here we use $t$ as the dimensionless time and $t'$ as dimensional time), $t = 1/\pi^2$ corresponds to:

| | | |
|---|---|---|
| $t' \approx 15$ minutes, | for a 1 m rod of copper | $(\kappa \approx 1.1 \text{ cm}^2 \text{ sec}^{-1})$ |
| $t' \approx 169$ minutes, | for a 1 m rod of steel | $(\kappa \approx 0.1 \text{ cm}^2 \text{ sec}^{-1})$ |
| $t' \approx 47$ hours, | for a 1 m rod of glass | $(\kappa \approx 0.006 \text{ cm}^2 \text{ sec}^{-1})$ |

At $t = 1/\pi^2$, the temperature at the center of the rod ($x = 1/2$) is, from (27),

$$u(x,t) \approx \frac{4}{\pi e} u_0 = 0.47 u_0.$$

Thus, after a (scaled) time $t = 1/\pi^2$, the temperature has decreased by a factor of 0.47 from the initial temperature $u_0$.

## 3.2 Geometrical visualization of the solution

[Sept 14, 2006]

To analyze qualitative features of the solution, we draw various types of curves in 2D:

1. Spatial temperature profile, given by

$$u = u(x, t_0)$$

where $t_0$ is a fixed value of $x$. These profiles are curves in the $ux$-plane.

2. Temperature profiles in time,

$$u = u(x_0, t)$$

where $x_0$ is a fixed value of $x$. These profiles are curves in the $ut$-plane.

3. Curves of constant temperature in the $xt$-plane (level curves),

$$u(x,t) = C$$

where $C$ is a constant.

Note that the solution $u = u(x,t)$ is a 2D surface in the 3D $uxt$-space. The above families of curves are the different cross-sections of this solution surface. Drawing the 2D cross-sections is much simpler than drawing the 3D solution surface.

Sketch typical curves: when sketching the curves in 1-3 above, we draw a few typical curves and any special cases. While math packages such as Matlab can be used to compute the curves from, say, 20 terms in the full power series solution (26), the emphasis in this course is to use simple considerations to get a rough idea of what the solution looks like. For example, one can use the first term approximation (27), simple physical considerations on heat transfer, and the fact that the solution $u(x,t)$ is continuous in $x$ and $t$, so that if $t_1$ is close to $t_1$, $u(x, t_1)$ is close to $u(x, t_2)$.

### 3.2.1 Spatial temperature profiles

For fixed $t = t_0$, the first term approximate solution (27) is

$$u(x,t) \approx \frac{4u_0}{\pi} e^{-\pi^2 t_0} \sin(\pi x), \qquad t \geq 1/\pi^2. \tag{28}$$

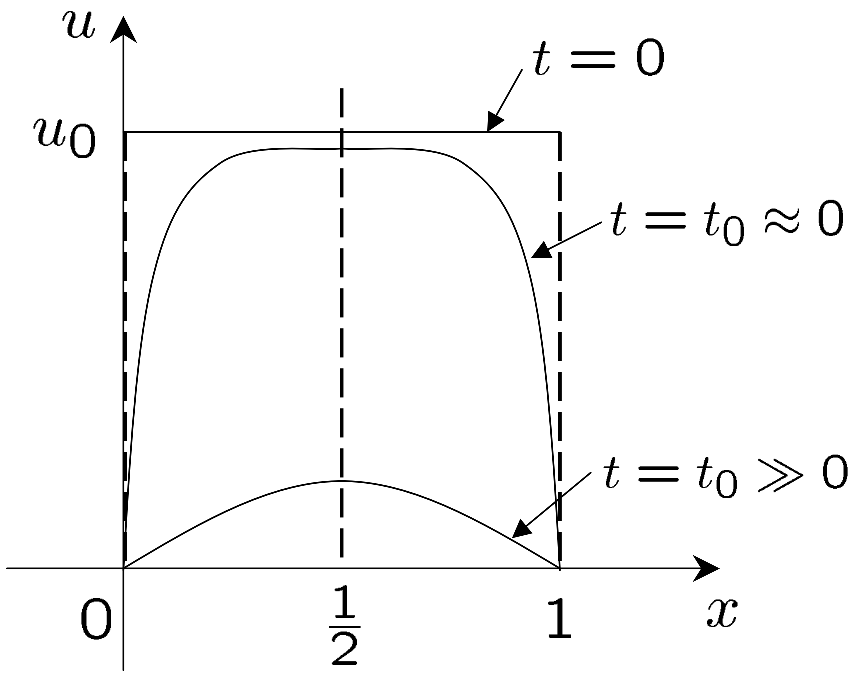


Figure 1: Spatial temperature profiles $u(x, t_0)$.

This suggests the center of the rod, $x = 1/2$, is a line of symmetry for $u\,(x, t)$, i.e.

$$u\left(\frac{1}{2} + s, t\right) = u\left(\frac{1}{2} - s, t\right)$$

and, for each fixed time, the location of the maximum (minimum) temperature if $u_0 > 0$ ($u_0 < 0$). We can prove the symmetry property by noting that the original PDE/BC/IC problem is invariant under the transformation $x \to 1 - x$. Note also that, from (26),

$$u_x\,(x, t) = 4u_0 \sum_{n=1}^{\infty} \cos\left((2n-1)\,\pi x\right) \exp\left(-\,(2n-1)^2\,\pi^2 t\right)$$

Thus $u_x\,(1/2, t) = 0$ and $u_{xx}\,(1/2, t) < 0$, so the 2nd derivative test implies that $x = 1/2$ is a local max.

In Figure 1, we have plotted two typical profiles, one at early times $t = t_0 \approx 0$ and the other at late times $t = t_0 \gg 0$, and two special profiles, the initial temperature at $t = 0$ ($u = u_0$) and the temperature as $t \to \infty$ ($u = 0$). The profile for $t = t_0 \gg 0$ is found from the first term approximation (28). The line of symmetry $x = 1/2$ is plotted as a dashed line for reference in Figure 1.

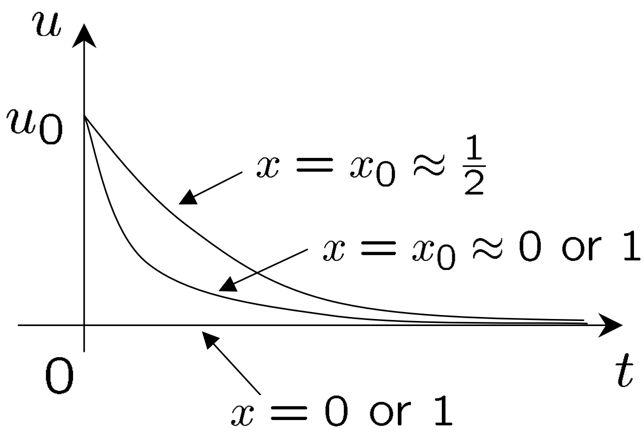


Figure 2: Temperature profiles in time $u\,(x_0, t)$.

#### 3.2.2 Temperature profiles in time

Setting $x = x_0$ in the approximate solution (27),

$$u\,(x_0, t) \approx \left(\frac{4u_0}{\pi}\sin\left(\pi x_0\right)\right) e^{-\pi^2 t}, \qquad t \geq 1/\pi^2.$$

Two typical profiles are sketched in Figure 2, one near the center of the rod ($x_0 \approx 1/2$) and one near the edges ($x_0 \approx 0$ or 1). To draw these we noted that the center of the rod cools more slowly than points near the ends. One special profile is plotted, namely the temperature at the rod ends ($x = 0, 1$).

#### 3.2.3 Curves of constant temperature (level curves of $u(x, t)$)

In Figure 3, we have drawn three typical level curves and two special ones, $u = 0$ (rod ends) and $u = u_0$ (the initial condition). For a fixed $x = x_0$, the temperature in the rod decreases as $t$ increases (motivated by the first term approximation (28)), as indicated by the points of intersection on the dashed line. The center of the rod ($x = 1/2$) is a line of symmetry, and at any time, the maximum temperature is at the center. Note that at $t = 0$, the temperature is discontinuous at $x = 0, 1$.

To draw the level curves, it is easiest to already have drawn the spatial temperature profiles. Draw a few horizontal broken lines across your $u$ vs. $x$ plot. Suppose you draw a horizontal line $u = u_1$. Suppose this line $u = u_1$ crosses one of your profiles $u\,(x, t_0)$ at position $x = x_1$. Then $(x_1, t_0)$ is a point on the level curve $u\,(x, t) = u_1$. Now plot this point in your level curve plot. By observing where the line $u = u_1$ crosses your various spatial profiles, you fill in the level curve $u\,(x, t) = u_1$. Repeat

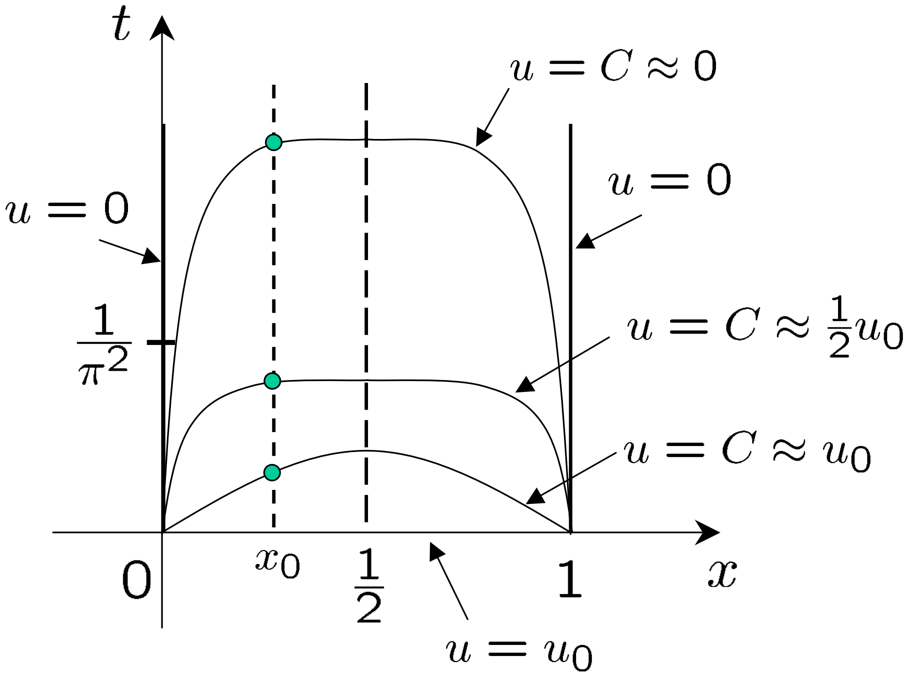


Figure 3: Curves of constant temperature $u(x,t) = c$, i.e., the level curves of $u(x,t)$.

this process for a few values of $u_1$ to obtain a few representative level curves. Plot also the special cases: $u(x,t) = u_0/w$, $u = 0$, etc.

When drawing visualization curves, the following result is also helpful.

### 3.2.4 Maximum Principle for the basic Heat Problem

Ref: Guenther & Lee §5.2, Myint-U & Debnath §8.2

This result is useful when plotting solutions: the extrema of the solution of the heat equation occurs on the space-time "boundary", i.e. the maximum of the initial condition and of the time-varying boundary conditions. More precisely, given the heat equation with some initial condition $f(x)$ and BCs $u(0,t)$, $u(1,t)$, then on a given time interval $[0,T]$, the solution $u(x,t)$ is bounded by

$$u_{\min} \le u(x,t) \le u_{\max}$$

where

$$\begin{aligned} u_{\max} &= \max\left\{ \max_{0<x<1} f(x), \quad \max_{0<t<T} u(0,t), \quad \max_{0<t<T} u(1,t) \right\} \\ u_{\min} &= \min\left\{ \min_{0<x<1} f(x), \quad \min_{0<t<T} u(0,t), \quad \min_{0<t<T} u(1,t) \right\} \end{aligned}$$

Thus, in the example above, $u_{\min} = 0$ and $u_{\max} = u_0$, hence for all $x \in [0,1]$ and $t \in [0,T]$, $0 \le u(x,t) \le u_0$.

# 4 Equilibrium temperature profile (steady-state)

Intuition tells us that if the ends of the rod are held at $0^o$ C and there are no heat sources or sinks in the rod, the temperature in the rod will eventually reach 0. The solution above confirms this. However, we do not have to solve the full problem to determine the asymptotic or long-time behavior of the solution.

Instead, the equilibrium or steady-state solution $u = u_E(x)$ must be independent of time, and will thus satisfy the PDE and BCs with $u_t = 0$,

$$u_E'' = 0, \quad 0 < x < 1; \quad u_E(0) = u_E(1) = 0.$$

The solution is $u_E = c_1 x + c_2$ and imposing the BCs implies $u_E(x) = 0$. In other words, regardless of the initial temperature distribution $u(x,0) = f(x)$ in the rod, the temperature eventually goes to zero.

## 4.1 Rate of decay of $u(x,t)$

How fast does $u$ approach $u_E = 0$? From our estimate (82) above,

$$|u_n(x,t)| \le B e^{-n^2\pi^2 t}, \qquad n = 1,2,3,...$$

where $B$ is a constant. Noting that $e^{-n^2\pi^2 t} \le e^{-n\pi^2 t} = \left(e^{-\pi^2 t}\right)^n$, we have

$$\left|\sum_{n=1}^{\infty} u_n(x,t)\right| \le \sum_{n=1}^{\infty} |u_n(x,t)| \le B \sum_{n=1}^{\infty} r^n = \frac{Br}{1-r}, \qquad r = e^{-\pi^2 t} < 1 \ (t > 0).$$

The last step is the geometric series result $\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}$, for $|r| < 1$. Thus

$$\left|\sum_{n=1}^{\infty} u_n(x,t)\right| \le \frac{Be^{-\pi^2 t}}{1 - e^{-\pi^2 t}}, \qquad t > 0. \tag{29}$$

Therefore

- $u(x,t)$ approaches the steady-state $u_E(x) = 0$ exponentially fast (i.e. the rod cools quickly)
- the first term in the series, $\sin(\pi x)\, e^{-\pi^2 t}$ (term with smallest eigenvalue $\lambda = \pi$) determines the rate of decay of $u(x,t)$
- the $B_n$'s may also affect the rate of approach to the steady-state, for other problems

## 4.2 Error of first-term approximation

Using the method in the previous section, we can compute the error between the first-term and the full solution (26),

$$\begin{aligned}\left|u(x,t)-u_1(x,t)\right| &= \left|\sum_{n=1}^{\infty}u_n(x,t)-u_1(x,t)\right| \\ &= \left|\sum_{n=2}^{\infty}u_n(x,t)\right| \le \sum_{n=2}^{\infty}\left|u_n(x,t)\right| \le B\sum_{n=2}^{\infty}r^n = \frac{Br^2}{1-r}\end{aligned}$$

where $r=e^{-\pi^2 t}$. Hence the solution $u(x,t)$ approaches the first term $u_1(x,t)$ exponentially fast,

$$\left|u(x,t)-u_1(x,t)\right| \le \frac{Be^{-2\pi^2 t}}{1-e^{-\pi^2 t}}, \qquad t>0. \tag{30}$$

With a little more work we can get a much tighter (i.e. better) upper bound. We consider the sum

$$\sum_{n=N}^{\infty}\left|u_n(x,t)\right|$$

Substituting for $u_n(x,t)$ gives

$$\sum_{n=N}^{\infty}\left|u_n(x,t)\right| \le B\sum_{n=N}^{\infty}e^{-n^2\pi^2 t} = Be^{-N^2\pi^2 t}\sum_{n=N}^{\infty}e^{-\left(n^2-N^2\right)\pi^2 t} \tag{31}$$

Here's the trick: for $n\ge N$,

$$n^2-N^2=(n+N)(n-N)\ge 2N(n-N)\ge 0.$$

Since $e^{-x}$ is a decreasing function, then

$$e^{-\left(n^2-N^2\right)\pi^2 t}\le e^{-2N(n-N)\pi^2 t} \tag{32}$$

for $t>0$. Using the inequality (32) in Eq. (31) gives

$$\begin{aligned}\sum_{n=N}^{\infty}\left|u_n(x,t)\right| &\le Be^{-N^2\pi^2 t}\sum_{n=N}^{\infty}e^{-2N(n-N)\pi^2 t} \\ &= Be^{N^2\pi^2 t}\sum_{n=N}^{\infty}\left(e^{-2N\pi^2 t}\right)^n = Be^{N^2\pi^2 t}\frac{\left(e^{-2N\pi^2 t}\right)^N}{1-e^{-2N\pi^2 t}}.\end{aligned}$$

Simplifying the expression on the right hand side yields

$$\sum_{n=N}^{\infty}\left|u_n(x,t)\right| \le \frac{Be^{-N^2\pi^2 t}}{1-e^{-2N\pi^2 t}} \tag{33}$$

For $N = 1$, (33) gives the temperature decay

$$|u(x,t)| = \left|\sum_{n=N}^{\infty} u_n(x,t)\right| \le \sum_{n=N}^{\infty} |u_n(x,t)| \le \frac{Be^{-\pi^2 t}}{1 - e^{-2\pi^2 t}}, \tag{34}$$

which is slightly tighter than (29). For $N = 2$, (33) gives the error between $u(x,t)$ and the first term approximation,

$$|u(x,t) - u_1(x,t)| = \left|\sum_{n=2}^{\infty} u_n(x,t)\right| \le \sum_{n=2}^{\infty} |u_n(x,t)| \le \frac{Be^{-4\pi^2 t}}{1 - e^{-4\pi^2 t}}$$

which is a much better upper bound than (30).

Note that for an initial condition $u(x,0) = f(x)$ that is is symmetric with respect to $x = 1/2$ (e.g. $f(x) = u_0$), then $u_{2n} = 0$ for all $n$, and hence error between $u(x,t)$ and the first term $u_1(x,t)$ is even smaller,

$$|u(x,t) - u_1(x,t)| = \left|\sum_{n=3}^{\infty} u_n(x,t)\right| \le \sum_{n=3}^{\infty} |u_n(x,t)|$$

Applying result (33) with $N = 3$ gives

$$|u(x,t) - u_1(x,t)| \le \frac{Be^{-9\pi^2 t}}{1 - e^{-6\pi^2 t}}.$$

Thus, in this case, the error between the solution $u(x,t)$ and the first term $u_1(x,t)$ decays as $e^{-9\pi^2 t}$ - very quickly!

# 5 Review of Fourier Series

Ref: Guenther & Lee, §3.1, Myint-U & Debnath §5.1–5.3, 5.5–5.6

[Sept 19, 2006]

Motivation: Recall that the initial temperature distribution satisfies

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x).$$

In the example above with a constant initial temperature distribution, $f(x) = u_0$, we have

$$u_0 = \frac{4u_0}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{2n-1}. \tag{35}$$

Note that at $x = 0$ and $x = 1$, the r.h.s. does NOT converge to $u_0 \neq 0$, but rather to 0 (the BCs). Note that the Fourier Sine Series of $f(x)$ is odd and 2-periodic in space and converges to the odd periodic extension of $f(x) = u_0$.

**Odd periodic extension** The odd periodic extension of a function $f(x)$ defined for $x \in [0,1]$, is the Fourier Sine Series of $f(x)$ evaluated at any $x \in \mathbb{R}$,

$$\tilde{f}(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x).$$

Note that since $\sin(n\pi x) = -\sin(-n\pi x)$ and $\sin(n\pi x) = \sin(n\pi(x+2))$, then $\tilde{f}(x) = -\tilde{f}(-x)$ and $\tilde{f}(x+2) = \tilde{f}(x)$. Thus $\tilde{f}(x)$ is odd, 2-periodic and $\tilde{f}(x)$ equals $f(x)$ on the open interval $(0,1)$. What conditions are necessary for $f(x)$ to equal $\tilde{f}(x)$ on the closed interval $[0,1]$? This is covered next.

Aside: cancelling $u_0$ from both sides of (35) gives a really complicated way of writing 1,

$$1 = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin((2n-1)\pi x)}{2n-1}.$$

## 5.1 Fourier Sine Series

Given an integrable function $f(x)$ on $[0,1]$, the Fourier Sine Series of $f(x)$ is

$$\sum_{n=1}^{\infty} B_n \sin(n\pi x) \tag{36}$$

where

$$B_n = 2\int_0^1 f(x)\sin(n\pi x)\, dx. \tag{37}$$

The associated orthogonality properties are

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)\, dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

## 5.2 Fourier Cosine Series

Given an integrable function $f(x)$ on $[0,1]$, the Fourier Cosine Series of $f(x)$ is

$$A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) \tag{38}$$

where

$$A_0 = \int_0^1 f(x)\, dx, \qquad \text{(average of } f(x)\text{)} \tag{39}$$

$$A_n = 2\int_0^1 f(x)\cos(n\pi x)\, dx, \qquad n \geq 1. \tag{40}$$

The associated orthogonality properties are

$$\int_0^1 \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} 1/2, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

e.g. The cosine series of $f(x) = u_0$ for $x \in [0,1]$ is just $u_0$. In other words, $A_0 = u_0$ and $A_n = 0$ for $n \geq 1$.

## 5.3 The (full) Fourier Series

The Full Fourier Series of an integrable function $f(x)$, now defined on $[-1,1]$, is

$$\hat{f}(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(n\pi x) + b_n \sin(n\pi x)\right) \tag{41}$$

where

$$\begin{aligned} a_0 &= \frac{1}{2}\int_{-1}^{1} f(x)\,dx \\ a_n &= \int_{-1}^{1} f(x)\cos(n\pi x)\,dx \\ b_n &= \int_{-1}^{1} f(x)\sin(n\pi x)\,dx \end{aligned}$$

The associated orthogonality properties of sin and cos on $[-1,1]$ are, for any $m, n = 1, 2, 3, ...$

$$\int_{-1}^{1} \sin(m\pi x)\cos(n\pi x)\,dx = 0, \qquad \text{all } m, n,$$

$$\int_{-1}^{1} \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 1, & m = n \neq 0, \\ 0, & m \neq n, \end{cases}$$

$$\int_{-1}^{1} \cos(m\pi x)\cos(n\pi x)\,dx = \begin{cases} 1, & m = n \neq 0, \\ 0, & m \neq n. \end{cases}$$

## 5.4 Piecewise Smooth

Ref: Guenther & Lee p. 50,

Provided a function $f(x)$ is integrable, its Fourier coefficients can be calculated. It does not follow, however, that the corresponding Fourier Series (Sine, Cosine or Full) converges or has the sum $f(x)$. In order to ensure this, $f(x)$ must satisfy some stronger conditions.

**Definition** Piecewise Smooth function: A function $f(x)$ defined on a closed interval $[a,b]$ is said to be piecewise smooth on $[a,b]$ if there is a partition of $[a,b]$,

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

such that $f$ has a continuous derivative (i.e. $\mathcal{C}^1$) on each closed subinterval $[x_m, x_{m+1}]$.

E.g. any function that is $\mathcal{C}^1$ on an interval $[a,b]$ is, of course, piecewise smooth on $[a,b]$.

E.g. the function

$$f(x) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 1/2, & 1/2 < x \le 1 \end{cases}$$

is piecewise smooth on $[0,1]$, but is not continuous on $[0,1]$.

E.g. the function $f(x) = |x|$ is both continuous and piecewise smooth on $[-1,1]$, despite $f'(x)$ not being defined at $x = 0$. This is because we partition $[-1,1]$ into two subintervals $[-1,0]$ and $[0,1]$. When worrying about $f'(x)$ near $x = 0$, note that on $[0,1]$ we only care about the left limit $f'(0^-)$ and for $[-1,0]$, we only care about the right limit $f'(0^+)$.

E.g. the function $f(x) = |x|^{1/2}$ is continuous on $[-1,1]$ but not piecewise smooth on $[-1,1]$, since $f'(0^-)$ and $f'(0^+)$ do not exist.

## 5.5 Convergence of Fourier Series

Ref: Guenther & Lee p. 49 and (optional) §3.3, Myint-U & Debnath §5.10

**Theorem** [Convergence of the Fourier Sine and Cosine Series]: If $f(x)$ is piecewise smooth on the closed interval $[0,1]$ and continuous on the open interval $(0,1)$, then the Fourier Sine and Cosine Series converge for all $x \in [0,1]$ and have the sum $f(x)$ for all $x \in (0,1)$.

Note: Suppose $f(x)$ is piecewise smooth on $[0,1]$ and is continuous on $(0,1)$ except at a jump discontinuity at $x = a$. Then the Fourier Sine and Cosine Series converge to $f(x)$ on $(0,1)$ and converge to the average of the left and right limits at $x = a$, i.e. $(f(a-) + f(a+))/2$. At the endpoints $x = 0, 1$, the Sine series converges to zero, since $\sin(n\pi x) = 0$ at $x = 0, 1$ for all $n$.

**Theorem** [Convergence of the Full Fourier Series]: If $f(x)$ is piecewise smooth on the closed interval $[-1,1]$ and continuous on the open interval $(-1,1)$, then the Full Fourier Series converges for all $x \in [-1,1]$ and has the sum $f(x)$ for all $x \in (-1,1)$.

## 5.6 Comments

Ref: see problems Guenther & Lee p. 53

Given a function $f(x)$ that is piecewise smooth on $[0,1]$ and continuous on $(0,1)$, the Fourier Sine and Cosine Series of $f(x)$ converge on $[0,1]$ and equal $f(x)$ on the open interval $(0,1)$ (i.e. perhaps excluding the endpoints). Thus, for any $x \in (0,1)$,

$$A_0 + \sum_{n=1}^{\infty} A_n \cos(n\pi x) = f(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x).$$

In other words, the Fourier Cosine Series (left hand side) and the Fourier Sine Series (right hand side) are two different representations for the same function $f(x)$, on the open interval $(0,1)$. The values at the endpoints $x = 0, 1$ may not be the same. The choice of Sine or Cosine series is determined from the type of eigenfunctions that give solutions to the Heat Equation and BCs.

# 6 Well-Posed Problems

Ref: Guenther & Lee §1.8 and §5.2 (in particular p. 160 and preceding pages, and p. 150)

Ref: Myint-U & Debnath §1.2, §6.5

We call a mathematical model or equation or problem well-posed if it satisfies the following 3 conditions:

1. [Existence of a solution]: The mathematical model has at least 1 solution. Physical interpretation: the system exists over at least some finite time interval.
2. [Uniqueness of solution]: The mathematical model has at most 1 solution. Physical interpretation: identical initial states of the system lead to the same outcome.
3. [Continuous dependence on parameters]: The solution of the mathematical model depends continuously on initial conditions and parameters. Physical interpretation: small changes in initial states (or parameters) of the system produce small changes in the outcome.

If an IVP (initial value problem) or BIVP (boundary initial value problem - e.g. Heat Problem) satisfies 1, 2, 3 then it is well-posed.

Example: for the basic Heat Problem, we showed 1 by construction a solution using the method of separation of variables. Continuous dependence is more difficult to show (need to know about norms), but it is true, and we will use this fact when sketching solutions. Also, when drawing level curves $u(x,t) = const$, small changes in parameters $(x,t)$ leads to a small change in $u$. We now prove the 2nd part of well-posedness, uniqueness of solution, for the basic heat problem.

## 6.1 Uniqueness of solution to the Heat Problem

**Definition** We define the two space-time sets

$$\begin{aligned}\mathcal{D} &= \{(x,t) : 0 \le x \le 1,\, t > 0\} \\ \overline{\mathcal{D}} &= \{(x,t) : 0 \le x \le 1,\, t \ge 0\}\end{aligned}$$

and the space of functions

$$\mathcal{C}^2\left(\overline{\mathcal{D}}\right) = \left\{u(x,t) : u_{xx} \text{ continuous in } \mathcal{D} \text{ and } u \text{ continuous in } \overline{\mathcal{D}}\right\}.$$

In other words, the space of functions that are twice continuously differentiable on $[0,1]$ for $t > 0$ and continuous on $[0,1]$ for $t \ge 0$.

**Theorem** The basic Heat Problem, i.e. the Heat Equation (8) with BC (10) and IC (9),

$$\begin{aligned}\text{PDE:} &\quad u_t = u_{xx}, && 0 < x < 1, \\ \text{IC:} &\quad u(x,0) = f(x), && 0 < x < 1, \\ \text{BC:} &\quad u(0,t) = u(1,t) = 0, && t > 0,\end{aligned}$$

has at most one solution in the space of functions $\mathcal{C}^2\left(\overline{\mathcal{D}}\right)$.

**Proof**: Consider two solutions $u_1,\ u_2 \in \mathcal{C}^2\left(\overline{\mathcal{D}}\right)$ to the Heat Problem. Let $v = u_1 - u_2$. We aim to show that $v = 0$ on $[0,1]$, which would prove that $u_1 = u_2$ and then solution to the Heat Equation (8) with BC (10) and IC (9) is unique. Since each of $u_1$, $u_2$ satisfies (8), (9), and (10), the function $v$ satisfies

$$\begin{aligned}v_t &= (u_1 - u_2)_t \qquad (42) \\ &= u_{1t} - u_{2t} \\ &= u_{1xx} - u_{2xx} \\ &= (u_1 - u_2)_{xx} \\ &= v_{xx}, \qquad 0 < x < 1,\end{aligned}$$

and similarly,

$$\text{IC:} \quad v(x,0) = u_1(x,0) - u_2(x,0) = f(x) - f(x) = 0, \qquad 0 < x < 1, \qquad (43)$$

$$\text{BC:} \quad v(0,t) = u_1(0,t) - u_2(0,t) = 0, \qquad v(1,t) = 0, \qquad t > 0. \qquad (44)$$

Define the function

$$\bar{V}(t) = \int_0^1 v^2(x,t)\,dx \ge 0, \qquad t \ge 0.$$

Showing that $v(x,t) = 0$ reduces to showing that $\bar{V}(t) = 0$, since $v(x,t)$ is continuous on $[0,1]$ for all $t \geq 0$ and if there was a point $x$ such that $v(x,t) \neq 0$, then $\bar{V}(t)$ would be strictly greater than 0. To show $\bar{V}(t) = 0$, we differentiate $\bar{V}(t)$ in time and substitute for $v_t$ from the PDE (42),

$$\frac{d\bar{V}}{dt} = 2\int_0^1 v v_t dx = 2\int_0^1 v v_{xx} dx$$

Integrating by parts (note: $v v_{xx} = (v v_x)_x - v_x^2$) gives

$$\frac{d\bar{V}}{dt} = 2\int_0^1 v v_{xx} dx = 2\left( v v_x|_{x=0}^1 - \int_0^1 v_x^2 dx \right)$$

Using the BCs (44) gives

$$\frac{d\bar{V}}{dt} = -2\int_0^1 v_x^2 dx \leq 0.$$

The IC (43) implies that

$$\bar{V}(0) = \int_0^1 v^2(x,0)\, dx = 0.$$

Thus, $\bar{V}(t) \geq 0$, $d\bar{V}/dt \leq 0$, and $\bar{V}(0) = 0$, i.e. $\bar{V}(t)$ is a non-negative, non-increasing function of time whose initial value is zero. Thus, for all time, $\bar{V}(t) = 0$ and $v(x,t) = 0$ for all $x \in [0,1]$, implying that $u_1 = u_2$. This proves that the solution to the Heat Equation (8), its IC (9), and BCs (10) is unique, i.e. there is at most one solution. Uniqueness proofs for other types of BCs follows in a similar manner.

# 7 Variations on the basic Heat Problem

[Sept 21, 2006]

We now consider variations to the basic Heat Problem, including different types of boundary conditions and the presence of sources and sinks.

## 7.1 Boundary conditions

### 7.1.1 Type I BCs (Dirichlet conditions)

Ref: Guenther & Lee p. 149

Type I, or Dirichlet, BCs specify the temperature $u(x,t)$ at the end points of the rod, for $t > 0$,

$$\begin{aligned} u(0,t) &= g_1(t), \\ u(1,t) &= g_2(t). \end{aligned}$$

Type I Homogeneous BCs are

$$\begin{aligned} u(0,t) &= 0, \\ u(1,t) &= 0. \end{aligned}$$

The physical significance of these BCs for the rod is that the ends are kept at $0^o$ C. The solution to the Heat Equation with Type I BCs was considered in class. After separation of variables $u(x,t) = X(x)T(t)$, the associated Sturm-Liouville Boundary Value Problem for $X(x)$ is

$$X'' + \lambda X = 0; \qquad X(0) = X(1) = 0.$$

The eigenfunctions are $X_n(x) = B_n \sin(n\pi x)$.

### 7.1.2 Type II BCs (Newmann conditions)

Ref: Guenther & Lee p. 152 problem 1

Type II, or Newmann, BCs specify the rate of change of temperature $\partial u/\partial x$ (or heat flux) at the ends of the rod, for $t > 0$,

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= g_1(t), \\ \frac{\partial u}{\partial x}(1,t) &= g_2(t). \end{aligned}$$

Type II Homogeneous BCs are

$$\begin{aligned} \frac{\partial u}{\partial x}(0,t) &= 0, \\ \frac{\partial u}{\partial x}(1,t) &= 0. \end{aligned}$$

The physical significance of these BCs for the rod is that the ends are insulated. These lead to another relatively simple solution involving a cosine series (see problem 6 on PS 1). After separation of variables $u(x,t) = X(x)T(t)$, the associated Sturm-Liouville Boundary Value Problem for $X(x)$ is

$$X'' + \lambda X = 0; \qquad X'(0) = X'(1) = 0.$$

The eigenfunctions are $X_0(x) = A_0 = const$ and $X_n(x) = A_n \cos(n\pi x)$.

### 7.1.3 Type III BCs (Mixed)

The general Type III BCs are a mixture of Type I and II, for $t > 0$,

$$\begin{aligned} \alpha_1 \frac{\partial u}{\partial x}(0,t) + \alpha_2 u(0,t) &= g_1(t), \\ \alpha_3 \frac{\partial u}{\partial x}(1,t) + \alpha_4 u(1,t) &= g_2(t). \end{aligned}$$

After separation of variables $u(x,t) = X(x)T(t)$, the associated Sturm-Liouville Boundary Value Problem for $X(x)$ is

$$X'' + \lambda X = 0,$$

$$\begin{aligned} \alpha_1 X'(0) + \alpha_2 X(0) &= 0 \\ \alpha_3 X'(0) + \alpha_4 X(0) &= 0 \end{aligned}$$

The associated eigenfunctions depend on the values of the constants $\alpha_{1,2,3,4}$.

Example 1. $\alpha_1 = \alpha_4 = 1$, $\alpha_2 = \alpha_3 = 0$. Then

$$X'' + \lambda X = 0; \qquad X'(0) = X(1) = 0$$

and the eigenfunctions are $X_n = A_n \cos\left(\frac{2n-1}{2}\pi x\right)$. Note: the constant $X_0 = A_0$ is not an eigenfunction here.

Example 2. $\alpha_1 = \alpha_4 = 0$, $\alpha_2 = \alpha_3 = 1$. Then

$$X'' + \lambda X = 0; \qquad X(0) = X'(1) = 0$$

and the eigenfunctions are $X_n = B_n \sin\left(\frac{2n-1}{2}\pi x\right)$. Note: the constant $X_0 = A_0$ is not an eigenfunction here.

Note: Starting with the BCs in Example 1 and rotating the rod about $x = 1/2$, you'd get the BCs in Example 2. It is not surprising then that under the change of variables $\tilde{x} = 1-x$, Example 1 becomes Example 2, and vice versa. The eigenfunctions also possess this symmetry, since

$$\sin\left(\frac{2n-1}{2}\pi(1-x)\right) = (-1)^n \cos\left(\frac{2n-1}{2}\pi x\right).$$

Since we can absorb the $(-1)^n$ into the constant $B_n$, the eigenfunctions of Example 1 become those of Example 2 under the transformation $\tilde{x} = 1-x$, and vice versa.

## 7.2 Solving the Heat Problem with Inhomogeneous (time-independent) BCs

Ref: Guenther & Lee p. 149

Consider the Heat Problem with inhomogeneous Type I BCs,

$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < 1 \\ u(0,t) &= 0, \qquad u(1,t) = u_1 = const, \qquad t > 0, \\ u(x,0) &= 0, \qquad 0 < x < 1. \end{aligned} \tag{45}$$

Directly applying separation of variables $u(x,t) = X(x)T(t)$ is not useful, because we'd obtain $X(1)T(t) = u_1$ for $t > 0$. The strategy is to rewrite the solution $u(x,t)$ in terms of a new variable $v(x,t)$ such that the new problem for $v$ has homogeneous BCs!

Step 1. Find the steady-state, or equilibrium solution $u_E(x)$, since this by definition must satisfy the PDE and the BCs,

$$\begin{aligned} u_E'' &= 0, \qquad 0 < x < 1 \\ u_E(0) &= 0, \qquad u_E(1) = u_1 = const. \end{aligned}$$

Solving for $u_E$ gives $u_E(x) = u_1 x$.

Step 2. Transform variables by introducing a new variable $v(x,t)$,

$$v(x,t) = u(x,t) - u_E(x) = u(x,t) - u_1 x. \tag{46}$$

Substituting this into the Heat Problem (45) gives a new Heat Problem,

$$\begin{aligned} v_t &= v_{xx}, \qquad 0 < x < 1 \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0, \\ v(x,0) &= -u_1 x, \qquad 0 < x < 1. \end{aligned} \tag{47}$$

Notice that the BCs are now homogeneous, and the IC is now inhomogeneous. Notice also that we know how to solve this - since it's the basic Heat Problem! Based on our work, we know that the solution to (47) is

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) e^{-n^2\pi^2 t}, \qquad B_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx \tag{48}$$

where $f(x) = -u_1 x$. Substituting for $f(x)$ and integrating by parts, we find

$$\begin{aligned} B_n &= -2u_1 \int_0^1 x \sin(n\pi x)\,dx \\ &= -2u_1 \left( -\left.\frac{x\cos(n\pi x)}{n\pi}\right|_{x=0}^{1} + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\,dx \right) \\ &= \frac{2u_1(-1)^n}{n\pi} \end{aligned} \tag{49}$$

Step 3. Transform back to $u(x,t)$, from (46), (48) and (49),

$$u(x,t) = u_E(x) + v(x,t) = u_1 x + \frac{2u_1}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(n\pi x) e^{-n^2\pi^2 t}. \tag{50}$$

The term $u_E(x)$ is the steady state and the term $v(x,t)$ is called the transient, since it exists initially to satisfy the initial condition but vanishes as $t \to \infty$. You can check for yourself by direct substitution that Eq. (50) is the solution to the inhomogeneous Heat Problem (45), i.e. the PDE, BCs and IC.

## 7.3 Heat sources

### 7.3.1 Derivation

Ref: Guenther & Lee p. 6 (Eq. 3-3)

To add a heat source to the derivation of the Heat Equation, we modify the energy balance equation to read,

$$\begin{array}{c} \text{change of} \\ \text{heat energy of} \\ \text{segment in time } \Delta t \end{array} = \begin{array}{c} \text{heat in from} \\ \text{left boundary} \end{array} - \begin{array}{c} \text{heat out from} \\ \text{right boundary} \end{array} + \begin{array}{c} \text{heat generated} \\ \text{in segment} \end{array}.$$

Let $Q(x,t)$ be the heat generated per unit time per unit volume at position $x$ in the rod. Then the last term in the energy balance equation is just $QA\Delta x\Delta t$. Applying Fourier's Law (1) gives

$$\begin{aligned} c\rho A\Delta x u(x,t+\Delta t) - c\rho A\Delta x u(x,t) &= \Delta t A\left(-K_0\frac{\partial u}{\partial x}\right)_x - \Delta t A\left(-K_0\frac{\partial u}{\partial x}\right)_{x+\Delta x} \\ &\quad + QA\Delta x\Delta t \end{aligned}$$

The last term is new; the others we had for the rod without sources. Dividing by $A\Delta x\Delta t$ and rearranging yields

$$\frac{u(x,t+\Delta t)-u(x,t)}{\Delta t} = \frac{K_0}{c\rho}\left(\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\Delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\Delta x}\right) + \frac{Q}{c\rho}.$$

Taking the limit $\Delta t, \Delta x \to 0$ gives the Heat Equation with a heat source,

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} + \frac{Q}{c\rho} \tag{51}$$

Introducing non-dimensional variables $\tilde{x} = x/l$, $\tilde{t} = \kappa t/l^2$ gives

$$\frac{\partial u}{\partial \tilde{t}} = \frac{\partial^2 u}{\partial \tilde{x}^2} + \frac{l^2 Q}{\kappa c\rho} \tag{52}$$

Defining the dimensionless source term $q = l^2 Q/(\kappa c\rho)$ and dropping tildes gives the dimensionless Heat Problem with a source,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + q \tag{53}$$

### 7.3.2 Solution method

Ref: Guenther & Lee p. 147 – 149, Myint-U & Debnath §6.7 (exercises)

The simplest case is that of a constant source $q = q(x)$ in the rod. The Heat Problem becomes,

$$\begin{aligned} u_t &= u_{xx} + q(x), \qquad 0 < x < 1, \\ u(0,t) &= b_1, \qquad u(1,t) = b_2, \qquad t > 0, \\ u(x,0) &= f(x), \qquad 0 < x < 1. \end{aligned} \tag{54}$$

If $q(x) > 0$, heat is generated at $x$ in the rod; if $q(x) < 0$, heat is absorbed.

The solution method is the same as that for inhomogeneous BCs: find the equilibrium solution $u_E(x)$ that satisfies the PDE and the BCs,

$$\begin{aligned} 0 &= u_E'' + q(x), \qquad 0 < x < 1, \\ u_E(0) &= b_1, \qquad u_E(1) = b_2. \end{aligned}$$

Then let

$$v(x,t) = u(x,t) - u_E(x).$$

Substituting $u(x,t) = v(x,t) + u_E(x)$ into (54) gives a problem for $v(x,t)$,

$$\begin{aligned} v_t &= v_{xx}, \qquad 0 < x < 1, \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0, \\ v(x,0) &= f(x) - u_E(x), \qquad 0 < x < 1. \end{aligned}$$

Note that $v(x,t)$ satisfies the homogeneous Heat Equation (PDE) and homogeneous BCs, i.e. the basic Heat Problem. Solve the Heat Problem for $v(x,t)$ and then obtain $u(x,t) = v(x,t) + u_E(x)$.

Note that things get complicated if the source is time-dependent - we won't see that in this course.

## 7.4 Periodic boundary conditions

Ref: Guenther & Lee p. 189-190; for alternate method, see Guether & Lee p. 149 and then p. 147

[Sept 26, 2006]

Above, we solved the heat problem with inhomogeneous, but time-independent, BCs by using the steady-state. We now show how to solve the heat problem with inhomogeneous, but time-varying, BCs. We consider the heat problem with an oscillatory (and periodic) BC,

$$\begin{aligned} u_t &= u_{xx}, \qquad 0 < x < 1 \\ u(0,t) &= A\cos\omega t, \qquad u(1,t) = 0, \qquad t > 0, \\ u(x,0) &= f(x), \qquad 0 < x < 1. \end{aligned} \tag{55}$$

The physical meaning of the BC $u(0,t) = A\cos\omega t$ is that we keep changing, in a periodic fashion, the temperature at the end $x = 0$ of the rod.

We don't expect the solution to be independent of time as $t \to \infty$, since we're changing the temperature periodically at one end. However, we do expect that after an initial transient time, the solution will become periodic with angular frequency $\omega$, i.e.

$$u(x,t) = v(x,t) + A(x)\cos(\omega t + \phi(x))$$

where $v(x,t) \to 0$ as $t \to \infty$ is the transient, $A(x)\cos(\omega t + \phi(x))$ is what we call the quasi-steady state, $A(x)$ and $\phi(x)$ are the amplitude and phase of the quasi steady state. To solve the problem, the goal is to first find $A(x)$ and $\phi(x)$, and then $v(x,t)$, if necessary. Often we might not care about the transient state, if we are more interested in the solution after "long times".

### 7.4.1 Complexify the problem

We use the notation $\mathrm{Re}\{z\}$, $\mathrm{Im}\{z\}$ to denote the real and imaginary parts of a complex number $z$. Note that

$$\begin{aligned} \mathrm{Re}\{z\} &= \frac{1}{2}(z + z^*), \qquad \cos\theta = \mathrm{Re}\left\{e^{i\theta}\right\} \\ \mathrm{Im}\{z\} &= \frac{1}{2i}(z - z^*), \qquad \sin\theta = \mathrm{Im}\left\{e^{i\theta}\right\} \end{aligned}$$

where asterisks denote the complex conjugate ($(x+iy)^* = x - iy$). Thus, we can write our quasi-steady solution in terms of complex exponentials,

$$A(x)\cos(\omega t + \phi(x)) = \mathrm{Re}\left\{A(x)e^{i\phi(x)}e^{i\omega t}\right\} = \mathrm{Re}\left\{U(x)e^{i\omega t}\right\}$$

where, for convenience, we have replaced $A(x)e^{i\phi(x)}$ with the complex function $U(x)$. We do this because complex exponentials are much easier to work with than $\cos(\omega t)$ and $\sin(\omega t)$. Note that $U(x)$ has magnitude $A(x) = |U(x)|$ and phase $\phi(x) = \arctan\left(\frac{\mathrm{Im}\{U(x)\}}{\mathrm{Re}\{U(x)\}}\right)$. The phase $\phi(x)$ delays the effects of what is happening at the end of the rod: if the end is heated at time $t = t_1$, the effect is not felt at the center until a later time $t = \phi(1/2)/\omega + t_1$. The following result will be useful.

**Lemma** [Zero sum of complex exponentials] If, for two complex constants $a$, $b$, we have

$$ae^{i\omega t} + be^{-i\omega t} = 0 \tag{56}$$

for all times $t$ in some open interval, then $a = b = 0$.

Proof: Differentiate (56) in time $t$,

$$i\omega \left( ae^{i\omega t} - be^{-i\omega t} \right) = 0. \tag{57}$$

Adding (56) to $1/i\omega \times$(57) gives

$$2ae^{i\omega t} = 0.$$

Since $e^{i\omega t}$ is never zero ($|e^{i\omega t}| = 1$), then $a = 0$. From (56), $be^{-i\omega t} = 0$ and hence $b = 0$.

Note that we could also use the Wronskian to show this:

$$W\left[e^{i\omega t}, e^{-i\omega t}\right] = \det \begin{bmatrix} e^{i\omega t} & e^{-i\omega t} \\ i\omega e^{i\omega t} & -i\omega e^{-i\omega t} \end{bmatrix} = -2i\omega \neq 0$$

and hence $e^{i\omega t}$ and $e^{-i\omega t}$ are linearly independent, meaning that $a = b = 0$.

■

### 7.4.2 ODE and ICs for quasi-steady state

Step 1. Find the quasi-steady state solution to the PDE and BCs of the Heat Problem (55) of the form

$$u_{SS}(x,t) = \operatorname{Re}\left\{U(x)\, e^{i\omega t}\right\} = \frac{1}{2}\left(U(x)\, e^{i\omega t} + U^*(x)\, e^{-i\omega t}\right) = A(x)\cos\left(\omega t + \phi(x)\right) \tag{58}$$

where $U(x)$ is a complex valued function. Substituting (58) for $u(x,t)$ into the PDE in (55) gives

$$\frac{1}{2}\left(i\omega U(x)\, e^{i\omega t} - i\omega U^*(x)\, e^{-i\omega t}\right) = \frac{1}{2}\left(U''(x)\, e^{i\omega t} + U''^*(x)\, e^{-i\omega t}\right),$$

for $0 < x < 1$ and $t > 0$. Multiplying both sides by 2 and re-grouping terms yields

$$\left(i\omega U(x) - U''(x)\right) e^{i\omega t} + \left(-i\omega U^*(x) - U''^*(x)\right) e^{-i\omega t} = 0, \qquad 0 < x < 1, \qquad t > 0. \tag{59}$$

Applying the Lemma to (59) gives

$$i\omega U(x) - U''(x) = 0 = -i\omega U^*(x) - U''^*(x) \tag{60}$$

Note that the left and right hand sides are the complex conjugates of one another, and hence they both say the same thing (so from now on we'll write one or the other). Substituting (58) into the BCs in (55) gives

$$\frac{1}{2}\left(U\left(0\right)e^{i\omega t}+U^{*}\left(0\right)e^{-i\omega t}\right)=\frac{A}{2}\left(e^{i\omega t}+e^{-i\omega t}\right),\qquad \frac{1}{2}\left(U\left(1\right)e^{i\omega t}+U^{*}\left(1\right)e^{-i\omega t}\right)=0, \tag{61}$$

for $t>0$. Grouping the coefficients of $e^{\pm i\omega t}$ and applying the Lemma yields

$$U\left(0\right)=A,\qquad U\left(1\right)=0. \tag{62}$$

To summarize, the problem for the complex amplitude $U\left(x\right)$ of the quasi-steady-state $u_{SS}\left(x,t\right)$ is, from (60) and (62),

$$U''\left(x\right)-i\omega U\left(x\right)=0;\qquad U\left(0\right)=A,\qquad U\left(1\right)=0. \tag{63}$$

Note that $\left(1+i\right)^{2}=2i$, and hence

$$i\omega=\frac{1}{2}\left(1+i\right)^{2}\omega=\left(\sqrt{\frac{\omega}{2}}\left(1+i\right)\right)^{2}.$$

Therefore, (63) can be rewritten as

$$U''-\left(\sqrt{\frac{\omega}{2}}\left(1+i\right)\right)^{2}U=0;\qquad U\left(0\right)=A,\qquad U\left(1\right)=0. \tag{64}$$

### 7.4.3 Solving for quasi-steady state

Solving the ODE (64) gives

$$U=c_{1}\exp\left(-\sqrt{\frac{\omega}{2}}\left(1+i\right)x\right)+c_{2}\exp\left(\sqrt{\frac{\omega}{2}}\left(1+i\right)x\right) \tag{65}$$

where $c_1$, $c_2$ are integration constants. Imposing the BCs gives

$$A=U\left(0\right)=c_{1}+c_{2},\qquad 0=U\left(1\right)=c_{1}\exp\left(-\sqrt{\frac{\omega}{2}}\left(1+i\right)\right)+c_{2}\exp\left(\sqrt{\frac{\omega}{2}}\left(1+i\right)\right).$$

Solving this set of linear equations for the unknowns $\left(c_{1},c_{2}\right)$ and substituting these back into gives

$$\begin{aligned} c_{1} &= \frac{A\exp\left(\sqrt{\frac{\omega}{2}}\left(1+i\right)\right)}{\exp\left(\sqrt{\frac{\omega}{2}}\left(1+i\right)\right)-\exp\left(-\sqrt{\frac{\omega}{2}}\left(1+i\right)\right)}\\ c_{2} &= A-c_{1}=-\frac{A\exp\left(-\sqrt{\frac{\omega}{2}}\left(1+i\right)\right)}{\exp\left(\sqrt{\frac{\omega}{2}}\left(1+i\right)\right)-\exp\left(-\sqrt{\frac{\omega}{2}}\left(1+i\right)\right)} \end{aligned}$$

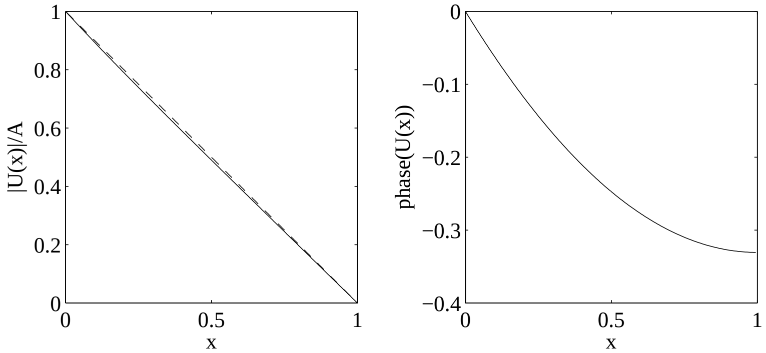


Figure 4: At left, the magnitude of $U(x)$ (solid) and $|U(x)|$ (dash). At right, the phase of $U(x)$.

Substituting these into (65) gives

$$U = A\frac{\exp\left(\sqrt{\frac{\omega}{2}}\,(1+i)\,(1-x)\right) - \exp\left(-\sqrt{\frac{\omega}{2}}\,(1+i)\,(1-x)\right)}{\exp\left(\sqrt{\frac{\omega}{2}}\,(1+i)\right) - \exp\left(-\sqrt{\frac{\omega}{2}}\,(1+i)\right)}.$$

Therefore, the quasi-steady-state solution to the heat problem is

$$u_{SS}\left(x,t\right) = \mathrm{Re}\left\{\frac{\exp\left(\sqrt{\frac{\omega}{2}}\,(1+i)\,(1-x)\right) - \exp\left(-\sqrt{\frac{\omega}{2}}\,(1+i)\,(1-x)\right)}{\exp\left(\sqrt{\frac{\omega}{2}}\,(1+i)\right) - \exp\left(-\sqrt{\frac{\omega}{2}}\,(1+i)\right)}Ae^{i\omega t}\right\}.$$

It is easy to check that $u_{SS}\left(x,t\right)$ satisfies the PDE and BCs in (55). Also, note that the square of the magnitude of the denominator is

$$\left|\exp\left(\sqrt{\frac{\omega}{2}}\,(1+i)\right) - \exp\left(-\sqrt{\frac{\omega}{2}}\,(1+i)\right)\right|^2 = 2\left(\cosh\sqrt{2\omega} - \cos\sqrt{2\omega}\right) > 0$$

which is greater than zero since $\omega > 0$ and hence $\cosh\sqrt{2\omega} > 1 \geq \cos\sqrt{2\omega}$. In Figure 4, the magnitude and phase of $U\left(x\right)$ are plotted as solid lines. The straight dashed line is drawn with $|U\left(x\right)|$ for comparison, illustrating that $|U\left(x\right)|$ is nearly linear in $x$. The phase of $U\left(x\right)$ is negative, indicating a delay between what happens at a point $x$ on the rod and what happens at the end $x = 0$.

### 7.4.4 Solving for the transient

Step 2. Solve for the transient, defined as before,

$$v\left(x,t\right) = u\left(x,t\right) - u_{SS}\left(x,t\right). \tag{66}$$

Substituting (66) into the heat problem (55), given that $u_{SS}(x,t)$ satisfies the PDE and BCs in (55), gives the following problem for $v(x,t)$,

$$\begin{aligned} v_t &= v_{xx}, \qquad 0 < x < 1 \\ v(0,t) &= 0, \qquad v(1,t) = 0, \qquad t > 0, \\ v(x,0) &= f_2(x), \qquad 0 < x < 1, \end{aligned} \tag{67}$$

where the initial condition $f_2(x)$ is given by

$$\begin{aligned} f_2(x) &= u(x,0) - u_{SS}(x,0) \\ &= f(x) - \mathrm{Re}\left\{ A \frac{\exp\left(\sqrt{\frac{\omega}{2}}(1+i)(1-x)\right) - \exp\left(-\sqrt{\frac{\omega}{2}}(1+i)(1-x)\right)}{\exp\left(\sqrt{\frac{\omega}{2}}(1+i)\right) - \exp\left(-\sqrt{\frac{\omega}{2}}(1+i)\right)} \right\}. \end{aligned}$$

The problem for $v(x,t)$ is the familiar basic Heat Problem whose solution is given by (25), (24) with $f(x)$ replaced by $f_2(x)$,

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}, \qquad B_n = 2\int_0^1 f_2(x) \sin(n\pi x)\, dx.$$

#### 7.4.5 Full solution

The full solution to the problem is

$$\begin{aligned} u(x,t) &= \mathrm{Re}\left\{ \frac{\exp\left(\sqrt{\frac{\omega}{2}}(1+i)(1-x)\right) - \exp\left(-\sqrt{\frac{\omega}{2}}(1+i)(1-x)\right)}{\exp\left(\sqrt{\frac{\omega}{2}}(1+i)\right) - \exp\left(-\sqrt{\frac{\omega}{2}}(1+i)\right)} A e^{i\omega t} \right\} \\ &\quad + \sum_{n=1}^{\infty} B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}. \end{aligned}$$

The first term is the quasi-steady state, whose amplitude at each $x$ is constant, plus a transient part $v(x,t)$ that decays exponentially as $t \to \infty$. If the IC $f(x)$ were given, then we could compute the $B_n$'s.

#### 7.4.6 Similar problem: heating/cooling of earth's surface

Consider a vertical column in the earth's crust that is cooled in the winter and heated in the summer, at the surface. We take the $x$-coordinate to be pointing vertically downward with $x = 0$ corresponding to the earth's surface. For simplicity, we model the column of earth by the semi-infinite line $0 \le x < \infty$. We crudely model the heating and cooling at the surface as $u(0,t) = A\cos\omega t$, where $\omega = 2\pi/\tau$ and the (scaled) period $\tau$ corresponds to 1 year. Under our scaling, $\tau = \kappa \times (1 \text{ year})/l^2$. The boundary condition as $x \to \infty$ is that the temperature $u$ is bounded ("$\infty$" is at the bottom of the earth's crust, still far away from the core, whose effects are neglected). What is the quasi-steady state?

The quasi-steady state satisfies the Heat Equation and the BCs,

$$\begin{aligned} (u_{SS})_t &= (u_{SS})_{xx}, \qquad 0 < x < \infty \\ u_{SS}(0,t) &= T_0 + T_1 \cos\omega t, \qquad u_{SS} \text{ bounded as } x \to \infty, \qquad t > 0. \end{aligned} \tag{68}$$

We use superposition: $u(x,t) = u_0 + u_1$, where

$$\begin{array}{ccc} (u_0)_t = (u_0)_{xx}, & (u_1)_t = (u_1)_{xx}, & 0 < x < \infty, \quad t > 0. \\ u_0(0,t) = T_0, & u_1(0,t) = T_1 \cos\omega t, & u_0,\, u_1 \text{ bounded as } x \to \infty, \end{array}$$

Obviously, $u_0(x,t) = T_0$ works, and by uniqueness, we know this is the only solution for $u_0(x,t)$. To solve for $u_1$, we proceed as before and let $u_1(x,t) = \mathrm{Re}\{U(x)\, e^{i\omega t}\}$ to obtain

$$\begin{aligned} U''(x) - i\omega U(x) &= 0, \qquad 0 < x < \infty \\ U(0) &= T_1, \qquad U \text{ bonded as } x \to \infty, \qquad t > 0. \end{aligned} \tag{69}$$

The general solution to the ODE (69) is

$$U = c_1 \exp\left(-\sqrt{\frac{\omega}{2}}(1+i)\,x\right) + c_2 \exp\left(\sqrt{\frac{\omega}{2}}(1+i)\,x\right).$$

The boundedness criterion gives $c_2 = 0$, since that term blows up as $x \to \infty$. The BC at the surface ($x = 0$) gives $c_1 = T_1$. Hence

$$U = T_1 \exp\left(-\sqrt{\frac{\omega}{2}}(1+i)\,x\right).$$

Putting things together, we have

$$\begin{aligned} u_{SS}(x,t) &= T_0 + \mathrm{Re}\left\{T_1 \exp\left(-\sqrt{\frac{\omega}{2}}(1+i)\,x\right) e^{i\omega t}\right\} \\ &= T_0 + T_1 e^{-\sqrt{\frac{\omega}{2}}x}\mathrm{Re}\left\{\exp\left(-i\sqrt{\frac{\omega}{2}}x + i\omega t\right)\right\} \\ &= T_0 + T_1 e^{-\sqrt{\frac{\omega}{2}}x} \cos\left(-\sqrt{\frac{\omega}{2}}x + \omega t\right). \end{aligned} \tag{70}$$

$u_{SS}(x,t)$ is plotted at various dimensionless times $\omega t/\pi = 0, 1/4, 1/2, 3/4, 1$ in Figure 5. Dashed lines give the amplitude $T_0 \pm T_1 e^{-\sqrt{\frac{\omega}{2}}x}$ of the quasi-steady-state $u_{SS}(x,t)$.

Physical questions: What is the ideal depth for a wine cellar? We want the wine to be relatively cool compared to the summer temperature and relatively warm to the

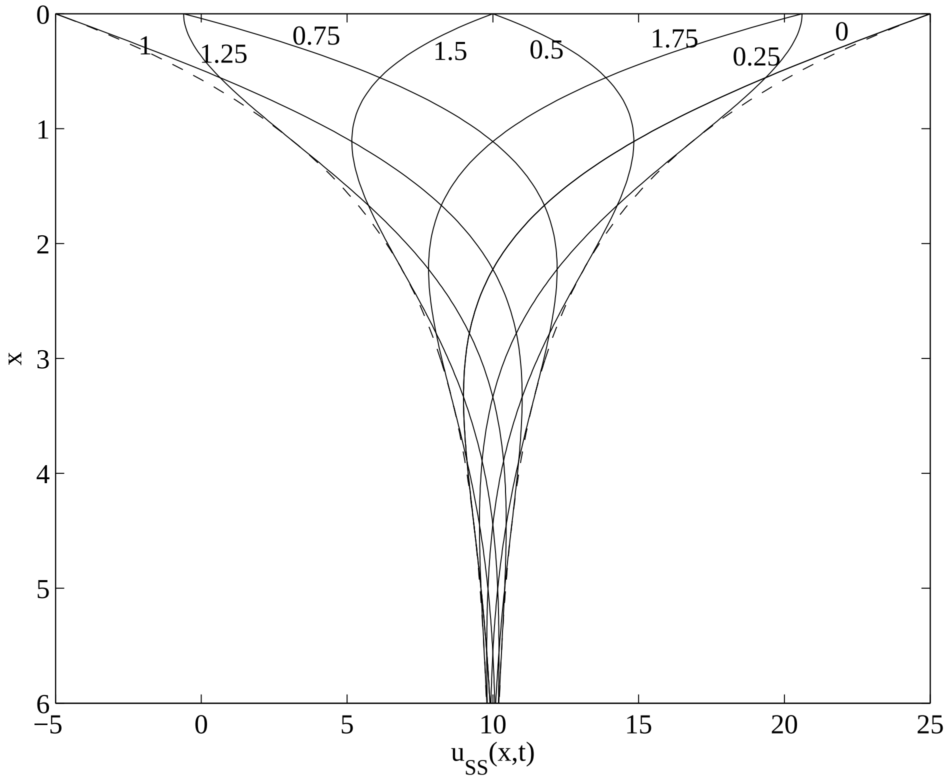


Figure 5: Plot of $u_{SS}\,(x,t)$ at various times. Numbers in figure indicate $\omega t/\pi$.

winter temperature, and yet we want the cellar close to the surface (to avoid climbing too many stairs). Then we must find the smallest depth $x$ such that the temperature $u_{SS}(x,t)$ will be opposite in phase to the surface temperature $u_{SS}(0,t)$. We take $\kappa = 2\times 10^{-3}$ cm$^2$/s and $l = 1$ m. Recall that the period is 1 year, $\tau = (\kappa/l^2)$ (1 year) and 1 year is $3.15 \times 10^7$ s. From the solution (70), the phase of $u_{SS}(x,t)$ is reversed when

$$\sqrt{\frac{\omega}{2}}x = \pi$$

Solving for $x$ gives

$$x = \pi\sqrt{2/\omega}$$

Returning to dimensional coordinates, we have

$$x' = lx = l\pi\sqrt{\frac{2}{2\pi}\tau} = \sqrt{\pi\kappa\,(1\text{ year})} = \sqrt{\pi\times(2\times 10^{-3}\text{ cm}^2)\times 3.15\times 10^7} = 4.45\text{ m}.$$

At this depth, the amplitude of temperature variation is

$$T_1 e^{-\sqrt{\frac{\omega}{2}}x} = T_1 e^{-\pi} \approx 0.04T_1$$

Thus, the temperature variations are only 4% of what they are at the surface. And being out of phase with the surface temperature, the temperature at $x' = 4.45$ m is cold in the summer and warm in the winter. This is the ideal depth of a wine cellar.

Note: for a different solution to this problem using Laplace transforms (not covered in this course), see Myint-U & Debnath Example 11.10.5.

# 8 Linearity, Homogeneity, and Superposition

Ref: Myint-U & Debnath §1.1, 1.3, 1.4

[Sept 28, 2006]

**Definition** Linear space: A set $\mathcal{V}$ is a linear space if, for any two elements[1] $v_1$, $v_2 \in \mathcal{V}$ and any scalar (i.e. number) $k \in \mathbb{R}$, the terms $v_1+v_2$ and $kv_1$ are also elements of $\mathcal{V}$.

E.g. Let $\mathcal{S}$ denote the set of functions that are $\mathcal{C}^2$ (twice-continuously differentiable) in $x$ and $\mathcal{C}^1$ (continuously differentiable) in $t$, for $x\in[0,1]$, $t\geq 0$. We write $\mathcal{S}$ as

$$\mathcal{S} = \{f(x,t) \mid f_{xx},\ f_t \text{ continuous for } x\in[0,1],\ t\geq 0\} \tag{71}$$

If $f_1, f_2 \in \mathcal{S}$, i.e. $f_1, f_2$ are functions that have continuous second derivatives in space and continuous derivatives in time, and $k\in\mathbb{R}$, then $f_1+f_2$ and $kf_1$ also have the

[1] Note that the symbol $\in$ means "an element of". So $x\in[0,l]$ means $x$ is in the interval $[0,l]$.

same, and hence are both elements of $\mathcal{S}$. Thus, by definition, $\mathcal{S}$ forms a linear space over the real numbers $\mathbb{R}$. For instance, $f_1(x,t) = \sin(\pi x)\, e^{-\pi^2 t}$ and $f_2 = x^2 + t^3$.

**Definition** Linear operator: An operator $\mathcal{L} : \mathcal{V} \to \mathcal{W}$ is linear if

$$\mathcal{L}(v_1 + v_2) = \mathcal{L}(v_1) + \mathcal{L}(v_2), \tag{72}$$

$$\mathcal{L}(kv_1) = k\mathcal{L}(v_1), \tag{73}$$

for all $v_1, v_2 \in \mathcal{V}$, $k \in \mathbb{R}$. The first property is the summation property; the second is the scalar multiplication property.

The term "operator" is very general. It could be a linear transformation of vectors in a vector space, or the derivative operation $\partial/\partial x$ acting on functions. Differential operators are just operators that contain derivatives. So $\partial/\partial x$ and $\partial/\partial x + \partial/\partial y$ are differential operators. These operators are functions that act on other functions. For example, the $x$-derivative operation is linear on a space of functions, even though the functions the derivative acts on might not be linear in $x$:

$$\frac{\partial}{\partial x}\left(\cos x + x^2\right) = \frac{\partial}{\partial x}(\cos x) + \frac{\partial}{\partial x}\left(x^2\right).$$

E.g. the identity operator $\mathbb{I}$, which maps each element to itself, i.e. for all elements $v \in \mathcal{V}$ we have $\mathbb{I}(v) = v$. Check for yourself that $\mathbb{I}$ satisfies (72) and (73) for an arbitrary linear space $\mathcal{V}$.

E.g. consider the partial derivative operator acting on $\mathcal{S}$ (defined in (71)),

$$\mathcal{D}(u) = \frac{\partial u}{\partial x}, \qquad u \in \mathcal{S}.$$

From the well-known properties of the derivative, namely

$$\frac{\partial}{\partial x}(u+v) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}, \qquad u, v \in \mathcal{S},$$

$$\frac{\partial}{\partial x}(ku) = k\frac{\partial u}{\partial x}, \qquad u \in \mathcal{S}, k \in \mathbb{R}$$

it follows that the operator $\mathcal{D}$ satisfies properties (72) and (73) and is therefore a linear operator.

E.g. Consider the operator that defines the Heat Equation

$$L = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$$

and acts over the linear space $\mathcal{S}$ of functions $\mathcal{C}^2$ in $x$, $\mathcal{C}^1$ in $t$. The Heat Equation can be written as

$$L(u) = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0.$$

For any two functions $u, v \in \mathcal{S}$ and a real $k \in \mathbb{R}$,

$$\begin{aligned} L(u+v) &= \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(u+v) \\ &= \frac{\partial}{\partial t}(u+v) - \frac{\partial^2}{\partial x^2}(u+v) = \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial t} - \frac{\partial^2 v}{\partial x^2} = L(u) + L(v) \end{aligned}$$

$$L(ku) = \left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)(ku) = \frac{\partial}{\partial t}(ku) - \frac{\partial^2}{\partial x^2}(ku) = k\left(\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right) = kL(u)$$

Thus, $L$ satisfies properties (72) and (73) and is therefore a linear operator.

## 8.1 Linear and homogeneous PDE, BC, IC

Consider a differential operator $\mathcal{L}$ and operators $\mathcal{B}_1$, $\mathcal{B}_2$, $\mathcal{I}$ that define the following problem:

$$\begin{aligned} \text{PDE:} &\quad \mathcal{L}(u) = h(x,t), \text{ with} \\ \text{BC:} &\quad \mathcal{B}_1(u(0,t)) = g_1(t), \qquad \mathcal{B}_2(u(1,t)) = g_2(t) \\ \text{IC:} &\quad \mathcal{I}(u(x,0)) = f(x). \end{aligned}$$

**Definition** The PDE is linear if the operator $\mathcal{L}$ is linear.

**Definition** The PDE is homogeneous if $h(x,t) = 0$.

**Definition** The BCs are linear if $\mathcal{B}_1$, $\mathcal{B}_2$ are linear. The BCs are homogeneous if $g_1(t) = g_2(t) = 0$.

**Definition** The IC is linear if $\mathcal{I}$ is a linear. The IC is homogeneous if $f(x) = 0$.

## 8.2 The Principle of Superposition

The Principle of Superposition has three parts, all which follow from the definition of a linear operator:

(1) If $\mathcal{L}(u) = 0$ is a linear, homogeneous PDE and $u_1$, $u_2$ are solutions, then $c_1u_1 + c_2u_2$ is also a solution, for all $c_1$, $c_2 \in \mathbb{R}$.

(2) If $u_1$ is a solution to $\mathcal{L}(u_1) = f_1$, $u_2$ is a solution to $\mathcal{L}(u_2) = f_2$, and $\mathcal{L}$ is linear, then $c_1u_1 + c_2u_2$ is a solution to

$$\mathcal{L}(u) = c_1f_1 + c_2f_2,$$

for all $c_1$, $c_2 \in \mathbb{R}$.

(3) If a PDE, its BCs, and its IC are all linear, then solutions can be superposed in a manner similar to (1) and (2).

The proofs of (1)-(3) follow from the definitions above. For example, the proof of (1) is

$$\begin{aligned}\mathcal{L}(c_1u_1+c_2u_2) &= \mathcal{L}(c_1u_1)+\mathcal{L}(c_2u_2)\\ &= c_1\mathcal{L}(u_1)+c_2\mathcal{L}(u_2)=c_1\cdot 0+c_2\cdot 0=0\end{aligned}$$

The first step follows from the addition rule of linear operators; the second from the scalar multiplication rule, and the third from the fact that $u_1$ and $u_2$ are solutions of $\mathcal{L}(u)=0$. The proofs of parts (2) and (3) are similar.

E.g. Suppose $u_1$ is a solution to

$$\begin{aligned}\mathcal{L}(u) &= u_t-u_{xx}=0\\ \mathcal{B}_1(u(0,t)) &= u(0,t)=0\\ \mathcal{B}_2(u(1,t)) &= u(1,t)=0\\ \mathcal{I}(u(x,0)) &= u(x,0)=100\end{aligned}$$

and $u_2$ is a solution to

$$\begin{aligned}\mathcal{L}(u) &= u_t-u_{xx}=0\\ \mathcal{B}_1(u(0,t)) &= u(0,t)=100\\ \mathcal{B}_2(u(1,t)) &= u(1,t)=0\\ \mathcal{I}(u(x,0)) &= u(x,0)=0\end{aligned}$$

Then $2u_1-u_2$ would solve

$$\begin{aligned}\mathcal{L}(u) &= u_t-u_{xx}=0\\ \mathcal{B}_1(u(0,t)) &= u(0,t)=-100\\ \mathcal{B}_2(u(1,t)) &= u(1,t)=0\\ \mathcal{I}(u(x,0)) &= u(x,0)=200\end{aligned}$$

To solve $\mathcal{L}(u)=g$ with inhomogeneous BCs using superposition, we can solve two simpler problems: $\mathcal{L}(u)=g$ with homogeneous BCs and $\mathcal{L}(u)=0$ with inhomogeneous BCs.

## 8.3 Application to the solution of the Heat Equation

Recall that we showed that $u_n(x,t)$ satisfied the PDE and BCs. Since the PDE (8) and BCs (10) are linear and homogeneous, then we apply the Principle of Superposition (1) repeatedly to find that the infinite sum $u(x,t)$ given in (80) is also a solution, provided of course that it can be differentiated term-by-term.

# 9 Uniform convergence, differentiation and integration of an infinite series

Ref: Guenther & Lee Ch 3 talk about these issue for the Fourier Series, in particular uniform convergence p. 50, differentiation and integration of an infinite series on p. 58 and 59, and the Weirstrauss M-Test on p. 59.

Ref: Myint-U & Debnath §5.10, 5.13 (problem 12 p. 136)

The infinite series solution (25) to the Heat Equation (8) only makes sense if it converges uniformly[2] on the interval $[0,1]$. The reason is that to satisfy the PDE, we must be able to integrate and differentiate the infinite series term-by-term. This can only be done if the infinite series AND its derivatives converge uniformly. The following results[3] dictate when we can differentiate and integrate an infinite series term-by-term.

**Definition** [Uniform convergence of a series] The series

$$\sum_{n=1}^{\infty} f_n(x)$$

of functions $f_n(x)$ defined on some interval $[a,b]$ converges if for every $\varepsilon > 0$, there exits an $N_0(\varepsilon) \geq 1$ such that

$$\left|\sum_{n=N}^{\infty} f_n(x)\right| < \varepsilon, \qquad \text{for all } N \geq N_0.$$

**Theorem** [Term-by-term differentiation] If, on an interval $x \in [a,b]$,

1. $f(x) = \sum_{n=1}^{\infty} f_n(x)$ converges uniformly,
2. $\sum_{n=1}^{\infty} f_n'(x)$ converges uniformly,
3. $f_n'(x)$ are continuous,

then the series may be differentiated term-by-term,

$$f'(x) = \sum_{n=1}^{\infty} f_n'(x).$$

[2]The precise definitions are outlined in §9.4 (optional reading).

[3]Proofs of these theorems [optional reading] can be found in any good text on real analysis [e.g. Rudin] - start by reading §9.4.

**Theorem** [Term-by-term integration] If, on an interval $x \in [a, b]$,

$$1. \qquad f(x) = \sum_{n=1}^{\infty} f_n(x) \quad \text{converges uniformly,}$$

$$2. \qquad f_n(x) \ \text{ are integrable,}$$

then the series may be integrated term-by-term,

$$\int_a^b f(x)\,dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\,dx.$$

Thus, uniform convergence of an infinite series of functions is an important property. To check that an infinite series has this property, we use the following tests in succession (examples to follow):

**Theorem** [Weirstrass M-Test for uniform convergence of a series of functions] Suppose $\{f_n(x)\}_{n=1}^{\infty}$ is a sequence of functions defined on an interval $[a, b]$, and suppose

$$|f_n(x)| \leq M_n \qquad (x \in [a, b], \quad n = 1, 2, 3, ...).$$

Then the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on $[a, b]$ if the series of numbers $\sum_{n=1}^{\infty} M_n$ converges absolutely.

**Theorem** [Ratio Test for convergence of a series of numbers] The series $\sum_{n=1}^{\infty} a_n$ converges absolutely if the ratio of successive terms is less than a constant $r < 1$, i.e.

$$\frac{|a_{n+1}|}{|a_n|} \leq r < 1, \tag{74}$$

for all $n \geq N \geq 1$.

Note: $N$ is present to allow the first $N - 1$ terms in the series not to obey the ratio rule (74).

**Theorem** [Convergence of an alternating series] Suppose

1. $|a_0| \geq |a_1| \geq |a_2| \geq \cdots$
2. $a_{2n-1} \geq 0, \quad a_{2n} \leq 0 \quad (n = 1, 2, 3 \ldots)$
3. $\lim_{n\to\infty} a_n = 0$

then the sum

$$\sum_{n=1}^{\infty} a_n$$

converges.

Note: This is not an absolute form of convergence, since the series $\sum_{n=1}^{\infty} (-1)^n / n$ converges, but $\sum_{n=1}^{\infty} 1/n$ does not.

## 9.1 Examples for the Ratio Test

E.g. Consider the infinite series of numbers

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \tag{75}$$

Writing this as a series $\sum_{n=1}^{\infty} a_n$, we identify $a_n = (1/2)^n$. We form the ratio of successive elements in the series,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{(1/2)^{n+1}}{(1/2)^n} = \frac{1}{2} < 1$$

Thus, the infinite series (75) satisfies the requirements of the Ratio Test with $r = 1/2$, and hence (75) converges absolutely.

E.g. Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n} \tag{76}$$

Writing this as a series $\sum_{n=1}^{\infty} a_n$, we identify $a_n = 1/n$. We form the ratio of successive elements in the series,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1/\left(n+1\right)}{1/n} = \frac{n}{n+1}$$

Note that $\lim_{n\to\infty} \frac{n}{n+1} = 1$, and hence there is no upper bound $r < 1$ that is greater than $|a_{n+1}| / |a_n|$ for ALL $n$. So the Ratio Test fails, i.e. it gives no information. It turns out that this series diverges, i.e. the sum is infinite.

E.g. Consider the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{77}$$

Again, writing this as a series $\sum_{n=1}^{\infty} a_n$, we identify $a_n = 1/n^2$. We form the ratio of successive elements in the series,

$$\frac{|a_{n+1}|}{|a_n|} = \frac{1/\left(n+1\right)^2}{1/n^2} = \frac{n^2}{\left(n+1\right)^2}$$

Note again that $\lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1$, and hence there is no $r < 1$ that is greater than $|a_{n+1}| / |a_n|$ for ALL $n$. So the Ratio Test gives no information again. However, it turns out that this series converges:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

So the fact that the Ratio Test fails does not imply anything about the convergence of the series!

Note that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} \tag{78}$$

converges for $p > 1$ and diverges (is infinite) for $p \leq 1$.

## 9.2 Examples of series of functions

Note,

$$\sum_{n=1}^{\infty} \sin (n\pi x)$$

does not converge at certain points in $[0, 1]$, and hence it cannot converge uniformly on the interval. In particular, at $x = 1/2$, we have

$$\sum_{n=1}^{\infty} \sin (n\pi x) = \sum_{n=1}^{\infty} \sin \left(\frac{n\pi}{2}\right) = 1 + 0 - 1 + 0 + 1 + 0 - 1 + \cdots$$

The partial sums (i.e. sums of the first $n$ terms) change from 1 to 0 forever. Thus the sum does not converge, since otherwise the more terms we add, the closer the sum must get to a single number.

Consider

$$\sum_{n=1}^{\infty} \frac{\sin (n\pi x)}{n^2} \tag{79}$$

Since the $n$'th term is bounded in absolute value by $1/n^2$, and since $\sum_{n=1}^{\infty} 1/n^2$ converges (absolutely), then the Weirstrass M-Test says the sum (79) converges uniformly on $[0, 1]$.

## 9.3 Application to the solution of the Heat Equation

Ref: Guenther & Lee p. 145 – 147

We now use the Weirstrass M-Test and the Ratio Test to show that the infinite series solution (25) to the Heat Equation converges uniformly,

$$u (x, t) = \sum_{n=1}^{\infty} u_n (x, t) = \sum_{n=1}^{\infty} B_n \sin (n\pi x)\, e^{-n^2\pi^2 t}, \tag{80}$$

for all positive time, i.e. $t \geq t_0 > 0$, and space $x \in [0, 1]$, provided the initial condition $f (x)$ is piecewise continuous.

To apply the M-Test, we need bounds on $|u_n(x,t)|$,

$$|u_n(x,t)| = \left|B_n \sin(n\pi x)\, e^{-n^2\pi^2 t}\right| \le |B_n|\, e^{-n^2\pi^2 t_0}, \qquad \text{for all } x \in [0,1]. \tag{81}$$

We now need a bound on the Fourier coefficients $|B_n|$. Note that from Eq. (24),

$$|B_m| = \left|2\int_0^1 \sin(m\pi x)\, f(x)\, dx\right| \le 2\int_0^1 |\sin(m\pi x)\, f(x)|\, dx \le 2\int_0^1 |f(x)|\, dx, \tag{82}$$

for all $x \in [0,1]$. To obtain the inequality (82), we used the fact that $|\sin(m\pi x)| \le 1$ and, for any integrable function $h(x)$,

$$\left|\int_a^b h(x)\, dx\right| \le \int_a^b |h(x)|\, dx.$$

You've seen this integral inequality, I hope, in past Calculus classes. We combine (81) and (82), to obtain

$$|u_n(x,t)| \le M_n \tag{83}$$

where

$$M_n = \left(2\int_0^1 |f(x)|\, dx\right) e^{-n^2\pi^2 t_0}.$$

To apply the Weirstrass M-Test, we first need to show that the infinite series of numbers $\sum_{n=1}^{\infty} M_n$ converges absolutely (we will use the Ratio Test). Forming the ratio of successive terms yields

$$\frac{M_{n+1}}{M_n} = \frac{e^{-(n+1)^2\pi^2 t_0}}{e^{-n^2\pi^2 t_0}} = e^{\left(n^2-(n+1)^2\right)\pi^2 t_0} = e^{-(2n+1)\pi^2 t_0} \le e^{-\pi^2 t_0} < 1, \qquad n = 1, 2, 3, ...$$

Thus, by the Ratio Test with $r = e^{-\pi^2 t_0} < 1$, the sum

$$\sum_{n=1}^{\infty} M_n$$

converges absolutely, and hence by Eq. (81) and the Weirstrass M-Test, $\sum_{n=1}^{\infty} u_n(x,t)$ converges uniformly for $x \in [0,1]$ and $t \ge t_0 > 0$.

A similar argument holds for the convergence of the derivatives $u_t$ and $u_{xx}$. Thus, for all $t \ge t_0 > 0$, the infinite series (80) for $u$ may be differentiated term-by-term and since each $u_n(x,t)$ satisfies the PDE and BCs, then so does $u(x,t)$. Later, after considering properties of Fourier Series, we will show that $u$ converges even at $t = 0$ (given conditions on the initial condition $f(x)$).

## 9.4 Background [optional]

[Note: You are not responsible for the material in this subsection 9.4 - it is only added for completeness]

Ref: Chapters 3 & 7 of "Principles of Mathematical Analysis", W. Rudin, McGraw-Hill, 1976.

**Definition** Convergence of a series of numbers: the series of real numbers $\sum_{n=1}^{\infty} a_n$ converges if for every $\varepsilon > 0$, there is an integer $N$ such that for any $n \geq N$,

$$\left| \sum_{m=n}^{\infty} a_m \right| < \varepsilon.$$

**Definition** Absolute convergence of a series of numbers: the series of real numbers $\sum_{n=1}^{\infty} a_n$ is said to converge absolutely if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

**Definition** Uniform convergence of a sequence of functions: A sequence of functions $\{f_n(x)\}_{n=1}^{\infty}$ defined on a subset $\mathbb{E} \subseteq \mathbb{R}$ converges uniformly on $\mathbb{E}$ to a function $f(x)$ if for every $\varepsilon > 0$ there is an integer $N$ such that for any $n \geq N$,

$$|f_n(x) - f(x)| < \varepsilon$$

for all $x \in \mathbb{E}$.

Note: pointwise convergence does not imply uniform convergence, i.e. in the definition, for each $\varepsilon$, one $N$ works for all $x$ in $\mathbb{E}$. For example, consider the sequence of functions $\{x^n\}_{n=1}^{\infty}$ on the interval $[0,1]$. These converge pointwise to the function

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1 \end{cases}$$

on $[0,1]$, but do not converge uniformly.

**Definition** Uniform convergence of a series of functions: A series of functions $\sum_{n=1}^{\infty} f_n(x)$ defined on a subset $\mathbb{E} \subseteq \mathbb{R}$ converges uniformly on $\mathbb{E}$ to a function $g(x)$ if the partial sums

$$s_m(x) = \sum_{n=1}^{m} f_n(x)$$

converge uniformly to $g(x)$ on $\mathbb{E}$.

Consider the following sum,

$$\sum_{n=1}^{\infty} \frac{\cos(n\pi x)}{n}$$

does converge, but this is a weaker form of convergence (pointwise, not uniform or absolute), by the Alternating Series Test (terms alternate in sign, absolute value of the terms goes to zero as $n \to \infty$). Consult Rudin for the Alternating Series Test, if desired.