Lecture 7

Quasilinear equations (minimal surface equation)

For any $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, the graph of f is $\{(x, f(x))\} \subset \mathbb{R}^{n+1}$.

The tangents of the graph is $(0, \cdots, 1, 0, \cdots, 0, f_i)$, where 1 is on the i^{th} slot. So the normal vector is $(-\nabla f, 1)$, and the unit normal vector is $\widehat{n} = \frac{1}{\sqrt{1+|\nabla f|^2}}(-\nabla f, 1)$.

The second fundamental form is a map $\prod(x) : TG_x \longrightarrow TG_x, \prod(x)(e_i) = \nabla_{e_i}\widehat{n}$. (Since $< \widehat{n}, \widehat{n} >= 1 \Longrightarrow \nabla_X < \widehat{n}, \widehat{n} >= 0 \Longrightarrow 2 < \nabla_X \widehat{n}, \widehat{n} >= 0 \Longrightarrow \nabla_X \widehat{n} \in TG$.)
We compute:

$$\begin{aligned}
\nabla_{e_i}\widehat{n} &= \frac{\partial}{\partial x^i}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}(-\nabla f, 1)\right) \\
&= \frac{\partial}{\partial x^i}\left(\frac{1}{\sqrt{1+|\nabla f|^2}}\right)(-\nabla f, 1) + \frac{1}{\sqrt{1+|\nabla f|^2}}\frac{\partial}{\partial x^i}((-\nabla f, 1)) \\
&= -\frac{1}{2}\frac{2 f_j f_{ji}}{(1+|\nabla f|^2)^{3/2}}(-\nabla f, 1) + \frac{1}{\sqrt{1+|\nabla f|^2}}(f_{1i}, \cdots, f_{ni}, 0) \\
&= a_{ij} e_j
\end{aligned}$$

where

$$a_{ij} = \frac{f_l f_{li} f_j}{(1+|\nabla f|^2)^{3/2}} - \frac{1}{\sqrt{1+|\nabla f|^2}} f_{ij}$$

(Assuming $T_x G = (\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}, 0)$ and $\hat{n} = e_{n+1}$.)

Minimal $\Leftrightarrow$ "0 mean curvature", i.e.

$$\begin{aligned}
& \sum a_{ii} = 0 \\
\Longrightarrow \quad & \frac{f_l f_{li} f_i}{(1+|\nabla f|^2)^{3/2}} = \frac{1}{\sqrt{1+|\nabla f|^2}}\Delta f \\
\Longrightarrow \quad & f_l f_{li} f_i = (1+|\nabla f|^2)\Delta f \\
\Longrightarrow \quad & div\left(\frac{1}{\sqrt{1+|\nabla f|^2}} f_i\right) = 0 \\
\Longrightarrow \quad & \partial_i\left(\frac{1}{\sqrt{1+|\nabla f|^2}} f_i\right) = 0
\end{aligned}$$

In general, the operator $L = a^{ij}(x, u, Du) D_{ij} u + \cdots$ is called **quasi-linear.**

Now we check that the surface is "minimal", i.e. has minimal area.
Denote $T : (x^1, \cdots, x^n) \longrightarrow (x^1, \cdots, x^n, f(x^1, \cdots, x^n))$. Since

$$T_* \partial_k = \sum_{j=1}^{n+1} a_{kj} \partial_j, \quad (a_{kj}) = \begin{pmatrix} I \\ \nabla f \end{pmatrix}_{(n+1)\times n},$$

we get

$$T^*g_{\mathbb{R}^{n+1}}(\partial_k,\partial_l) = g_{\mathbb{R}^{n+1}}(T_*\partial_k, T_*\partial_l) = (A^TA)_{kl} = (I+\nabla f\nabla f^T)_{kl} = \delta_{kl} + f_kf_l.$$

The matrix $I+\nabla f\nabla f^T$ can be diagonized to $diag\{1+|\nabla f|^2, 1, \cdots, 1\}$, so the area of graph of f is

$$A(f) = \int_{\mathbb{R}^n}\sqrt{det(g_{ij})}dx = \int_{\mathbb{R}^n}\sqrt{det(I+\nabla f\nabla f^T)}dx = \int_{\mathbb{R}^n}(1+|\nabla f|^2)^{1/2}dx.$$

Thus

$$\begin{aligned} A'(f)h &= \frac{d}{dt}A(f+th)|_{t=0} \\ &= \int_{\mathbb{R}^n}\frac{1}{2}(1+|\nabla f|^2)^{-1/2}\cdot 2 < \nabla f, \nabla h > dx \\ &= \int_{\mathbb{R}^n} < \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}, \nabla h > dx \\ &= -\int_{\mathbb{R}^n} h\cdot div(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}})dx. \end{aligned}$$

Thus Minimal $\Longleftrightarrow div(\frac{\nabla f}{\sqrt{1+|\nabla f|^2}}) = 0.$

Fully nonlinear equations (Monge-Ampère equation).

Suppose $\Omega\subset\mathbb{R}^n$. Now we consider the differential equations like

$$F[u] = F(x,u,Du,D^2u) = 0,$$

where $F:\Omega\times\mathbb{R}\times\mathbb{R}^n\times S(n)\longrightarrow\mathbb{R}$, and $S(n)$ is the set of all symmetric $n\times n$ matrices.

Definition 1 *F is* **elliptic** *in some subset $\Gamma\subset\Omega\times\mathbb{R}\times\mathbb{R}^n\times S(n)$ if $(\frac{\partial F}{\partial r_{ij}})(\gamma) > 0$, $\forall\gamma = (x,z,p,r)\in\Gamma$.*

If $\exists\Lambda,\lambda > 0$ such that $\Lambda I > (\frac{\partial F}{\partial r_{ij}}) > \lambda I$ for all $\gamma\in\Gamma$, then we say F **uniformly elliptic***.*

If $u\in C^2(\Omega)$, and F is elliptic on range of $x\to(x,u,Du,D^2u)$, then we say F is **elliptic with respect to** *u.*

Example: Monge-Ampère Equation

$$F[u] = detD^2u - f(x) = 0.$$

(Note that $\Delta u = trace(D^2u)$).

We do some computation:

$$det\ r_{ij} = \sum_{\sigma \in S_n} (-1)^{sign \sigma} r_{1\sigma(1)} r_{2\sigma(2)} \cdots r_{n\sigma(n)},$$

$F_{ij}(r) = \frac{\partial F}{\partial r_{ij}} = (i,j)-$cofactor of r,
$(r^{-1})_{ij} = \frac{1}{det\ r} F_{ij}(r)$,
$F_{ij}(r) = det\ r \cdot (r^{-1})^{ij}$.

So F is elliptic when r is positive definite, and thus $F[u]$ is elliptic if u is strictly convex.

More generally, $F[u] = det\ D^2 u - f(x, u, Du) = 0$ is elliptic for strictly convex functions.

Given $F[u]$, define the **linearization** of F at a function u to be

$$F'[u] : C^2(\Omega) \to \mathbb{R}, \quad h \longmapsto \frac{d}{dt} F[u + th]|_{t=0}.$$

$$\begin{aligned} F'[u](h) &= \frac{d}{dt} F[x, u + th, Du + tDh, D^2 u + tD^2 h]|_{t=0} \\ &= F_z(u) h + F_{p_i} h_i + F_{r_{ij}}(u) D_{ij} h \\ &= (F_{r_{ij}}(u) D_{ij} + F_{P_i}(u) D_i + F_z(u)) h \\ &= Lh \end{aligned}$$

So our definition of elliptic at $u \Leftrightarrow$ linearization of F at u is an elliptic operator.

Example: Linearization of Monge-Ampère:

$$F[u] = det D^2 u - f(x).$$

$$F'[u](h) = F_{r_{ij}}(D^2 u) D_{ij} h$$

Let λ_i be eigenvalues of $D^2 u$, then eigenvalues of $F_{r_{ij}}$ are
$$\lambda_2 \cdots \lambda_n,\ \lambda_1 \lambda_3 \cdots \lambda_n,\ \cdots,\ \lambda_1 \cdots \lambda_{n-1}$$
Certainly F is not uniformly elliptic.

Elementary Symmetric Function: $\sigma_k(D^2 u) =$ Sum of principal $k \times k$ matrix.

$$\sigma_k(\lambda_1, \cdots, \lambda_n) = \sum_{i_1 < \cdots < i_k} \lambda_{i_1} \cdots \lambda_{i_k},$$

Now for $F[u] = det D^2 u - f(x)$, $F'[u](h) = F_{r_{ij}}(D^2 u) D_{ij} h$, when is it elliptic?

Theorem 1 *If* $\sigma_k > 0, \sigma_{k-1} > 0, \cdots, \sigma_1 > 0$, *then* $F_{r_{ij}} > 0$.

$\Gamma_k = \{component of \sigma_k > 0\}$.

Example: $n = 3$.

$$\begin{aligned} &det = \lambda_1\lambda_2\lambda_3, \\ &\sigma_2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3, \\ &\Delta = \lambda_1 + \lambda_2 + \lambda_3. \end{aligned}$$

$\Gamma_3 = \{\text{positive cone}\}$.

For Γ_2, $\sigma_2 = 0$ is a cone, so $\{\sigma_2 > 0\}$ has two components, $\Gamma_2^+ = \{x_2 > 0\} \cap \{\sigma_1 > 0\}$, e.v. of $F_{r_{ij}}$ on $(\lambda_2 + \lambda_3, \lambda_1 + \lambda_3, \lambda_1 + \lambda_2)$,

$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$,

$\lambda_1 + \lambda_2 + \lambda_3 > 0$.

Claim: If $\lambda_1 \geq \lambda_2 \geq \lambda_3$, then $\lambda_2 > 0$.

In fact, by $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 > 0$, we get

$$\lambda_1(\lambda_2 + \lambda_3) + \lambda_2\lambda_3 > 0, \quad i.e. \quad \lambda_1(\lambda_2 + \lambda_3) > -\lambda_2\lambda_3.$$

If $\lambda_2 \leq 0$,then $\lambda_2 + \lambda_3 < 0$, thus we get

$$\begin{aligned} &-\lambda_2 - \lambda_3 < \lambda_1 \leq \frac{-\lambda_2\lambda_3}{\lambda_2 + \lambda_3} \\ \Longrightarrow &-\lambda_2 - \lambda_3 < \frac{-\lambda_2\lambda_3}{\lambda_2 + \lambda_3} \\ \Longrightarrow &\lambda_2 + \lambda_3 > \frac{\lambda_2\lambda_3}{\lambda_2 + \lambda_3} \\ \Longrightarrow &(\lambda_2 + \lambda_3)^2 < \lambda_2\lambda_3 \\ \Longrightarrow &\lambda_2^2 + \lambda_2\lambda_3 + \lambda_3^2 < 0 \end{aligned}$$

which is a contradiction.

So we have $\lambda_1, \lambda_2 > 0$, thus $\lambda_1 + \lambda_2 > 0$.

If $\lambda_1 + \lambda_3 \leq 0$, then $\lambda_1\lambda_3 < 0$, which contradicts with $\lambda_2(\lambda_1 + \lambda_3) + \lambda_1\lambda_3 > 0$. Thus $\lambda_1 + \lambda_3 > 0$.

Also from $\lambda_1(\lambda_2 + \lambda_3) + \lambda_2\lambda_3 > 0$, we can get $\lambda_2 + \lambda_3 > 0$ by the same way.

Theorem 2 *$\sigma_2(D^2u) = f(x)$ is elliptic if $f(x) > 0$ and $\Delta u \geq 0$.*

$\sigma_k(D^2u) = f(x)$ is elliptic if $f(x) > 0, D^2u \in \Gamma_k^+$, and $\sigma_k > 0, \sigma_{k-1} > 0, \cdots, \sigma_1 > 0$.

Comparison principle for nonlinear equations.

First we give a maximum principle.

Theorem 3 *Let* $u, v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ *satisfy* $F[u] \geq F[v]$ *in* Ω, $u \leq v$ *on* $\partial\Omega$, *and*
(i) F *is elliptic along the straight line path* $tu + (1-t)v$,
(ii) $F_z \leq 0$.
Then $u \leq v$ *in* Ω.

Proof:

$$\begin{aligned} F[u] - F[v] &= \int_0^1 \frac{d}{dt} F[tu + (1-t)v]dt \\ &= \int_0^1 F_{r_{ij}} \frac{d}{dt}(tD^2u + (1-t)D^2v) + F_{p_i}(D_iu - D_iv) + F_z(u-v)dt \\ &= (\int_0^1 F_{r_{ij}} dt) D^2_{ij}(u-v) + (\int_0^1 F_{p_i} dt) D_i(u-v) + (\int_0^1 F_z dt)(u-v) \\ &= L(u-v) \\ &> 0, \end{aligned}$$

but $u \leq v$ on $\partial\Omega$. Since elliptic on path, we get $a^{ij} > 0$ and $c \leq 0$, thus $u \leq v$ in Ω. ■

Corollary 1 *Suppose* $u, v \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ *satisfy* $F[u] = F[v]$ *in* Ω, *and (i),(ii) hold, with* $u = v$ *on* $\partial\Omega$. *Then* $u = v$ *in* Ω.

Example: Monge-Ampère.

$detD^2u = f(x) > 0$, $detD^2v = f(x)$, with u, v strictly convex, $tu + (1-t)v$ is also strictly convex. So (i) works. For (ii), there is no z dependence. So $u = v$ on $\partial\Omega$ implies $u = v$ on Ω.

Similarly for σ_k.

Result also works for Minimal surface.

Theorem 4 *Suppose* $u \in C^2(\Omega), F[u] = 0$, *and* F *elliptic with respect to* u. *Also suppose* F *is* C^∞, *(e.g.* $detD^2u = f(x) > 0 \in C^\infty$*). Then* $u \in C^\infty(\Omega)$.

Proof: Use difference quotients. Fix coordinate vector e_1.

Let $v(x) = u(x + he_1), h \in \mathbb{R}$, and $u_t = tv + (1-t)u, 0 \leq t \leq 1$.

$$\int_0^1 \frac{d}{dt} F(x + the_1, u_t, Du_t, D^u_t))dt = F(x + he_1, v, Dv, D^2v) - F(x, u, Du, D^2u) = 0,$$

$$\int_0^1 F_{x_1}(\cdot)h + \int_0^1 F_z(\cdot)(v-u) + \int_0^1 F_{p_i}(\cdot)D_i(v-u) + \int_0^1 F_{r_{ij}}(\cdot)D_{ij}(v-u) = 0.$$

We can write this to be

$$L(v-u) = -f \cdot h.$$

Thus

$$L(\frac{v-u}{h}) = L(\Delta_h{'}u) = -f = \int_0^1 F_{x_1}(x + the_1, u_t, Du_t, D^2u_t)dt.$$

So

$$\Delta_h{}'u \in W^{2,p}, \forall p \Longrightarrow u \in W^{3,p}.$$

(We will prove this later.)

By Sobolev embedding, $u \in C^{2,\alpha}$.

Then $f \in C^{\alpha} \Longrightarrow \Delta_h u \in C^{2,\alpha} \Longrightarrow u \in C^{3,\alpha}$

$\Longrightarrow f \in C^{1,\alpha} \Longrightarrow \Delta_h u \in C^{3,\alpha} \Longrightarrow u \in C^{4,\alpha}$.

Go on with this procedure, we get C^{∞} at last. ■