Lecture 6 : Elliptic operators + friends

* 1d "Sturm-Liouville" operator:

$$\hat{A} = \underbrace{\frac{1}{w(x)}}_{>0} \left[-\frac{\partial}{\partial x} \underbrace{c(x) \frac{\partial}{\partial x} + p(x)}_{\text{real}} \right], \quad u(x) \text{ on } [0, L] = \Omega$$

Dirichlet boundaries $u\big|_{\partial\Omega} = 0$

let $\langle u, v \rangle = \int_0^L w(x) \overline{u(x)} v(x) \, dx = \int_\Omega w \bar{u} v$

$$\Rightarrow \langle u, \hat{A} v \rangle = \int_\Omega \cancel{w} \bar{u} \frac{1}{\cancel{w}} \left[-\frac{\partial}{\partial x} (c v') + p v \right]$$

$$= -\int_\Omega \bar{u} (c v')' + \int_\Omega \bar{u} p v$$

$$= \int_\Omega \bar{u}' c v' + \int_\Omega \overline{p u} \, v = -\int_\Omega \overline{(c u')'} \, v + \int_\Omega \overline{p u} \, v$$

$$- \cancelto{0}{\bar{u} c v' \Big|_{\partial\Omega}} \qquad\qquad + \cancelto{0}{c \bar{u}' v \Big|_{\partial\Omega}}$$

$$= \int_\Omega w \overline{\frac{1}{w} \left[-(c u')' + p u \right]} \, v = \langle \hat{A} u, v \rangle$$

$$\Rightarrow \hat{A} = \hat{A}^* \Longrightarrow \text{real } \lambda, \text{ orthogonal eigenfunctions}$$

(+ "diagonalizable" for "reasonable" w, c, p)

= "Sturm-Liouville theory"

$$\langle u, \hat{A} u \rangle \underset{\text{(same steps)}}{=} \cdots = \int_\Omega \Big(\underbrace{c|u'|^2}_{\substack{>0 \\ \text{for } u' \neq 0 \\ \Updownarrow \\ u \neq 0 \\ \text{since } u = \text{constant} \Rightarrow u = 0}} + \underbrace{p|u|^2}_{\substack{>0 \\ \text{for } u \neq 0}} \Big)$$

> 0 for $u \neq 0$

if $c > 0$, $p \geq 0$

$\Rightarrow \hat{A}$ positive-definite for $c > 0$, $p \geq 0$

= "<u>elliptic operator</u>" (also elliptic if negative-definite)

* Higher dimensions:

- even more useful to do such analysis in $> 1d$, since analytical solution are even harder, so ability to say general things from $\hat{A}$ is crucial to <u>understanding</u>

a "simple" case: $\hat{A} = -\nabla^2 = -\nabla \cdot \nabla = -\text{div. grad.}$
(still very hard in $> 1d$!)

a generalization: (non-uniform media)

$$\boxed{\hat{A} = \underbrace{\frac{1}{w(\vec{x})}}_{>0} \Big[-\nabla \cdot \underbrace{c(\vec{x})}_{\text{real}} \nabla + \underbrace{p(\vec{x})}_{\text{real}} \Big]}$$

on functions $u(\vec{x})$ on some (finite) domain Ω

+ Dirichlet boundaries (for now): $\boxed{u\big|_{\partial\Omega} = 0}$

(even more general: c could be a self-adjoint matrix) consider: $\langle u, v \rangle = \int_\Omega w \bar{u} v$

$$\Rightarrow \langle u, \hat{A} v \rangle = \underbrace{-\int_\Omega \bar{u} \nabla \cdot (c \nabla v)}_{\text{need to integrate by parts}} + \underbrace{\int_\Omega \bar{u} p v}_{= \int_\Omega w \overline{\frac{1}{w} p u}\, v}$$

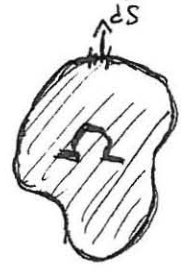
$\Rightarrow p$ term is self-adjoint

☆ review of integration by parts:

1d: $$\int_\Omega f g' = \int_\Omega \Big[\underbrace{(fg)'}_{\substack{\text{integral of}\\ \text{derivative}\\ = \text{easy}}} - \underbrace{f' g}_{\substack{\text{from}\\ \text{product}\\ \text{rule}}} \Big] = fg \Big|_{\partial\Omega} - \int f' g$$

> 1d: $$\int_\Omega f \nabla \cdot \vec{g} = \int_\Omega \Big[\underbrace{\nabla \cdot (f \vec{g})}_{\substack{\text{integral}\\ \text{of divergence}\\ = \text{easy by}\\ \text{divergence theorem}}} - \underbrace{(\nabla f) \cdot \vec{g}}_{\substack{\text{from "product rule"}\\ \nabla \cdot (f\vec{g}) =\\ (\nabla f)\cdot \vec{g} + f \nabla \cdot g}} \Big]$$

$$= \underbrace{\oint\!\!\oint_{\partial\Omega}}_{\substack{\text{surface}\\ \text{integral}\\ \text{(1 less dimension)}}} f \vec{g} \cdot \underbrace{d\vec{S}}_{\substack{\text{outward}\\ \text{normal}}} - \int_\Omega (\nabla f) \cdot \vec{g}$$



$$\Rightarrow \; -\int_\Omega \overbrace{\bar{u}}^{f}\, \nabla\cdot \overbrace{(c\nabla v)}^{\vec{g}} = -\cancel{\oint_{\partial\Omega} \bar{u}(c\nabla v)\cdot d\vec{S}} + \int_\Omega \overline{c\nabla u}\cdot\nabla v$$

new: $\vec{g}$ f

$\cancel{\oint} \to 0$ if $u|_{\partial\Omega}=0$

(or if $(\nabla v)\cdot d\vec{S}\big|_{\partial\Omega}=0$)

$$= \cancel{\oint_{\partial\Omega} v\,\overline{c\nabla u}\cdot d\vec{S}}^{\,0} - \int_\Omega v\, \overline{\nabla\cdot(c\nabla u)\cdot w\cdot \frac{1}{w}}$$

$$\Rightarrow \langle u, \hat{A} v\rangle = \langle \hat{A}u, v\rangle$$

$$\Rightarrow \boxed{\hat{A} = \hat{A}^*} \Rightarrow \boxed{\text{real } \lambda, \text{ orthogonal eigenvectors (+ usually "diagonalizable")}}$$

also: $\langle u, \hat{A}u\rangle = \underbrace{\cdots}_{\substack{\text{same steps,}\\ \text{stopping}\\ \text{halfway}}} = \int_\Omega \Big[\underbrace{c|\nabla u|^2}_{\substack{>0 \text{ for}\\ u\neq \text{constant}\\ =0 \text{ by Dirichlet}}} + \underbrace{p|u|^2}_{\substack{>0\\ \text{for } u\neq 0}}\Big]$

> 0 for $u \neq 0$

if $c > 0$, $p \geq 0$

$$\Rightarrow \boxed{\hat{A} \text{ ("elliptic") positive-definite for } c>0, p\geq 0} \Rightarrow \boxed{\lambda > 0, \; N(\hat{A}) = \{0\}}$$

examples:

heat/diffusion: $\underbrace{\frac{1}{w}}_{w \sim \text{heat capacity} > 0} \nabla \cdot (\underbrace{c}_{\substack{\text{thermal conductivity} \\ > 0}} \nabla \underbrace{u}_{\text{temperature}}) = \frac{\partial u}{\partial t} \quad (+ \underbrace{f(\vec{x},t)}_{\text{sources/sinks}})$

$= \boxed{\text{"parabolic equation"}}: \hat{A} u = \frac{\partial u}{\partial t}$ $\quad \hat{A}$ negative-definite

$$\left(u(\vec{x},t) = \sum_{n=1}^{\infty} \langle u_n, u \rangle\Big|_{t=0} u_n(\vec{x}) e^{\lambda_n t} \qquad \lambda_n < 0 \right.$$

$$\left. \text{normalization} \quad \langle u_n, u_m \rangle = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases} \right)$$

$\Rightarrow$ exponentially decaying solutions (smoothing: faster spatial oscillations decay faster)

Poisson: $\underbrace{\frac{1}{w}}_{>0} \nabla \cdot (\underbrace{c}_{>0 \text{ or } <0} \nabla u) = f$ $\quad$ (eg. steady-state heat eq. $\frac{\partial u}{\partial t} = 0$)

$= \boxed{\text{"elliptic equation"}}: \hat{A} u = f$ $\quad \hat{A}$ definite (pos. or neg.)

$N(\hat{A}) = \{0\} \Rightarrow$ unique solution (if any: existence if f in span of eigenfunctions)

$$\left(u(\vec{x}) = \sum_{n=1}^{\infty} \frac{\langle u_n, f \rangle u_n(\vec{x})}{\lambda_n} \right)$$

note "smoothing" property:

u is smoother than f since large λ = fast spatial oscillations are suppressed

<u>scalar wave equation:</u> $\underbrace{\frac{1}{w}}_{\sim \frac{1}{\text{density}} > 0} \nabla \cdot (\underbrace{c}_{\substack{\sim \text{springiness} \\ > 0}} \nabla u) = \frac{\partial^2 u}{\partial t^2} \quad (+ \underbrace{f(\vec{x},t)}_{\substack{\text{external} \\ \text{force}}})$

(e.g. pressure waves)

$= \boxed{\text{"hyperbolic equation"}}$: $\hat{A} u = \frac{\partial^2 u}{\partial t^2}$ $\hat{A}$ negative definite (or maybe semidefinite)

$\Longrightarrow$ <u>oscillating solutions:</u>

$$\hat{A} u_n = \underbrace{\lambda_n}_{<0} u_n = -\omega_n^2 u_n$$

$$(\omega_n = \sqrt{-\lambda_n})$$

choose $\langle u_n, u_m \rangle = \begin{cases} 1 & n=m \\ 0 & n \neq m \end{cases}$

$$\Longrightarrow u(\vec{x},t) = \sum_{n=1}^{\infty} \underbrace{\left[\langle u_n, u|_{t=0} \rangle \cos(\omega_n t) + \frac{\langle u_n, \dot{u}|_{t=0} \rangle}{\omega_n} \sin(\omega_n t) \right] u_n(\vec{x})}_{\text{"normal modes"}}$$

<u>Laplace's equation:</u>

$$\underbrace{\frac{1}{w}}_{>0} \nabla \cdot (\underbrace{c}_{>0} \nabla u) = 0$$

: e.g. heat equation for $\frac{\partial u}{\partial t} = 0$
or wave equation for $\frac{\partial^2 u}{\partial t^2} = 0$
and no external force or sources: $f = 0$

$\Longrightarrow u = 0$ (boring!)
for $u|_{\partial\Omega} = 0$

... more interesting if $u|_{\partial\Omega} \neq 0$ ...