Theory of functions of one complex variable

(i) continuity: $f(z)$ is continuous in some region S if

$$\lim_{z \to z_0} f(z) = f(z_0) \text{ for any path in } S.$$

Theorem: $f(z) = u + iv$: continuous $\overset{\text{if and only if}}{\Longleftrightarrow}$ $u(x,y)$, $v(x,y)$: continuous.

(ii) "differentiability" of $f(z)$:

<u>real case:</u> $f(x)$ is differentiable at x_0 if

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) : \text{~~finite (existent)~~}$$

<u>complex:</u> $f(z)$ is differentiable at z_0 if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = f'(z_0)$$ exists and is finite and is <u>independent of the path</u> [region does not include boundaries (open sets)]

(iii) analytic function $f(z)$ in region S: (only defined in regions, not on boundary)

$f(z)$: differentiable in S, and single valued

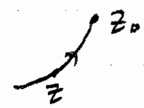
Theorem: $\boxed{\begin{array}{l}\underline{\text{Cauchy-Riemann equations}} \\ u=u(x,y),\ v=v(x,y):\text{real},\quad z=x+iy \\ \text{If } f(z)=u+iv \text{ is analytic,} \\ \dfrac{\partial u}{\partial x}=\dfrac{\partial v}{\partial y} \quad \text{and} \quad \dfrac{\partial u}{\partial y}=\dfrac{-\partial v}{\partial x}\end{array}}$

"proof": $z_0=x_0+iy_0$, $u_0=u(x_0,y_0)$, $v_0=v(x_0,y_0)$

$\Delta x = x-x_0 \qquad \Delta y = y-y_0$

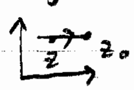
$\Delta u = u-u_0 \qquad \Delta v = v-v_0$

$$f'(z_0)=\lim_{\substack{z\to z_0 \\ (\Delta x\to 0 \\ \Delta y \to 0)}} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{\substack{(\Delta x\to 0 \\ \Delta y\to 0)}} \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$

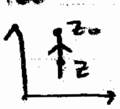


Two ways to go to z_0 (path independence):

a) $\Delta y = 0$



b) $\Delta x = 0$



$$f'(z_0)=\frac{\partial u}{\partial x}+\frac{i\partial v}{\partial x} \qquad\qquad f'(z_0)=-i\frac{\partial u}{\partial y}+\frac{\partial v}{\partial y}$$

$$\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=\frac{\partial v}{\partial y}-i\frac{\partial u}{\partial y}$$

$$\boxed{\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}}$$

u, v: harmonic functions (real and imaginary parts of analytic functions)

$$\rightarrow \begin{cases} \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}=0 \\ \dfrac{\partial^2 v}{\partial x^2}+\dfrac{\partial^2 v}{\partial y^2}=0 \end{cases} \quad \text{(laplace's equation)}$$

Proof:

$$\left\{\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 v}{\partial x\partial y}\right\}+\left\{\frac{\partial^2 u}{\partial y^2}=\frac{-\partial^2 v}{\partial x\partial y}\right\}=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

(similar for v)

ex Show that $f(z)=\bar{z}$ is not analytic.

① C-R equations: $f(z) = x - iy$ $(z = x+iy)$ $u = x,\ v = -y$

$$1 = \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} = -1 \quad \therefore f(z) \text{ cannot be analytic.}$$

② Is $f'(z_0)$ independent of the path?

$$\frac{\bar{z}-\bar{z}_0}{z-z_0} = \frac{(x-iy)-(x_0-iy_0)}{(x+iy)-(x+iy)} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}$$

$$f'(z_0) = \lim_{z\to z_0} \frac{\bar{z}-\bar{z}_0}{z-z_0} = \lim_{\substack{\Delta x\to 0\\ \Delta y \to 0}} \frac{1-i(\Delta y/\Delta x)}{1+i(\Delta y/\Delta x)}\ ; \quad m = \Delta y/\Delta x \text{ (slope as we go to } z_0)$$

$f'(z_0)$ depends on m $\rightarrow f(z)=\bar{z}$ is not analytic.

ex $f(z) = u + iv$, $v = 2xy + y$; find u, $f(z)$ so that $f(z)$: analytic

① $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ ② $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ $\quad \frac{\partial v}{\partial y} = 2x+1 \quad \frac{\partial v}{\partial x} = 2y$

$$\frac{\partial u}{\partial x} = 2x+1 \rightarrow u(x,y) = \int (2x+1)dx + \underbrace{C(y)}_{\text{arbitrary function of } y} = x^2 + x + C(y)$$

② $C'(y) = -2y \rightarrow \quad C(y) = -y^2 + K \leftarrow$ arbitrary constant

$u(x,y) = x^2 + x - y^2 + K \qquad v = 2xy + y$

$f(z) = u + iv = x^2 + x - y^2 + K + i(2xy + y)$

$$= \underbrace{x^2 - y^2 + 2ixy}_{(x+iy)^2} + x + iy + K = \boxed{z^2 + z + K} \quad (K \text{ is real})$$

Alternatively,

$z = x + iy$ $\quad x = \frac{z+\bar{z}}{2}$

$\bar{z} = x - iy$ $\quad y = \frac{z-\bar{z}}{2i}$

If done correctly, $\bar{z}$ should cancel out.