18.091 Lab 1

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1 Experiment

The objective of this experiment is to observe the behavior of the iterates of a few different functions on the computer platform. We utilize Mathematica to generate and graph the orbits of these functions with various seeds. A secondary goal along with these observations is to identify the limitations of computer precision over the course of a large number of calculations.

For each function we calculate 100 iterates for each seed and determine whether there is a pattern. The first function is $f(x) = x^2 - 2$, the quadratic map with $c = -2$. With seeds chosen from $(-2, 2)$, the important characteristic is that $f(x) \in (-2, 2)$ if $x \in (-2, 2)$. As can be seen below in the data table, generic values chosen within that interval exhibit chaotic behavior on iteration. The fixed points of the function are those x such that $x^2 - 2 = x$ or $x = 2, -1$, so these do not exhibit chaotic behavior. One can also calculate analytically the n-cycles of f, for example, the two non-fixed 2-cycles of f are those solutions to $(x^2 - 2)^2 - 2 = x$, namely $x = \frac{1}{2}(-1 \pm \sqrt{5})$. This is not necessary for the present purposes.

When $c < -2$ for the quadratic map, different behavior is observed, since the function values are no longer restricted to $(-2, 2)$ like $x^2 - 2$. All of the chosen points tend to infinity. Graphically, when points exit the $[-2, 2] \times [-2, 2]$ window, the orbits spiral out to infinity. This increase can be observed simply, since if $|x| > |c| > 2$, then $x^2 - c > |x|$, and so forth.

The doubling function $f(x) \equiv 2x \pmod 1$ has perhaps the most interesting behavior. Rational seeds behave as predicted via Mathematica's exact rational arithmetic, though, somewhat curiously, decimal seeds die off (as in, drop suddenly to 0) after about 50 iterations. This actually has a very simple explanation, having to do with the way in which decimal numbers are approximated in binary representation. First, we analyze the iterates of any number of the following form under the doubling function:

$$\frac{b_1}{2} + \frac{b_2}{2^2} + \frac{b_3}{2^3} + \cdots + \frac{b_k}{2^k},$$

where each $b_i \in \{0, 1\}$. This is equivalent to the binary number $0.b_1 b_2 b_3 \ldots b_k$. Each application of the doubling function shifts the digits by one place, then taking the resulting number modulo 1 truncates the leading b_i. The result is that after k iterations of the doubling function, the last digit b_k is truncated, leaving zero. The conclusion drawn from that is that the binary representation of decimal numbers is finite, and after a particular number of applications of the doubling function (which by our calculations is about 50), the value "suddenly" drops to 0. Observe this in a graph below.

This experiment offers a first glimpse at the behavior of the quadratic map and also provides a method to test the arithmetic precision of a computer algebra system. While in the $c = -2$ case for the quadratic map, chaotic, though restricted behavior occurred, changing c by very little resulted in orbits tending to infinity. The doubling map iterated numbers in the particular way that exposed the way in which decimal numbers are approximated in memory.

2 Data and Graphs

$$f(x) = x^2 - 2$$

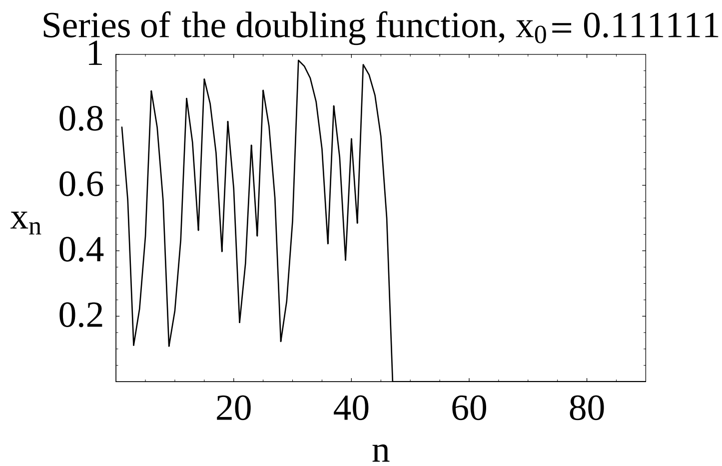
Initial Seed x_0	**Behavior**
0.1	chaotic
0.35	chaotic
1.2	chaotic
-0.5	chaotic
0.7	chaotic
0.99	chaotic
1.1	chaotic
-1.5	chaotic
0.057	chaotic
-0.6	chaotic
1.9	chaotic
2.1	overflow

$$f(x) = x^2 - 4$$

Initial Seed x_0	**Behavior**
1.01	overflow
-0.5	overflow
-3	overflow
1.5	overflow
0.3	overflow
1.7	overflow
-1.4	overflow
1.9	overflow
0.015	overflow
-0.2	overflow

$$f(x) \equiv 2x \pmod 1$$

Initial Seed x_0	**Behavior**
1/5	regular and periodic
0.2	drops to 0 after 45 iterations
0.7	drops to 0
1/9	regular and periodic
.111111	drops to 0
1/21	periodic
1/11	periodic
-0.07	drops to 0
0.001	drops to 0 after 51 iterations
1/17	periodic


Series of the doubling function, x0 = 0.111111
1
0.8
0.6
0.4
0.2
xn
20
40
60
80
n