18.102 Introduction to Functional Analysis
Spring 2009

Lecture 20. Thursday, April 23: Spectral theorem for compact self-adjoint operators

Let $A \in \mathcal{K}(\mathcal{H})$ be a compact operator on a separable Hilbert space. We know of course, even without assuming that A is compact, that

$$\operatorname{Nul}(A) \subset \mathcal{H} \tag{20.1}$$

is a closed subspace, so $\operatorname{Nul}(A)^\perp$ is a Hilbert space – although it could be finite-dimensional (or even 0-dimensional in the uninteresting case that $A = 0$).

Theorem 15. *If $A \in \mathcal{K}(\mathcal{H})$ is a self-adjoint, compact operator on a separable Hilbert space, so $A^* = A$, then $\operatorname{null}(A)^\perp$ has an orthonormal basis consisting of eigenfunctions of A, u_j such that*

$$Au_j = \lambda_j u_j,\ \ \lambda_j \in \mathbb{R} \setminus \{0\}, \tag{20.2}$$

arranged so that $|\lambda_j|$ is a non-increasing sequence satisfying $\lambda_j \to 0$ as $j \to \infty$ (in case $\operatorname{Nul}(A)^\perp$ is finite dimensional, this sequence is finite).

Before going to the proof, let's notice some useful conclusions. One is called 'Fredholm's alternative'.

Corollary 4. *If $A \in \mathcal{K}(\mathcal{H})$ is a compact self-adjoint operator on a separable Hilbert space then the equation*

$$u - Au = f \tag{20.3}$$

either has a unique solution for each $f \in \mathcal{H}$ or else there is a non-trivial finite dimensional space of solutions to

$$u - Au = 0 \tag{20.4}$$

and then (20.3) has a solution if and only if f is orthogonal to all these solutions.

Proof. This is just saying that the null space of $\operatorname{Id} - A$ is a complement to the range – which is closed. So, either $\operatorname{Id} - A$ is invertible or if not then the range is precisely the orthocomplement of $\operatorname{Nul}(\operatorname{Id} - A)$. You might say there is not much alternative from this point of view, since it just says the range is *always* the orthocomplement of the null space. □

Let me separate off the heart of the argument from the bookkeeping.

Lemma 14. *If $A \in \mathcal{K}(\mathcal{H})$ is a self-adjoint compact operator on a separable (possibly finite-dimensional) Hilbert space then*

$$F(u) = (Au, u),\ F : \{u \in \mathcal{H}; \|u\| = 1\} \longrightarrow \mathbb{R} \tag{20.5}$$

is a continuous function on the unit sphere which attains its supremum and infimum. Furthermore, if the maximum or minimum is non-zero it is attained at an eivenvector of A with this as eigenvalue.

Proof. So, this is just like function in finite dimensions, except that it is not. First observe that F is real-valued, which follows from the self-adjointness of A since

$$\overline{(Au, u)} = (u, Au) = (A^*u, u) = (Au, u). \tag{20.6}$$

Moreover, continuity of F follows from continuity of A and of the inner product so

$$|F(u) - F(u')| \le |(Au, u) - (Au, u')| + |(Au, u') - (Au', u')| \le 2\|A\|\|u - u'\| \tag{20.7}$$

since both u and u' have norm one.

If we were in finite dimensions this finishes the proof, since the sphere is then compact and a continuous function on a compact set attains its sup and inf. In the general case we need to use the compactness of A. Certainly F is bounded,

$$|F(u)| \leq \sup_{\|u\|=1} |(Au, u)| \leq \|A\|. \tag{20.8}$$

Thus, there is a sequence u_n^+ such that $F(u_n^+) \to \sup F$ and another u_n^- such that $F(u_n^-) \to \inf F$. The *weak* compactness of the unit sphere means that we can pass to a subsequence in each case, and so assume that $u_n^\pm \rightharpoonup u^\pm$ converges weakly. Then, by the compactness of A, $Au_n^\pm \to Au^\pm$ converges strongly, i.e. in norm. But then we can write

$$\begin{aligned} |F(u_n^\pm) - F(u^\pm)| &\leq |(A(u_n^\pm - u^\pm), u_n^\pm)| + |(Au^\pm, u_n^\pm - u^\pm)| \\ &= |(A(u_n^\pm - u^\pm), u_n^\pm)| + |(u^\pm, A(u_n^\pm - u^\pm))| \leq 2\|Au_n^\pm - Au^\pm\| \end{aligned} \tag{20.9}$$

to deduce that $F(u^\pm) = \lim F(u_n^\pm)$ are respectively the sup and inf of F. Thus indeed, as in the finite dimensional case, the sup and inf are attained, as in max and min.

So, suppose that $\Lambda^+ = \sup F > 0$. Then for any $v \in \mathcal{H}$ with $v \perp u^+$ the curve

$$L_v : (-\pi, \pi) \ni \theta \longmapsto \cos\theta u^+ + \sin\theta v \tag{20.10}$$

lies in the unit sphere. Computing out

$$\begin{aligned} F(L_v(\theta)) &= \\ &(AL_v(\theta), L_v(\theta)) = \cos^2\theta F(u^+) + 2\sin(2\theta)\operatorname{Re}(Au^+, v) + \sin^2(\theta)F(v) \end{aligned} \tag{20.11}$$

we know that this function must take its maximum at $\theta = 0$. The derivative there (it is certainly continuously differentiable on $(-\pi, \pi)$) is $\operatorname{Re}(Au^+, v)$ which must therefore vanish. The same is true for iv in place of v so in fact

$$(Au^+, v) = 0 \ \forall\ v \perp u^+,\ \|v\| = 1. \tag{20.12}$$

Taking the span of these v's it follows that $(Au^+, v) = 0$ for all $v \perp u^+$ so A^+u must be a multiple of u^+ itself. Inserting this into the definition of F it follows that $Au^+ = \Lambda^+ u^+$ is an eigenvector with eigenvalue $\Lambda^+ = \sup F$.

The same argument applies to $\inf F$ if it is negative, for instance by replacing A by $-A$. This completes the proof of the Lemma. □

Proof of Theorem 15. First consider the Hilbert space $\mathcal{H}_0 = \operatorname{Nul}(A)^\perp \subset \mathcal{H}$. Then A maps $\mathcal{H}_0$ into itself, since

$$(Au, v) = (u, Av) = 0 \ \forall\ u \in \mathcal{H}_0,\ v \in \operatorname{Nul}(A) \Longrightarrow Au \in \mathcal{H}_0. \tag{20.13}$$

Moreover, A_0, which is A restricted to $\mathcal{H}_0$, is again a compact self-adjoint operator – where the compactness follows from the fact that $A(B(0,1))$ for $B(0,1) \subset \mathcal{H}_0$ is smaller than (actually of course equal to) the whole image of the unit ball.

Thus we can apply the Lemma above to A_0, with quadratic form F_0, and find an eigenvector. Let's agree to take the one associated to $\sup F_{A_0}$ unless $\sup_{A_0} < -\inf F_0$ in which case we take one associated to the inf . Now, what can go wrong here? Nothing except if $F_0 \equiv 0$. However,

Lemma 15. *In general for a self-adjoint operator on a Hilbert space*

$$F \equiv 0 \Longleftrightarrow A \equiv 0. \tag{20.14}$$

Proof. In principle F is only defined on the unit ball, but of course we can recover (Au, u) for all $u \in \mathcal{H}$ from it. Namely, if $u = 0$ it vanishes of course and otherwise

$$(Au, u) = \|u\|^2 F(\frac{u}{\|u\|}). \tag{20.15}$$

Thenxs we can recover A by 'polarization'. Since

$$2(Au, v) = (A(u+v), u+v) + i(A(u+iv, u+iv). \tag{20.16}$$

Thus if $F \equiv 0$ then $A \equiv 0$. □

So, we know that we can find an eigenvector unless $A \equiv 0$ which would imply $\mathrm{Nul}(A) = \mathcal{H}$. Now we proceed by induction. Suppose we have found N mutually orthogonal eigenvectors e_j for A all with norm 1 and eigenvectors λ_j – an orthonormal set of eigenvectors and all in $\mathcal{H}_0$. Then we consider

$$\mathcal{H}_N = \{u \in \mathcal{H}_0 = \mathrm{Nul}(A)^\perp; (u, e_j) = 0,\ j = 1, \dots, N\}. \tag{20.17}$$

From the argument above, A maps $\mathcal{H}_N$ into itself, since

$$(Au, e_j) = (u, Ae_j) = \lambda_j(u, e_j) = 0 \text{ if } u \in \mathcal{H}_N \Longrightarrow Au \in \mathcal{H}_N. \tag{20.18}$$

Moreover this restricted operator is self-adjoint and compact on $\mathcal{H}_N$ as before so we can again find an eigenvector, with eigenvalue either the max of min of the new F for $\mathcal{H}_N$. The only problem arises if $F \equiv 0$ at some stage, but then $A \equiv 0$ on $\mathcal{H}_N$ and since $\mathcal{H}_N \perp \mathrm{Nul}(A)$ this implies $\mathcal{H}_N = \{0\}$ so $\mathcal{H}_0$ must have been finite dimensional.

Thus, either $\mathcal{H}_0$ is finite dimensional or we can grind out an infinite orthonormal sequence e_i of eigenvectors of A in $\mathcal{H}_0$ with the corresponding sequence of eigenvalues such that $|\lambda_i|$ is non-increasing – since the successive F_N's are restrictions of the previous ones the max and min are getting closer to (or at least no further from) 0. In fact it follows that $\lambda_j \to 0$ in this case, since otherwise there must be one eigenvalue $\lambda \neq 0$ for which the space of eigenvectors is infinte dimensional – ruled out by the fact that $\lambda(\mathrm{Id} - \lambda^{-1}A)$ has finite dimensional null space as shown last time.

Finally then, why must this orthonormal sequence be an orthonormal basis of $\mathcal{H}_0$? If not, then we can form the closure of the span of the e_i we have constructed, $\mathcal{H}'$, and its orthocomplement in $\mathcal{H}_0$ – which would have to be non-trivial. However, as before F restricts to this space to be F' for the restriction of A' to it, which is again a compact self-adjoint operator. So, if F' is not identically zero we can again construct an eigenfunction, with non-zero eigenvalue, which contracdicts the fact the we are always choosing a largest eigenvalue, in absolute value at least. Thus in fact $F' \equiv 0$ so $A' \equiv 0$ and the eigenvectors form and orthonormal basis of $\mathrm{Nul}(A)^\perp$. This completes the proof of the theorem. □