18.102 Introduction to Functional Analysis
Spring 2009

SOLUTIONS TO PROBLEM SET 5 FOR 18.102, SPRING 2009 WAS DUE 11AM TUESDAY 17 MAR.

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You should be thinking about using Lebesgue's dominated convergence at several points below.

Problem 5.1

Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be an element of $\mathcal{L}^1(\mathbb{R})$. Define

$$f_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Show that $f_L \in \mathcal{L}^1(\mathbb{R})$ and that $\int |f_L - f| \to 0$ as $L \to \infty$.

Solution. If χ_L is the characteristic function of $[-N, N]$ then $f_L = f\chi_L$. If f_n is an absolutely summable series of step functions converging a.e. to f then $f_n\chi_L$ is absolutely summable, since $\int |f_n\chi_L| \le \int |f_n|$ and converges a.e. to f_L, so $f_L \int \mathcal{L}^1(\mathbb{R})$. Certainly $|f_L(x) - f(x)| \to 0$ for each x as $L \to \infty$ and $|f_L(x) - f(x)| \le |f_l(x)| + |f(x)| \le 2|f(x)|$ so by Lebesgue's dominated convergence, $\int |f - f_L| \to 0$.

Problem 5.2

Consider a real-valued function $f : \mathbb{R} \longrightarrow \mathbb{R}$ which is locally integrable in the sense that

$$g_L(x) = \begin{cases} f(x) & x \in [-L, L] \\ 0 & x \in \mathbb{R} \setminus [-L, L] \end{cases} \tag{5.2}$$

is Lebesgue integrable of each $L \in \mathbb{N}$.

(1) Show that for each fixed L the function

$$g_L^{(N)}(x) = \begin{cases} g_L(x) & \text{if } g_L(x) \in [-N, N] \\ N & \text{if } g_L(x) > N \\ -N & \text{if } g_L(x) < -N \end{cases} \tag{5.3}$$

is Lebesgue integrable.

(2) Show that $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$.

(3) Show that there is a sequence, h_n, of step functions such that

$$h_n(x) \to f(x) \text{ a.e. in } \mathbb{R}. \tag{5.4}$$

(4) Defining

$$h_{n,L}^{(N)} = \begin{cases} 0 & x \notin [-L, L] \\ h_n(x) & \text{if } h_n(x) \in [-N, N],\ x \in [-L, L] \\ N & \text{if } h_n(x) > N,\ x \in [-L, L] \\ -N & \text{if } h_n(x) < -N,\ x \in [-L, L] \end{cases}. \tag{5.5}$$

Show that $\int |h_{n,L}^{(N)} - g_L^{(N)}| \to 0$ as $n \to \infty$.

Solution:

(1) By definition $g_L^{(N)} = \max(-N\chi_L, \min(N\chi_L, g_L))$ where χ_L is the characteristic funciton of $-[L, L]$, thus it is in $\mathcal{L}^1(\mathbb{R})$.
(2) Clearly $g_L^{(N)}(x) \to g_L(x)$ for every x and $|g_L^{(N)}(x)| \le |g_L(x)|$ so by Dominated Convergence, $g_L^{(N)} \to g_L$ in L^1, i.e. $\int |g_L^{(N)} - g_L| \to 0$ as $N \to \infty$ since the sequence converges to 0 pointwise and is bounded by $2|g(x)|$.
(3) Let $S_{L,n}$ be a sequence of step functions converging a.e. to g_L – for example the sequence of partial sums of an absolutely summable series of step functions converging to g_L which exists by the assumed integrability. Then replacing $S_{L,n}$ by $S_{L,n}\chi_L$ we can assume that the elements all vanish outside $[-N, N]$ but still have convergence a.e. to g_L. Now take the sequence

$$h_n(x) = \begin{cases} S_{k,n-k} & \text{on } [k, -k] \setminus [(k-1), -(k-1)],\ 1 \le k \le n, \\ 0 & \text{on } \mathbb{R} \setminus [-n, n]. \end{cases} \tag{5.6}$$

This is certainly a sequence of step functions – since it is a finite sum of step functions for each n – and on $[-L, L] \setminus [-(L-1), (L-1)]$ for large integral L is just $S_{L,n-L} \to g_L$. Thus $h_n(x) \to f(x)$ outside a countable union of sets of measure zero, so also almost everywhere.
(4) This is repetition of the first problem, $h_{n,L}^{(N)}(x) \to g_L^{(N)}$ almost everywhere and $|h_{n,L}^{(N)}| \le N\chi_L$ so $g_L^{(N)} \in \mathcal{L}^1(\mathbb{R})$ and $\int |h_{n,L}^{(N)} - g_L^{(N)}| \to 0$ as $n \to \infty$.

Problem 5.3

Show that $\mathcal{L}^2(\mathbb{R})$ is a Hilbert space – since it is rather central to the course I wanted you to go through the details carefully!

First working with real functions, define $\mathcal{L}^2(\mathbb{R})$ as the set of functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ which are locally integrable and such that $|f|^2$ is integrable.

(1) For such f choose h_n and define g_L, $g_L^{(N)}$ and $h_n^{(N)}$ by (5.2), (5.3) and (5.5).
(2) Show using the sequence $h_{n,L}^{(N)}$ for fixed N and L that $g_L^{(N)}$ and $(g_L^{(N)})^2$ are in $\mathcal{L}^1(\mathbb{R})$ and that $\int |(h_{n,L}^{(N)})^2 - (g_L^{(N)})^2| \to 0$ as $n \to \infty$.
(3) Show that $(g_L)^2 \in \mathcal{L}^1(\mathbb{R})$ and that $\int |(g_L^{(N)})^2 - (g_L)^2| \to 0$ as $N \to \infty$.
(4) Show that $\int |(g_L)^2 - f^2| \to 0$ as $L \to \infty$.
(5) Show that $f, g \in \mathcal{L}^2(\mathbb{R})$ then $fg \in \mathcal{L}^1(\mathbb{R})$ and that

$$|\int fg| \le \int |fg| \le \|f\|_{L^2}\|g\|_{L^2},\ \|f\|_{L^2}^2 = \int |f|^2. \tag{5.7}$$

(6) Use these constructions to show that $\mathcal{L}^2(\mathbb{R})$ is a linear space.
(7) Conclude that the quotient space $L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N}$, where $\mathcal{N}$ is the space of null functions, is a real Hilbert space.
(8) Extend the arguments to the case of complex-valued functions.

Solution:

(1) Done. I think it should have been $h_{n,L}^{(N)}$.

(2) We already checked that $g_L^{(N)} \in \mathcal{L}^1(\mathbb{R})$ and the same argument applies to $(g_L^{(N)})$, namely $(h_{n,L}^{(N)})^2 \to g_L^{(N)}$ almost everywhere and both are bounded by $N^2\chi_L$ so by dominated convergence

$$(5.8)\quad \begin{gathered} (h_{n,L}^{(N)})^2 \to g_L^{(N)})^2 \le N^2\chi_L \text{ a.e. } \Longrightarrow g_L^{(N)})^2 \in \mathcal{L}^1(\mathbb{R}) \text{ and} \\ |h_{n,L}^{(N)})^2 - g_L^{(N)})^2| \to 0 \text{ a.e. }, \\ |h_{n,L}^{(N)})^2 - g_L^{(N)})^2| \le 2N^2\chi_L \Longrightarrow \int |h_{n,L}^{(N)})^2 - g_L^{(N)})^2| \to 0. \end{gathered}$$

(3) Now, as $N \to \infty$, $(g_L^{(N)})^2 \to (g_L)^2$ a.e. and $(g_L^{(N)})^2 \to (g_L)^2 \le f^2$ so by dominated convergence, $(g_L)^2 \in \mathcal{L}^1$ and $\int |(g_L^{(N)})^2 - (g_L)^2| \to 0$ as $N \to \infty$.

(4) The same argument of dominated convergence shows now that $g_L^2 \to f^2$ and $\int |g_L^2 - f^2| \to 0$ using the bound by $f^2 \in \mathcal{L}^1(\mathbb{R})$.

(5) What this is all for is to show that $fg \in \mathcal{L}^1(\mathbb{R})$ if f, $F = g \in \mathcal{L}^2(\mathbb{R})$ (for easier notation). Approximate each of them by sequences of step functions as above, $h_{n,L}^{(N)}$ for f and $H_{n,L}^{(N)}$ for g. Then the product sequence is in $\mathcal{L}^1$ – being a sequence of step functions – and

$$(5.9)\quad h_{n,L}^{(N)}(x)H_{n,L}^{(N)}(x) \to g_L^{(N)}(x)G_L^{(N)}(x)$$

almost everywhere and with absolute value bounded by $N^2\chi_L$. Thus by dominated convergence $g_L^{(N)}G_L^{(N)} \in \mathcal{L}^1(\mathbb{R})$. Now, let $N \to \infty$; this sequence converges almost everywhere to $g_L(x)G_L(x)$ and we have the bound

$$(5.10)\quad |g_L^{(N)}(x)G_L^{(N)}(x)| \le |f(x)F(x)|\frac{1}{2}(f^2 + F^2)$$

so as always by dominated convergence, the limit $g_L G_L \in \mathcal{L}^1$. Finally, letting $L \to \infty$ the same argument shows that $fF \in \mathcal{L}^1(\mathbb{R})$. Moreover, $|fF| \in \mathcal{L}^1(\mathbb{R})$ and

$$(5.11)\quad |\int fF| \le \int |fF| \le \|f\|_{L^2}\|F\|_{L^2}$$

where the last inequality follows from Cauchy's inequality – if you wish, first for the approximating sequences and then taking limits.

(6) So if f, $g \in \mathcal{L}^2(\mathbb{R})$ are real-value, $f + g$ is certainly locally integrable and

$$(5.12)\quad (f+g)^2 = f^2 + 2fg + g^2 \in \mathcal{L}^1(\mathbb{R})$$

by the discussion above. For constants $f \in \mathcal{L}^2(\mathbb{R})$ implies $cf \in \mathcal{L}^2(\mathbb{R})$ is directly true.

(7) The argument is the same as for $\mathcal{L}^1$ versus L^1. Namely $\int f^2 = 0$ implies that $f^2 = 0$ almost everywhere which is equivalent to $f = 0$ a@è. Then the norm is the same for all $f + h$ where h is a null function since fh and h^2 are null so $(f+h)^2 = f^2 + 2fh + h^2$. The same is true for the inner product so it follows that the quotient by null functions

$$(5.13)\quad L^2(\mathbb{R}) = \mathcal{L}^2(\mathbb{R})/\mathcal{N}$$

is a preHilbert space.

However, it remains to show completeness. Suppose $\{[f_n]\}$ is an absolutely summable series in $L^2(\mathbb{R})$ which means that $\sum_n \|f_n\|_{L^2} < \infty$. It follows that the cut-off series $f_n\chi_L$ is absolutely summable in the L^1 sense since

$$\int |f_n\chi_L| \le L^{\frac12}(\int f_n^2)^{\frac12} \tag{5.14}$$

by Cauchy's inequality. Thus if we set $F_n = \sum\limits_{k-1}^{n} f_k$ then $F_n(x)\chi_L$ converges almost everywhere for each L so in fact

$$F_n(x) \to f(x) \text{ converges almost everywhere.} \tag{5.15}$$

We want to show that $f \in \mathcal{L}^2(\mathbb{R})$ where it follows already that f is locally integrable by the completeness of L^1. Now consider the series

$$g_1 = F_1^2,\ g_n = F_n^2 - F_{n-1}^2. \tag{5.16}$$

The elements are in $\mathcal{L}^1(\mathbb{R})$ and by Cauchy's inequality for $n > 1$,

$$\int |g_n| = \int |F_n^2 - F_{n-1}|^2 \le \|F_n - F_{n-1}\|_{L^2}\|F_n + F_{n-1}\|_{L^2} \le \|f_n\|_{L^2} 2\sum_k \|f_k\|_{L^2} \tag{5.17}$$

where the triangle inequality has been used. Thus in fact the series g_n is absolutely summable in $\mathcal{L}^1$

$$\sum_n \int |g_n| \le 2(\sum_n \|f_n\|_{L^2})^2. \tag{5.18}$$

So indeed the sequence of partial sums, the F_n^2 converge to $f^2 \in \mathcal{L}^1(\mathbb{R})$. Thus $f \in \mathcal{L}^2(\mathbb{R})$ and moroever

$$\int (F_n - f)^2 = \int F_n^2 + \int f^2 - 2\int F_n f \to 0 \text{ as } n \to \infty. \tag{5.19}$$

Indeed the first term converges to $\int f^2$ and, by Cauchys inequality, the series of products $f_n f$ is absulutely summable in L^1 with limit f^2 so the third term converges to $-2\int f^2$. Thus in fact $[F_n] \to [f]$ in $L^2(\mathbb{R})$ and we have proved completeness.

(8) For the complex case we need to check linearity, assuming f is locally integrable and $|f|^2 \in \mathcal{L}^1(\mathbb{R})$. The real part of f is locally integrable and the approximation $F_L^{(N)}$ discussed above is square integrable with $(F_L^{(N)})^2 \le |f|^2$ so by dominated convergence, letting first $N \to \infty$ and then $L \to \infty$ the real part is in $\mathcal{L}^2(\mathbb{R})$. Now linearity and completeness follow from the real case.

Problem 5.4

Consider the sequence space

$$h^{2,1} = \left\{ c : \mathbb{N} \ni j \longmapsto c_j \in \mathbb{C}; \sum_j (1+j^2)|c_j|^2 < \infty \right\}. \tag{5.20}$$

(1) Show that

$$h^{2,1} \times h^{2,1} \ni (c,d) \longmapsto \langle c,d\rangle = \sum_j (1+j^2)c_j\overline{d_j} \tag{5.21}$$

is an Hermitian inner form which turns $h^{2,1}$ into a Hilbert space.

(2) Denoting the norm on this space by $\|\cdot\|_{2,1}$ and the norm on l^2 by $\|\cdot\|_2$, show that

$$h^{2,1} \subset l^2,\ \|c\|_2 \le \|c\|_{2,1}\ \forall\ c \in h^{2,1}. \tag{5.22}$$

Solution:

(1) The inner product is well defined since the series defining it converges absolutely by Cauchy's inequality:

$$\begin{gathered}\langle c, d\rangle = \sum_j (1+j^2)^{\frac12} c_j \overline{(1+j^2)^{\frac12} d_j},\\ \sum_j |(1+j^2)^{\frac12} c_j \overline{(1+j^2)^{\frac12} d_j}| \le (\sum_j (1+j^2)|c_j|^2)^{\frac12} (\sum_j (1+j^2)|d_j|^2)^{\frac12}.\end{gathered} \tag{5.23}$$

It is sesquilinear and positive definite since

$$\|c\|_{2,1} = (\sum_j (1+j^2)|c_j|^2)^{\frac12} \tag{5.24}$$

only vanishes if all c_j vanish. Completeness follows as for l^2 – if $c^{(n)}$ is a Cauchy sequence then each component $c_j^{(n)}$ converges, since $(1+j)^{\frac12} c_j^{(n)}$ is Cauchy. The limits c_j define an element of $h^{2,1}$ since the sequence is bounded and

$$\sum_{j=1}^N (1+j^2)^{\frac12} |c_j|^2 = \lim_{n\to\infty} \sum_{j=1}^N (1+j^2)|c_j^{(n)}|^2 \le A \tag{5.25}$$

where A is a bound on the norms. Then from the Cauchy condition $c^{(n)} \to c$ in $h^{2,1}$ by passing to the limit as $m \to \infty$ in $\|c^{(n)} - c^{(m)}\|_{2,1} \le \epsilon$.

(2) Clearly $h^{2,2} \subset l^2$ since for any finite N

$$\sum_{j=1}^N |c_j|^2 \sum_{j=1}^N (1+j)^2 |c_j|^2 \le \|c\|_{2,1}^2 \tag{5.26}$$

and we may pass to the limit as $N \to \infty$ to see that

$$\|c\|_{l^2} \le \|c\|_{2,1}. \tag{5.27}$$

Problem 5.5

In the separable case, prove Riesz Representation Theorem directly.

Choose an orthonormal basis $\{e_i\}$ of the separable Hilbert space H. Suppose $T : H \longrightarrow \mathbb{C}$ is a bounded linear functional. Define a sequence

$$w_i = \overline{T(e_i)},\ i \in \mathbb{N}. \tag{5.28}$$

(1) Now, recall that $|Tu| \le C\|u\|_H$ for some constant C. Show that for every finite N,

$$\sum_{j=1}^N |w_i|^2 \le C^2. \tag{5.29}$$

(2) Conclude that $\{w_i\} \in l^2$ and that

$$w = \sum_i w_i e_i \in H. \tag{5.30}$$

(3) Show that

$$T(u) = \langle u, w\rangle_H \ \forall \ u \in H \text{ and } \|T\| = \|w\|_H. \tag{5.31}$$

Solution:

(1) The finite sum $w_N = \sum_{i=1}^{N} w_i e_i$ is an element of the Hilbert space with norm $\|w_N\|_N^2 = \sum_{i=1}^{N} |w_i|^2$ by Bessel's identity. Expanding out

$$T(w_N) = T(\sum_{i=1}^{N} w_i e_i) = \sum_{i=1}^{n} w_i T(e_i) = \sum_{i=1}^{N} |w_i|^2 \tag{5.32}$$

and from the continuity of T,

$$|T(w_N)| \le C\|w_N\|_H \Longrightarrow \|w_N\|_H^2 \le C\|w_N\|_H \Longrightarrow \|w_N\|^2 \le C^2 \tag{5.33}$$

which is the desired inequality.

(2) Letting $N \to \infty$ it follows that the infinite sum converges and

$$\sum_i |w_i|^2 \le C^2 \Longrightarrow w = \sum_i w_i e_i \in H \tag{5.34}$$

since $\|w_N - w\| \le \sum_{j>N} |w_i|^2$ tends to zero with N.

(3) For any $u \in H$ $u_N = \sum_{i=1}^{N} \langle u, e_i\rangle e_i$ by the completness of the $\{e_i\}$ so from the continuity of T

$$\begin{aligned} T(u) = \lim_{N\to\infty} T(u_N) &= \lim_{N\to\infty} \sum_{i=1}^{N} \langle u, e_i\rangle T(e_i) \\ &= \lim_{N\to\infty} \sum_{i=1}^{N} \langle u, w_i e_i\rangle = \lim_{N\to\infty} \langle u, w_N\rangle = \langle u, w\rangle \end{aligned} \tag{5.35}$$

where the continuity of the inner product has been used. From this and Cauchy's inequality it follows that $\|T\| = \sup_{\|u\|_H=1} |T(u)| \le \|w\|$. The converse follows from the fact that $T(w) = \|w\|_H^2$.

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