Lab 1

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1 Introduction

The goal of this experiment was to discover the behavior of various functions under iteration under different seeds using Mathematica. The first function studied was the “doubling function,” which was defined as $F(x) = 2x \pmod 2$. The rest of the functions were the family of functions $F(x) = x^2 - c$, where c is any positive constant.

In each case, I attempted to classify the orbits as fixed, periodic, tending towards a certain limit or chaotic. To this end, I ran twenty iterations of each function suing Mathematica to see if a pattern seemed to be appearing. If unsure, I ran more iterations.

2 The Doubling Function

For testing the doubling function, I divided possible seeds into three groups. The first group was rational numbers, written in the form $p/q, p, q \epsilon \mathbb{Z}^+$. The second was rational numbers written as a decimal expansion. The last group was irrational numbers, such as the square root of 2.

2.1 Rational Numbers p/q

In the case of rational numbers written in fractional form, one of three results always showed up. Zero returned the fixed point orbit $F(x) = 0$. If $q = x^n$, for some n, the orbit became eventually fixed at 0. All other orbits were periodic or eventually periodic.

2.2 Decimal Expansions of Rational Numbers

Decimal inputs returned incorrect results due to rounding errors. Because the computer stores these numbers in the form

$$\frac{a_1}{2} + \frac{a_2}{2^2} + \frac{a_3}{2^3} + \ldots + \frac{a_n}{2^n} \tag{1}$$

after k iterations, the k^{th} term becomes equivalent to $0 \pmod 1$. Therefore, the computer falsely stated that all decimal inputs became eventually fixed at 0. For example, $x_0 = 2/5$ returned a periodic orbit, whereas $x_0 = 0.2$ became fixed.

2.3 Irrational Numbers

Irrational numbers appeared to be chaotic, at least until the rounding errors compounded enough to send the orbit to zero. When I calculated the orbit without rounding, the orbits remained chaotic.

3 $F(x) = x^2 - c, c > 0$

I started by testing $c = 2$. Once I had a feel for how the orbit behaved for different seeds, I started varying c, attempting to find a pattern that flowed between all values of c. As with the doubling function, I looked for orbits that tended towards periodicity or towards a specific orbit.

3.1 Conditions For Divergence

The first obvious characteristic of the entire family of functions was that if the seed was outside some specific interval, then it would tend towards infinity. For c=2, the interval of convergence was [-2,2]. For c=1, the range was $[\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}]$.

Theorem 1. *Let $F(x) = x^2 - c$, $c > 0$. Let $p = \frac{1+\sqrt{4c+1}}{2}$ and $q = \frac{1-\sqrt{4c+1}}{2}$. Then if*

i $x_0 > p$, $F(x_0)$ diverges under iteration.

ii $x_0 < q - 1$, $F(x_0)$ diverges under iteration.

Proof (i). Since $x_0 > p, \exists \varepsilon > 0, \varepsilon = x_0 - p$.

$$F(x_0) = F(p + \varepsilon) = (p + \varepsilon)^2 - c = p^2 + 2p\varepsilon + \varepsilon^2 - c$$

However, since $p^2 - c = p$, this simplifies to

$$F(x_0) = p + 2p\varepsilon + \varepsilon^2 > p + \varepsilon = x_0$$

Therefor, $F(x_0)$ is monotonically increasing under iteration. Furthermore, since $F(x_0) - x_0 > \varepsilon$ and ε gets bigger with every iteration, $F(x)$ is not bounded. Therefore, $F(x_0)$ diverges. ☐

(ii). Since $x_0 < q - 1 and q < 0, \exists \varepsilon > 1, \varepsilon = q - x_0$.

$$F(x_0) = F(q - \varepsilon) = (q - \varepsilon)^2 - c = q^2 - 2q\varepsilon + \varepsilon^2 - c$$

Since $q = p - \sqrt{4c+1}$ and $q^2 - c = q$,

$$F(x_0) = q - 2q\varepsilon + \varepsilon = p - \sqrt{4c+1} - 2q\varepsilon + \varepsilon^2 = p + (\varepsilon - 1)(\sqrt{4c+1} + \varepsilon)$$

Since $\varepsilon > 1$, $F(x_0) > p$, so by (i), $F(x_0)$ diverges. ☐

If $c > 2$, then all orbits tended towards infinity, unless they were exactly periodic. Because of this, all remaining discussion will be limited to $c < 2$ and values of x_0 such that $F(x) \not\to \infty$

3.2 Behavior Under Various Values of c

In general, there always existed k-cycles which could be found by solving $F^k(x) = x$. For values not lying on one of these cycles, there were three cases.

When $0 < c < 1$, the orbit tended to converge to the fixed point $p = \frac{1-\sqrt{4c+1}}{2}$, an attracting fixed point. Note that the other fixed point is repelling, so any point within the convergent interval has to go towards this attracting point.

When $c = 1$, all orbits tend towards the 2-cycle (0,1,0,1,...). As long as $|x_0| < 1.68$, the computer could not tell the difference.

The last case was $1 < c < 2$. Within this interval, all orbits appeared to become chaotic.

Table 1: Orbits of $F(x) = x^2 - c$, x_0 not on a k-cycle

Range	Behavior
$0 < c < 1$	towards a fixed point
1	towards a cycle
$1 < c < 2$	Chaotic
$c > 2$	towards infinity

4 Summary

The most important result of this experiment was the rounding issue that arose when a decimal was used as the seed of the doubling function. Unless very careful analysis is used as to how the computer is treating the number and how errors are propagated through the iterations, computer results must be taken with a large grain of salt.

¿From a stand-point of generating and studying chaos, the main result was how easily very simple, predictable systems became chaotic. The doubling function, which was completely predictable for all rational numbers, showed no patterns for irrational numbers. For the family of functions $F(x) = x^2 - c$, orbits were predictable (excluding individual k-cycles) for all values of c outside the range (1,2]. Within that range, unless the orbit was periodic, it was always chaotic.

Table 2: Doubling Function $F(x) = 2x \pmod 2$

Input	Correct Orbit	Computer's Output
0.2	periodic	goes to zero
1/5	periodic	periodic
1/9	periodic	periodic
0.23	periodic	goes to zero
5/17	periodic	periodic
0	fixed point	fixed point
1/2	eventually fixed	eventually fixed
1/10	eventually periodic	eventually periodic
1/4	eventually fixed	eventually fixed
$x \in \mathbb{R} - \mathbb{Q}$ (10 trials)	no pattern	no pattern

Table 3: $x^2 - 2$

Input	Eventual Behavior
2	Fixed
0	Eventually Fixed
-2	Eventually Fixed
-1	Fixed
1	Eventually Fixed
1/2	Chaotic
$\sqrt{2}$	eventually fixed
$\sqrt{3}$	eventually fixed
8 other tests	chaotic

Data

Table 4: $x^2 - 3$

Input	Eventual Behavior
$\sqrt{2}$	Periodic
$\sqrt{3}$	to infinity
0	to infinity
2	periodic
$\frac{1+\sqrt{13}}{2}$	fixed
1	periodic

Table 5: Summary of Data for Other Values of c

c	behavior
4	2 fixed points, all other tests to infinity
1.5	chaotic or towards infinity
1	towards 2-cycle or towards infinity
.5	towards .366025 or towards infinity